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ON

GLOBAL GEOMETRIC AND TOPOLOGICAL METHODS IN ANALYSIS

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CHANGE OF STRUCTURE GROUP IN FIBRE BUNDLES.

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One important way of constructing fibre bundles is as follows. Let G be a topological group and $H \subset G$ a closed subgroup. The space G/H of left cosets is a left G-space, the action being given by $g\cdot(g'H) = (gg')H$.

Def. A map $p: E \to B$ has a local cross section at $x \in B$ if there is a neighbourhood U of x and a map $\lambda: U \to E$ with $p \circ \lambda = id_U$

Lemma If $p: G \to G/H$ has a local cross section at H, then $p: G \to G/H$ is a locally trivial principal H-bundle.

Pf. We first produce a local cross section at any point gH in G/H. Let $\lambda: U \to G$ be a local cross section at H. Then gU is a neighbourhood of gH and $\lambda_g: gU \to G$ can be given by $\lambda_g(g'H) = g\cdot\lambda(g^{-1}g'H)$. Clearly λ_g is continuous and

$$p(\lambda_g(g'H)) = p(g\,\lambda(g^{-1}g'H)) = g\,p\,\lambda(g^{-1}g'H) = g(g^{-1}g'H) = g'H.$$

The result now follows, since the existence of a (local) section of a principal bundle is equivalent to the triviality. ∎

If further $K \subset H$ is also a closed subgroup, then H/K is a left H-space and we can construct the fibre bundle $G \overset{H}{\times} (H/K)$ over G/H with fibre H/K. But since $G \overset{H}{\times} (H/K) \cong G/K$ as H-spaces (simply send $[g, hK]$ in $G \overset{H}{\times} (G/H)$ to ghK in G/K; the inverse is $gK \longmapsto [g, K]$), one has

Theorem If $p: G \to G/H$ has a local cross section at H, then for any closed subgroup $K \subset H$, the natural projection $p': G/K \to G/H$ is a locally trivial fibre bundle with fibre H/K.

Examples.

(1) Let $O(n)$ denote the orthogonal group of $n \times n$ matrices. We have inclusions $O(n-k) \subset O(n)$, $A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I_k \end{array}\right)$

$O(n) \subset \mathbb{R}^{n^2}$ is compact and $O(n-k) \subset O(n)$ is closed.

Recall that an k-frame in $\mathbb{R}^n$ is a set of k lin. indep. vectors of $\mathbb{R}^n$. The space of all k-frames in $\mathbb{R}^n$ is the Stiefel manifold $V_k(\mathbb{R}^n)$.

Let $\varphi: O(n)/O(n-k) \longrightarrow V_k(\mathbb{R}^n)$ be defined by $\varphi(A\cdot O(n-k))$ = the last k-columns of A. Then φ can be shown to be a homeomorphism.

For instance, $O(n+1)/O(n) \cong S^n$.

If $k \le \ell$, then the diagram

$$\begin{tikzcd} O(n)/O(n-\ell) \arrow[r, "\cong"'] \arrow[d, "p'"'] & V_\ell(\mathbb{R}^n) \arrow[d, "q"] \\ O(n)/O(n-k) \arrow[r, "\cong"'] & V_k(\mathbb{R}^n) \end{tikzcd}$$

commutes, where $q(v_1,\dots,v_\ell) = (v_{\ell-k+1},\dots,v_\ell)$

We wish to show that $p: O(n) \to O(n-k)$ has a *local cross section* at $O(n-k)$. We do this by constructing a local cross section of $q: V_n(\mathbb{R}^n) \to V_k(\mathbb{R}^n)$ at $(e_{n-k+1},\dots,e_n)$. Let U be the open set of $V_k(\mathbb{R}^n)$ consisting of frames $(v_1,\dots,v_k)$ so that $(e_1,\dots,e_{n-k},v_1,\dots,v_k)$ is lin. indep. Then $\lambda: U \to V_n(\mathbb{R}^n) = O(n)$ is given as follows: using the Gram-Schmidt process we obtain an orthonormal basis $(e_1', e_2', \dots, e_{n-k}', v_1', \dots, v_k') = \lambda(v_1,\dots,v_k)$.

$$\left(v_n' = \frac{v_n}{\|v_n\|},\quad v_{n-1}' = \frac{v_{n-1} - \langle v_n', v_{n-1}\rangle \cdot v_n'}{\|\quad\|},\ \text{\& so on.} \right)$$

But clearly $v_k' = v_k, \dots, v_1' = v_1$, it follows that $q \circ \lambda = id$.

