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SMR.378/2

WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

EXISTENCE RESULTS FOR THE EULER AND NAVIER-STOKES EQUATIONS FOR NON-HOMOGENEOUS AND COMPRESSIBLE FLUIDS

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Existence results for the Euler and Navier-Stokes equations for non-homogeneous and compressible fluids (*)

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1. The equations in general form.

Let us begin by writing the equations which describe the motion of a fluid in Eulerian coordinates (see for instance Serrin [32], pag.135, 132, 177)

(1.1) $\rho[v_t + (v\cdot\nabla)v - f] = \operatorname{div} T$ (conservation of momentum),

(1.2) $\rho_t + \operatorname{div}(\rho v) = 0$ (conservation of mass),

(1.3) $\rho[e_t + v\cdot\nabla e - r] = T{:}D - \operatorname{div} q$ (conservation of energy),

where

$$(\operatorname{div} T)_i \equiv \sum_j D_j T_{ji} ,$$
$$T{:}D \equiv \sum_{i,j} T_{ij} D_{ij} ,$$

ρ is the density of the fluid, v the velocity and e the internal energy per unit mass; T is the stress tensor and D is the deformation tensor

$$D_{ij} \equiv (1/2)(D_j v_i + D_i v_j) ;$$

q is the heat flux; f and r are the (assigned) external force field per unit mass and the (assigned) heat supply per unit mass per unit time, respectively.
Moreover, the following constitutive equations are assumed:

(*) Lectures held at EVEQ 88 in Štoky (Czechoslovakia), May 16-20, 1988.

(1.4) $T_{ij} = [-p + (\zeta - 2\mu/3)\operatorname{div} v]\,\delta_{ij} + 2\mu D_{ij}$ (Stokesian fluid linearly dependent on D_{ij}),

(1.5) $q = -\chi\nabla\theta$ (Fourier's law),

where p is the pressure, μ and ζ are the shear and bulk viscosity coefficients, respectively, θ is the (absolute) temperature and χ is heat conductivity coefficient.

1.A. Compressible fluids.

Choosing as thermodynamic unknowns the density ρ and the temperature θ, we must add the following constitutive equations

(1.6) $p = P(\rho,\theta)$,

(1.7) $e = E(\rho,\theta)$,

(1.8) $\mu = \mu_*(\rho,\theta)$, $\zeta = \zeta_*(\rho,\theta)$, $\chi = \chi_*(\rho,\theta)$,

where P , E , μ_* , ζ_* and χ_* are known functions subjected to the thermodynamic restrictions (Clausius-Duhem inequalities)

(1.9) $\mu_* \geq 0$, $\zeta_* \geq 0$, $\chi_* \geq 0$.

Moreover, from the well-known relation

(1.10) $dE = \theta dS - Pd(\rho^{-1})$, S specific entropy,

E and P must satisfy the compatibility condition

(1.11) $E_\rho = \rho^{-2}(P - \theta P_\theta)$.

Hence we can rewrite the equations for the unknowns ρ , v , θ :

(1.12) $$\rho[v_t + (v\cdot\nabla)v - f] = -\nabla P + \sum_j D_j(\mu_* D_j v + \mu_* \nabla v_j) + \nabla[(\zeta_* - 2\mu_*/3)\operatorname{div} v] ,$$

(1.13) $\rho_t + \operatorname{div}(\rho v) = 0$,

(1.14) $$\rho E_\theta(\theta_t + v\cdot\nabla\theta) = -\theta P_\theta \operatorname{div} v + \rho r + \operatorname{div}(\chi_*\nabla\theta) + (\mu_*/2)\sum_{i,j}(D_i v_j + D_j v_i)^2 + (\zeta_* - 2\mu_*/3)(\operatorname{div} v)^2 .$$

It must be noticed that if P, μ^* and ζ^* do not depend on θ, then equation (1.14) can be separated from (1.12), (1.13).

1.B. Incompressible fluids.

In this case pressure (and density) no longer is a thermodynamical variable (see Serrin [32], pag. 177 and 234). Hence one has to assume

(1.15) $$e = E(\theta)$$

instead of (1.7), and moreover

(1.16) $$\mathrm{div}\, v = 0$$

instead of (1.6). (The constitutive equations $\mu = \mu^*(\rho,\theta)$ and $\chi = \chi^*(\rho,\theta)$ are also supposed to hold, of course: remark on the contrary that ζ no more appears in the equations).

This last equation (1.16) describes the incompressibility of the fluid, i.e. the assumption that any given amount of fluid does not change its volume during the motion. (Hence it would be better to write incompressible flow instead of incompressible fluid).

Finally, the specific entropy S is related to E and θ by

(1.17) $$dE = \theta dS\,,$$

which substitutes equation (1.10).

One obtains in this way the following equations for the unknowns ρ, v, θ, p:

(1.18) $$\rho[v_t + (v\cdot\nabla)v - f] = -\nabla p + \sum_j D_j(\mu^* D_j v + \mu^* \nabla v_j)\,,$$

(1.19) $$\mathrm{div}\, v = 0\,,$$

(1.20) $$\rho_t + v\cdot\nabla\rho = 0\,,$$

(1.21) $$\rho E_\theta(\theta_t + v\cdot\nabla\theta) = \rho r + \mathrm{div}(\chi^*\nabla\theta) + (\mu^*/2)\sum_{i,j}(D_i v_j + D_j v_i)^2\,.$$

It must be noticed that if μ^* does not depend on θ, then equation (1.21) can be separated from (1.18)-(1.20).

Moreover, if μ^* is a constant, (1.18) takes the well-known form

(1.22) $$\rho[v_t + (v\cdot\nabla)v - f] = -\nabla p + \mu^*\Delta v\,.$$

1.C. Boundary conditions.

Several boundary conditions could be considered with respect to different physical situations. Here we want to present the most frequently used, describing the motion of a fluid in a rigid container Ω, a bounded connected domain of R^m, $m \geq 1$.

One has to distinguish between viscous and inviscid fluids.

(i) Case $\mu^* > 0$, $\zeta^* \geq 0$ (viscous fluids).

The no-slip condition

(1.23) $$v = 0 \quad \text{on } \partial\Omega$$

is assumed.

(ii) Case $\mu^* = 0$, $\zeta^* > 0$ (bulk-viscous fluids).

The slip boundary condition

(1.24) $$v\cdot n = 0 \quad \text{on } \partial\Omega$$

is assumed (here and in the sequel $n = n(x)$ denotes the unit outward normal vector to $\partial\Omega$).

It must be noticed that this case has a meaning only for compressible fluids.

(iii) Case $\mu^* = 0$, $\zeta^* = 0$ (inviscid fluids).

Condition (1.24) is assumed.

The boundary condition for the absolute temperature is different in the two cases $\chi^* > 0$ and $\chi^* = 0$.

(iv) Case $\chi^* > 0$ (conductive fluids).

One can impose

(1.25) $$\theta = \theta^* \qquad \text{on } \partial\Omega \text{ (Dirichlet)}$$

or

$$\chi_* \frac{\partial \theta}{\partial n} = q^* \qquad \text{on } \partial\Omega \text{ (Neumann)} \tag{1.26}$$

or

$$\chi_* \frac{\partial \theta}{\partial n} + k\theta = k\theta^* \qquad \text{on } \partial\Omega \text{ (third type)}, \tag{1.27}$$

where k is a given positive constant, and $\theta^* > 0$ and q^* are known functions.

(v) Case $\chi_* = 0$ (non conductive fluids).
No boundary condition has to be imposed on θ if (1.23) or (1.24) is satisfied.

1.D. Initial conditions.

If we are concerned with non-stationary problems, some initial conditions have to be added. Looking at the preceding equations (1.12)-(1.14) or (1.18)-(1.21) we see at once that one has to assign

$$v_{|t=0} = v_0(x)\,, \quad \rho_{|t=0} = \rho_0(x) > 0\,, \quad \theta_{|t=0} = \theta_0(x) > 0. \tag{1.28}$$