So one obtains fibre bundles

$O(n) \longrightarrow O(n)/O(n-k) = V_k(\mathbb{R}^n)$ with fibre $O(n-k)$

and more generally

$V_\ell(\mathbb{R}^n) \longrightarrow V_k(\mathbb{R}^n)$ with fibre $O(n-k)/O(n-\ell) = V_{\ell-k}(\mathbb{R}^{n-k})$

(2) $O(n-k) \times O(k) \subset O(n)$ given by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$

where $A \in O(n-k)$, $B \in O(k)$.

There is an homeomorphism

$$\psi: O(n)/O(n-k)\times O(k) \longrightarrow G_k(\mathbb{R}^n) -$$

given by $\psi(A \cdot (O(n-k)\times O(k))) =$ subspace generated by the last k columns of $A \in O(n)$. In fact, one has a commutative diagram

$$\begin{tikzcd} O(n)/O(n-k) \arrow[r, "q"] \arrow[d, "p'"'] & V_k(\mathbb{R}^n) \arrow[d, "q'"] \\ O(n)/O(n-k)\times O(k) \arrow[r, "\psi"] & G_k(\mathbb{R}^n) \end{tikzcd}$$

A local cross section can be constructed for q' at the subspace spanned by $(e_{n-k+1},\dots,e_n)$ using again Gram-Schmidt. Hence $V_k(\mathbb{R}^n) \longrightarrow G_k(\mathbb{R}^n)$ is a fibre bundle with fibre $O(n-k)\times O(k)/O(n-k) \cong O(k)$. This is the principal $O(k)$-bundle associated with $\gamma^k(\mathbb{R}^n)$.

By considering $U(n)$ the group of unitary $n\times n$ matrices and $U(n-k)\subset U(n)$ given also by $A\longmapsto\left(\begin{array}{c|c}A&0\\\hline 0&I\end{array}\right)$

one also obtains $U(n)/U(n-k)\cong V_k(\mathbb{C}^n)$ the space of k-complex frames in $\mathbb{C}^n$. Locally trivial fibre bundles can be obtained analogously:

$$U(n)\longrightarrow U(n)/U(n-k)=V_k(\mathbb{C}^n) \quad \text{with fibre } U(n-k)$$

If $l\geq k$, $\quad V_l(\mathbb{C}^n)\longrightarrow V_k(\mathbb{C}^n)$ - with fibre $U(n-k)/U(n-l)$

$$V_k(\mathbb{C}^n)\longrightarrow G_k(\mathbb{C}^n) \quad \text{with fibre } U(k).$$

Here $G_k(\mathbb{C}^n)$ is the complex Grassmann manifold of k-planes in $\mathbb{C}^n$-space.

(3) Let again ξ denote an $\mathbb{R}^n$-Riemannian vector bundle. and let $P(\xi)$ denote the associated principal $O(n)$-bundle.

The subgroup $SO(n)$ of $O(n)$ is normal and $O(n)/SO(n)\cong\mathbb{Z}_2$.

$P(\xi)$ admits a reduction to $SO(n)$ iff there is a section of the double covering $P(\xi)\times^{O(n)}(O(n)/SO(n))\cong P(\xi)\times^{O(n)}\mathbb{Z}_2$.

Such a section would trivialize the covering. The fibre over b consists of the two orientations of $F_b(\xi)$. So a reduction to $SO(n)$ amounts to choose continuously an orientation of $F_b(\xi)$.

(4) As a final application of the theorem, let us show how any $GL(n,\mathbb{R})$-bundle admits a reduction to $O(n)$, if B is a CW-comp

<u>Lemma</u> $O(n,\mathbb{R})\subset GL(n,\mathbb{R})$ is a strong def. retract.