2. Something about the existence theorems.

Let us make precise now which problems we want to study in the sequel. For simplicity, assume that (1.14) or (1.21) can be separated from the other equations (i.e. P, μ_* and ζ_* do not depend on θ in the compressible case, or μ_* does not depend on θ in the incompressible one).
First of all, we want to underline that the theories for compressible and incompressible fluids and for viscous and inviscid fluids are strongly different.
Roughly speaking, one can say that the equations for viscous fluids (*Navier-Stokes equations*) are parabolic and that the equations for inviscid fluids (*Euler equations*) are hyperbolic.
But it is necessary to look more deeply to the equations. In fact, equations (1.13) and (1.20) concerning the density ρ are both of hyperbolic type, regardless of the viscosity.
Hence one can say more precisely that Navier-Stokes equations are hyperbolic-parabolic (or incompletely parabolic, following the definition of Belov-Yanenko [9] and Strikwerda [37]).
Let us underline now the main difference between compressible and incompressible fluids. In this second case, one has to remark that the solution of (1.20), $(1.28)_2$

(where v has to be considered as an assigned vector satisfying (1.23) or (1.24)) is given by

$$\rho(t,x) = \rho_0(U(0,t,x)), \tag{2.1}$$

where $U(t,s,x)$ is the solution of

$$\begin{cases} U(t,s,x)_t = v(t,U(t,s,x)) \\ U(s,s,x) = x. \end{cases} \tag{2.2}$$

($U(t,s,x)$ is usually called the flow of the vector field $v(t,x)$).
Hence directly from (2.1) one obtains that $0 < \min\rho_0 \le \rho(t,x) \le \max\rho_0$. This fact has several consequences: the most important is that equation (1.18) does not degenerates; secondly, if $\rho_0(x) = \rho_*$, a positive constant, then $\rho(t,x) = \rho_*$ for each (t,x), and (1.18) becomes

$$v_t + v\cdot\nabla v - f = -\nabla p^* + \nu_*\Delta v\,, \tag{2.3}$$

where $p^* \equiv p/\rho_*$ and $\nu_* \equiv \mu_*(\rho_*)/\rho^*$.
Equations (2.3) and (1.19) are usually called Navier-Stokes (if $\mu_* > 0$) or Euler (if $\mu_* = 0$) equations for *homogeneous* incompressible fluids.
When we consider the compressible case, from (1.13), $(1.28)_2$ we get

$$\rho(t,x) = \rho_0(U(0,t,x)) \exp\Big[-\int_0^t (\operatorname{div} v)(s,U(s,t,x))ds\Big]\;; \tag{2.4}$$

hence ρ can degenerate at the finite time t^* in the point x if

$$\Big|\int_0^{t^*} (\operatorname{div} v)(s,U(s,t^*,x))ds\Big| = +\infty.$$

We can affirm that the principal problem concerning compressible fluids is to find a-priori estimates assuring the non-degeneration of the density ρ. This is obviously easier locally in time, or for small data, as we will see in the sequel.
We want to present some results concerning the existence of a (unique) solution for these non-stationary problems. Due to the lack of time, we will just give without proof the principal results for incompressible viscous fluids and compressible inviscid fluids, while for incompressible inviscid fluids and compressible viscous fluids we will enter in the proofs more in detail.

3. Incompressible viscous fluids (div $v = 0$, $\mu^* > 0$).

The principal results are due to Kazhikhov [16], Ladyzhenskaya-Solonnikov [20]; see also Simon [34], Okamoto [25], Kim [19].

Theorem A. Suppose that the viscosity coefficient μ^* is a positive constant, and that the bounded domain $\Omega \subset \mathbf{R}^3$, the initial data v_0 and ρ_0, the external force f are regular enough. Assume moreover that $\inf \rho_0(x) > 0$. Then there exists a (unique) local in time solution (v,ρ,p) to (1.19), (1.20), (1.22), (1.23), $(1.28)_{1,2}$. Moreover, $\inf \rho(t,x) = \inf \rho_0(x) > 0$. If $\Omega \subset \mathbf{R}^2$ or if v_0 and f are small enough, then the solution is global in time.

In particular, the well-known results concerning Navier-Stokes equations for homogeneous incompressible fluids are contained in this theorem (take ρ_0 equal to a positive constant).

4. Compressible inviscid fluids ($p = P(\rho)$, $\mu^* = 0$, $\zeta^* = 0$).

The principal results are due to Beirão da Veiga [2], Agemi [1]; see also Ebin [11].

Theorem B. Suppose that the bounded domain $\Omega \subset \mathbf{R}^3$, the initial data v_0 and ρ_0, the external force f and the function P are regular enough. Assume moreover that $\inf \rho_0(x) > 0$ and $P'(\xi) > 0$ for $\xi > 0$. Then there exists a (unique) local in time solution (v,ρ) to (1.12) (with $\mu^* = 0 = \zeta^*$), (1.13), (1.24), $(1.28)_{1,2}$. Moreover, $\inf \rho(t,x) > 0$.

This result has been extended to the complete system (1.12) (with $\mu^* = 0 = \zeta^*$), (1.13), (1.14) (with $\chi^* = 0$) by Schochet [29], assuming $f = 0$ and $r = 0$.

5. Incompressible inviscid fluids (div $v = 0$, $\mu^* = 0$).

Let us rewrite the system of equations

(5.1)	$\rho[v_t + (v\cdot\nabla)v - f] = -\nabla p$	in $Q_T \equiv]0,T[\times \Omega$,
(5.2)	$\rho_t + v\cdot\nabla\rho = 0$	in Q_T,
(5.3)	$\operatorname{div} v = 0$	in Q_T,
(5.4)	$v\cdot n_{\mid\partial\Omega} = 0$	on $\Sigma_T \equiv]0,T[\times \partial\Omega$,
(5.5)	$v_{\mid t=0} = v_0(x)$, $\rho_{\mid t=0} = \rho_0(x) > 0$	in Ω.

We want to present a method due to Valli-Zajaczkowski [44]. Other results were obtained by Marsden [22], Beirão da Veiga-Valli [6], [7], [8], Delort [10].

Theorem C. Let $\Omega \subset \mathbf{R}^m$ $(m \geq 2)$ be a bounded domain with $\partial\Omega \in C^{k+2}$, $v_0 \in W^{k,r}(\Omega)$, $\operatorname{div} v_0 = 0$, $v_0\cdot n_{\mid\partial\Omega} = 0$, $\rho_0 \in W^{k,r}(\Omega)$, $\inf \rho_0(x) > 0$, $f \in L^1(0,T_0;W^{k,r}(\Omega))$, $k \in N$, $k > 1+m/r$, $1<r<+\infty$. Then there exist $T^* \in]0,T_0]$ sufficiently small,

$$v \in C^0([0,T^*];W^{k,r}(\Omega)) \cap W^{1,1}(0,T^*;W^{k-1,r}(\Omega)),$$
$$\rho \in C^0([0,T^*];W^{k,r}(\Omega)) \cap C^1([0,T^*];W^{k-1,r}(\Omega)),$$
$$p \in L^1(0,T^*;W^{k+1,r}(\Omega)),$$

such that (v,ρ,p) is the (unique) solution of (5.1)-(5.5) in Q_{T^*}. Moreover, $\inf \rho(t,x) = \inf \rho_0(x) > 0$. If $f \in C^0([0,T_0];W^{k,r}(\Omega))$, then $v \in C^1([0,T^*];W^{k-1,r}(\Omega))$ and $p \in C^0([0,T^*];W^{k+1,r}(\Omega))$, hence (v,ρ,p) is a classical solution.

(Here and in the sequel $W^{k,r}(\Omega)$ denotes the usual Sobolev space of (classes of equivalence of) functions u having distributional derivatives $D^\alpha u$ $(|\alpha| \leq k)$ of r-th power summable in Ω. The corresponding norm will be denoted by $\|\cdot\|_{k,r}$ ($\|\cdot\|_k$ if $r=2$)).

Uniqueness was proved by Graffi [13] (see also Beirão da Veiga-Valli [5]).

For simplicity, let us assume that $m=k=3$, $r=2$ and $f = 0$. The general case can be treated in the same way. We will solve a sequence of linear problems, followed by a fixed point argument.

<u>Step 1.</u>

Assign a vector $v \in L^\infty(0,T;H^3(\Omega))$, $T \in]0,T_0]$, with $\operatorname{div} v = 0$ and $v\cdot n_{\mid\partial\Omega} = 0$, and solve the hyperbolic problem

(5.6)	$\rho_t + v\cdot\nabla\rho = 0$	in Q_T,
(5.7)	$\rho_{\mid t=0} = \rho_0$	in Ω.