<u>Proof.</u> Let $A\in GL(n,\mathbb{R})$. By applying the Gram-Schmidt process to A one obtains an ortogonal matrix $O(A)$. Moreover, $H(A)=O(A)^{-1}\cdot A$. is a triangular matrix $\begin{pmatrix}1 & & \alpha_{ij}\\ & \ddots & \\ 0 & & 1\end{pmatrix}$

Write $A=O(A)\cdot H(A)$

$$F: GL(n,\mathbb{R})\times I\longrightarrow GL(n,\mathbb{R}) \text{ def. by } (A,t)\longmapsto O(A)\cdot\begin{pmatrix}1 & & (1-t)\alpha_{ij}\\ & \ddots & \\ 0 & & 1\end{pmatrix}$$

is a homotopy that fixes $O(n)$. ∎

The bundle $GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})/O(n)$ gives rise to an exact sequence

$$\pi_i(O(n)) \xrightarrow{\cong} \pi_i(GL(n,\mathbb{R})) \xrightarrow[0]{} \pi_i(GL(n,\mathbb{R})/O(n)) \xrightarrow[0]{} \pi_{i-1}(O(n)) \xrightarrow{\cong}$$

implying(*) $\pi_i(GL(n,\mathbb{R})/O(n)) = 0$ for all i.

If $P \longrightarrow B$ is a principal $GL(n,\mathbb{R})$-bundle, then by obstruction theory, (*) implies that the bundle $P \overset{GL}{\times} (GL(n,\mathbb{R})/O(n))$ admits a section.

Reference Dale Husemöller, Fibre bundles. Chapters 6 & 7.

Let $H \subset G$ be closed subgroup. If $p': P' \longrightarrow B$ is a principal H-bundle, then as H acts on G one has an associated bundle $P' \overset{H}{\times} G$ over B with fibre G. In fact $P' \overset{H}{\times} G$ can be made into a right G-space, namely, $(x', g)\cdot g' = (x', gg')$

Def. A principal G-bundle $p: P \longrightarrow B$ admits a <u>reduction</u> to H if and only if there exists a principal H-bundle $p': P' \longrightarrow B$ with $P' \overset{H}{\times} G \cong P$ as G-bundles.

This means that $P' \subset P$, and in fact,
$$\begin{tikzcd} P \overset{B}{\times} P \arrow[r, "\tau"] & G \\ P' \overset{B}{\times} P' \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\tau'"] & H \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$
commutes.

Theorem. Let H be a closed subgroup of G. A principal G-bundle $p: P \longrightarrow B$ admits a reduction to H iff $q: P \overset{G}{\times} (G/H) \longrightarrow B$ admits a section.

Proof $\Rightarrow$) It is enough to produce an G-equivariant map $P \longrightarrow G/H$.

$$P \cong P' \overset{H}{\times} (G/H) \xrightarrow[\text{proj}]{} G/H$$

$\Leftarrow$) Given s;
$$\begin{tikzcd} P \overset{G}{\times} (G/H) \arrow[d] \\ B \arrow[u, bend left, "s"] \end{tikzcd}$$
This corresponds to a G-equivariant map $\varphi: P \longrightarrow G/H$.

Let $P' = \varphi^{-1}(H)$. Then P' is the required H-bundle. ∎

We now give some examples.

(1) Let $H=\{1\}$ be the trivial group. If $p: P \to B$ admits a reduction to $H=\{1\}$, then P is trivial. For the theorem would imply the existence of a section to $P \overset{G}{\times} G/_{\{1\}} \equiv P$.

(2) Let $E(\xi) \longrightarrow B$ be a $\mathbb{R}^n$ Riemannian vector bundle and let $P(\xi) \to B$ denote its associated $O(n)$-principal bundle. If $P(\xi)$ admits a reduction to $O(n-1)$, then the bundle $P(\xi) \overset{O(n)}{\times} (O(n)/O(n-1)) = P(\xi) \overset{O(n)}{\times} S^{n-1}$ admits a section. But this bundle is precisely the <u>sphere bundle</u> $S(\xi)$ of all vectors of ξ of unit length. Therefore ξ admits a non-vanishing section. More generally, $P(\xi)$ admits a reduction to $O(n-k)$ iff $P(\xi) \overset{O(n)}{\times} \left(O(n)/O(n-k)\right) = P(\xi) \overset{O(n)}{\times} V_k(\mathbb{R}^n)$ admits a section. This is the bundle of <u>k-frames of ξ.</u>

The fibre over B is the set of all k-frames of $F_b(\xi)$.

A single section to this bundle produces k-orthonormal sections of ξ.

<u>Remark</u>

These problems are very difficult to solve.