The solution is given by (2.1). Moreover, multiplying by ρ and integrating in Ω one gets

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 = -\int_\Omega (v\cdot\nabla\rho)\rho = -\frac{1}{2}\int_\Omega v\cdot\nabla(\rho^2) = 0.$$

Repeating the same argument for the successive derivatives one obtains

$$\frac{d}{dt}\|\rho\|_3^2 \le c\,\|v\|_3\|\rho\|_3^2 .$$

Hence by Gronwall's lemma

$$\|\rho(t)\|_3 \le \|\rho_0\|_3 \exp\Big(c\int_0^t \|v(s)\|_3 ds\Big). \tag{5.8}$$

Step 2.

Solve now the Neumann problem

$$\mathrm{div}(\rho^{-1}\nabla p) = -\sum_{i,s} D_i v_s D_s v_i \equiv F \qquad \text{in } Q_T, \tag{5.9}$$

$$\rho^{-1}\nabla p\cdot n = \sum_{i,s} v_i v_s D_i n_s \equiv G \qquad \text{on } \Sigma_T, \tag{5.10}$$

$$\int_\Omega p = 0 . \tag{5.11}$$

The compatibility condition is satisfied, since

$$\mathrm{div}[(v\cdot\nabla)v] = \sum_{i,s} D_i v_s D_s v_i ,$$

$$[(v\cdot\nabla)v]\cdot n = \sum_{i,s} v_i (D_i v_s) n_s - \sum_i v_i D_i\Big(\sum_s v_s n_s\Big) = -\sum_{i,s} v_i v_s D_i n_s ,$$

as $\mathrm{div}\, v = 0$ and $\nabla(v\cdot n)$ is parallel to n on $\partial\Omega$.
Multiplying by p and integrating in Ω it is easy to find

$$\|\nabla p\|_0 \le c(\|F\|_0 + \|G\|_1).$$

Moreover, by estimating Δp in Ω and $\nabla p\cdot n$ on $\partial\Omega$

$$\|p\|_4 \le c(\|\rho F + \rho^{-1}\nabla\rho\cdot\nabla p\|_2 + \|\rho G\|_3).$$

By using an interpolation inequality to evaluate $\|\nabla p\|_2$ in terms of $\|\nabla p\|_0$ and $\|p\|_4$, one has

$$\|p\|_4 \le c(\|\rho\|_3)(\|F\|_2 + \|G\|_3) \le c_1(\|\rho\|_3)\|v\|_3^2 , \tag{5.12}$$

where c_1 is a non-decreasing function of $\|\rho\|_3$.

Step 3.

Solve, similarly to step 1,

$$w_t + (v\cdot\nabla)w = -\rho^{-1}\nabla p \qquad \text{in } Q_T, \tag{5.13}$$

$$w_{|t=0} = v_0 \qquad \text{in } \Omega . \tag{5.14}$$

One obtains as in (5.8)

$$\|w(t)\|_3 \le \Big[\|v_0\|_3 + \int_0^t \|(\rho^{-1}\nabla p)(s)\|_3 ds\Big] \exp\Big(c\int_0^t \|v(s)\|_3 ds\Big). \tag{5.15}$$

Remark now that at this level we are not able to infer that $\mathrm{div}\, w = 0$ and $w\cdot n_{|\partial\Omega} = 0$. We need another step.

Step 4.

Project w on $H \equiv \{u\in L^2(\Omega) \mid \mathrm{div}\, u = 0 \text{ and } u\cdot n_{|\partial\Omega} = 0\}$. Call πw this projection. It satisfies the same assumption imposed on v in step 1. Moreover, if $\|v(t)\|_3 \le A$ for each $t\in[0,T]$, by (5.12) and (5.15) we have

$$\|\pi w(t)\|_3 \le c_2\|w(t)\|_3 \le c_2[\|v_0\|_3 + c_3(\|\rho\|_3)A^2T]\exp(cAT),$$

where $c_2 = c_2(\Omega)$ and c_3 is a non-decreasing function of $\|\rho\|_3$. By choosing $A > c_2\|v_0\|_3$ and T small enough, we get $\|\pi w(t)\|_3 \le A$ for each $t\in[0,T]$.
We are now in a position to find a fixed point of the map $\phi: v \to \pi w$. Define

$$K_T \equiv \{v\in L^\infty(0,T;H^3(\Omega)) \mid \|v(t)\|_3 \le A \text{ a.e. in } [0,T],\ \pi v = v\}.$$

We have $\phi(K_T)\subset K_T$ if T is small enough, say $T = T^*$; moreover K_{T^*} is convex, closed in $X \equiv C^0([0,T^*];H^2(\Omega))$, and $\Phi(K_{T^*})$ is relatively compact in X. As a consequence Φ is a compact map and from Schauder's theorem there exists a fixed point $v = \pi w$.
Remark that we have not yet solved the original problem (5.1)-(5.5), but

$$w_t + (\pi w\cdot\nabla)w = -\rho^{-1}\nabla p \qquad \text{in } Q_{T^*}, \tag{5.16}$$

$$(5.17)\qquad \rho_t + \pi w\cdot\nabla\rho = 0 \qquad \text{in } Q_{T^*},$$

$$(5.18)\qquad w_{|t=0} = v_0\ ,\ \ \rho_{|t=0} = \rho_0 \qquad \text{in } \Omega,$$

$$(5.19)\qquad \mathrm{div}[(\pi w\cdot\nabla)\pi w + \rho^{-1}\nabla p] = 0 \qquad \text{in } Q_{T^*},$$

$$(5.20)\qquad [(\pi w\cdot\nabla)\pi w + \rho^{-1}\nabla p]\cdot n = 0 \qquad \text{on } \Sigma_{T^*}.$$

Hence we need to prove that $w = \pi w$. Let us show that $Qw = 0$ $(Q \equiv I - \pi)$. Apply Q to (5.16):

$$(5.21)\qquad (Qw)_t + Q[(\pi w\cdot\nabla)Qw] + Q[(\pi w\cdot\nabla)\pi w + \rho^{-1}\nabla p] = 0.$$

As (5.19), (5.20) mean $Q[(\pi w\cdot\nabla)\pi w + \rho^{-1}\nabla p] = 0$, multiplying (5.22) by Qw and integrating in Ω one has

$$(5.22)\qquad \frac{1}{2}\frac{d}{dt}\int_\Omega |Qw|^2 = 0,$$

since

$$\int_\Omega Q[(\pi w\cdot\nabla)Qw]\cdot Qw = \frac{1}{2}\int_\Omega \pi w\cdot\nabla(|Qw|^2) = 0.$$

Having assumed $Qv_0 = 0$, (5.22) gives $Qw(t) = 0$ for each $t\in[0,T^*]$.

Let us say now a few words about some <u>open problems</u>, which are probably difficult to be solved (or <u>even false</u>):

(i) global existence for $\Omega\subset R^2$, extending to non-homogeneous fluids what is already known if $\rho_0(x) = \rho_*$, a positive constant. However, one has to remark that the vorticity $\omega \equiv \mathrm{rot}\ v$ is not conserved when ρ is not constant, and the same happens for any function $f(\rho)\omega$ and $\mathrm{rot}[f(\rho)v]$. (The conservation of ω is the crucial point in the proof of global existence for homogeneous fluids, see Wolibner [45], Schaeffer [28], Yudovič [46], Kato [15]).

(ii) free boundary problems, even for homogeneous fluids. In this last case, the result is true in the class of analytic functions (see Reeder-Shinbrot [27]) in an horizontal slab. A lack of regularity appears in the approximating problem, hence it is not clear how to apply the same method for Sobolev or Hölder spaces. Moreover, admitting existence, Ebin [12] proved that the free boundary problem is not well-posed (in contrast with the fixed boundary one).

6. Compressible viscous fluids ($p = P(\rho)$, $\mu^* > 0$, $\zeta^* \geq 0$).

We want to present a global in time existence theorem which is due to Valli [41], extending some ideas of Matsumura-Nishida [23], [24]. Other local existence theorems were obtained some time before by Solonnikov [35], Tani [38], Valli [40].
It is useful to introduce the mean density

$$\rho^* \equiv \frac{1}{|\Omega|}\int_\Omega \rho(t,x), \qquad (|\Omega| \equiv \mathrm{meas}\,(\Omega)),$$

which is a positive constant in consequence of (1.13), (1.23) and $(1.28)_2$.

Theorem D. Let $\Omega\subset R^m$ $(m\leq 3)$ be a bounded domain with $\partial\Omega\in C^3$, $v_0\in H^2(\Omega)\cap H^1_0(\Omega)$, $\rho_0\in H^2(\Omega)$, $\inf\rho_0(x)>0$, $f\in L^\infty(R^+;H^1(\Omega))$, $f_t\in L^\infty(R^+;H^{-1}(\Omega))$, $P\in C^2(R^+)$, $\mu_*\in C^2(R^+)$, $\zeta_*\in C^2(R^+)$, $P'(\xi) > 0$, $\mu_*(\xi) > 0$ and $\zeta_*(\xi) \geq 0$ for $\xi > 0$. Assume that $\|v_0\|_2$, $\|\rho_0-\rho^*\|_2$ and $\sup_{R^+}(\|f(t)\|_1 + \|f_t(t)\|_{-1})$ are small enough. Then there exist

$v\in L^2_{loc}(R^+;H^3(\Omega))\cap C^0_B(R_+;H^2(\Omega))$, $v_t\in L^2_{loc}(R^+;H^1(\Omega))\cap C^0_B(R_+;L^2(\Omega))$

$\rho\in C^0_B(R_+;H^2(\Omega))$, $\rho_t\in C^0_B(R_+;H^1(\Omega))$

such that (v,ρ) is the (unique) solution of (1.12), (1.13), (1.23), $(1.28)_{1,2}$. Moreover, $\inf_{Q_\infty}\rho(t,x) > 0$.

Here $C^0_B(R_+;X)$ means the space of continuous and bounded functions on $R_+ \equiv [0,+\infty)$ valued in the Banach space X, and $R^+ \equiv (0,+\infty)$.
This results has been extended to the complete system (1.12)-(1.14) by Valli-Zajaczkowski [43], where some results on inflow-outflow problems (i.e. $v\cdot n_{|\partial\Omega} \neq 0$) can also be found.
First of all, let us assume for simplicity that $\mu_* = 1$, $\zeta_* = 2/3$, $\rho^* = 1$, $P(\xi) = \xi$, and consider the new unknown

$$(6.1)\qquad \sigma(t,x) \equiv \rho(t,x) - 1,$$

which satisfies $\int_\Omega \sigma = 0$.

We can rewrite the equations in this form

$$(6.2)\qquad v_t + (v\cdot\nabla)v - f = (\sigma + 1)^{-1}(-\nabla\sigma + \Delta v + \nabla\mathrm{div}\ v) \qquad \text{in } Q_\infty,$$

(6.3) $\sigma_t + \operatorname{div}(\sigma v) + \operatorname{div} v = 0$ in Q_∞ ,

(6.4) $v_{|\partial\Omega} = 0$ on Σ_∞ ,

(6.5) $v_{|t=0} = v_0(x)$, $\sigma_{|t=0} = \rho_0(x) - 1 > -1$ in Ω .

Let us just show how to get some global a-priori estimates in $L^\infty(R^+;H^2(\Omega))$, since the proof of the existence of a local solution can be obtained by linearization plus a fixed point argument, and is not difficult in concept, though precise estimates on the linear problem and several calculations are needed.

Consider the problem

(6.6) $v_t - \Delta v - \nabla \operatorname{div} v + \nabla\sigma = H$ in Q_∞ ,

(6.7) $\sigma_t + \operatorname{div} v = L$ in Q_∞ ,

plus (6.4) and (6.5), where

(6.8) $H \equiv -(v\cdot\nabla)v - (\sigma+1)^{-1}\sigma(-\nabla\sigma + \Delta v + \nabla \operatorname{div} v) + f$,

(6.9) $L \equiv -\sigma \operatorname{div} v - v\cdot\nabla\sigma$.

In this paragraph we will indicate by $\|\cdot\|_k$ each equivalent norm in $H^k(\Omega)$.

<u>Step 1.</u>

Eliminate the terms $\nabla\sigma$ and $\operatorname{div} v$ multiplying (6.6) by v and (6.7) by σ and integrating in Ω . Adding the two equations, from

(6.10) $$\int_\Omega \nabla\sigma\cdot v = -\int_\Omega \sigma \operatorname{div} v$$

one obtains

(6.11) $$\frac{d}{dt}(\|v\|_0^2 + \|\sigma\|_0^2) + \|\nabla v\|_0^2 \le c\,\|f\|_{-1}^2 + \text{N.L.}\ ,$$

where N.L. means some norms related to nonlinear terms (which we expect to be "good", since they will be smaller then the linear terms because we assume small initial data). This procedure can be repeated for v_t (and for tangential and "interior" derivatives of v up to order two), since in all these cases the boundary conditions permit integration by parts as in (6.10). Remark that the term $v\cdot\nabla\sigma$ contained in L (and the others similar to it which will appear in the sequel) must be integrated by parts in this way

$$\int_\Omega (v\cdot\nabla D^\alpha\sigma)D^\alpha\sigma = -\frac{1}{2}\int_\Omega \operatorname{div} v\,(D^\alpha\sigma)^2\ ,\quad |\alpha| \le 2\ .$$

Moreover, by considering separately equations (6.6) and (6.7), one obtains.

(6.12) $$\frac{d}{dt}\|v\|_1^2 + \|v\|_2^2 \le c(\|\sigma\|_1^2 + \|f\|_0^2) + \text{N.L.}\ ,$$

(6.13) $$\frac{d}{dt}\|\sigma\|_2^2 \le c\,\|v\|_3^2 + \text{N.L.}\ ,$$

Finally, $\|\sigma_t\|_1$ can be estimated directly from (6.7).

Adding all these estimates we (essentially) get

(6.14) $$\frac{d}{dt}(\|v\|_1^2 + \|\sigma\|_2^2 + \|v_t\|_0^2 + \|\sigma_t\|_0^2) + \|v\|_3^2 + \|\sigma\|_2^2 + \|v_t\|_1^2 + \|\sigma_t\|_1^2 \le$$
$$\le c(\|v\|_3^2 + \|\sigma\|_2^2) + \text{N.L.} + c(\|f\|_0^2 + \|f_t\|_{-1}^2)\ .$$

<u>Step 2.</u>

We need to estimate only one of the norms $\|v\|_3^2$ and $\|\sigma\|_2^2$ (since they are connected by (6.6)). However, as we said, we can control "interior" and tangential derivatives of v , but the normal derivatives give some difficulties. On the other hand, we shall see that the normal derivatives of σ can be estimated, whereas it is not clear how to do the same for tangential derivatives. The trick is to consider (6.6), (6.7) as a Stokes problem, i.e. to utilize the estimate

(6.15) $$\|v\|_3^2 + \|\sigma\|_2^2 \le c(\|v_t\|_1^2 + \|\operatorname{div} v\|_2^2 + \|f\|_1^2) + \text{N.L.}\ .$$

Since v_t is already controlled (see (6.11) for v_t and σ_t), the crucial point is to estimate

$$\|\operatorname{div} v\|_2^2.$$

<u>Step 3.</u>

As we said before, it is only necessary to estimate the normal derivatives of $\operatorname{div} v$. Let us begin observing that on $\partial\Omega$ we have essentially

(6.16) $$\Delta v\cdot n \cong (\nabla \operatorname{div} v)\cdot n$$

(up to first order terms and second order terms containing only one normal derivative at most).
Hence by adding to (6.6) the gradient of (6.7) multiplied by $2n$ we get

$$2(\nabla\sigma\cdot n)_t + (\nabla\sigma\cdot n) \cong - v_t\cdot n + f\cdot n + \text{N.L.} .$$

In this way we estimate $\nabla\sigma\cdot n$ and we repeat the same argument for the second order normal derivative of σ. On the other hand, by (6.6) and (6.16),

$$2\,(\nabla \mathrm{div}\, v)\cdot n \cong \nabla\sigma\cdot n + v_t\cdot n - f\cdot n ,$$

hence we have obtained "good" estimates for the normal derivatives of $\mathrm{div}\, v$.
By considering (6.14), (6.15) and these estimates on $\mathrm{div}\, v$ (plus Stokes problem in local coordinates near the boundary for controlling the tangential derivatives of D^2v), we get

$$\varphi_t + \psi \le \text{N.L.} + c(\|f\|_1^2 + \|f_t\|_{-1}^2) , \tag{6.17}$$

where

$$\varphi = \|v\|_1^2 + \|\sigma\|_2^2 + \|v_t\|_0^2 + \|\sigma_t\|_0^2 ,$$
$$\psi = \|v\|_3^2 + \|\sigma\|_2^2 + \|v_t\|_1^2 + \|\sigma_t\|_1^2 .$$

(Remark that $\psi \ge \varphi$).

<u>Step 4.</u>
It is essential now to estimate in a "good" way the nonlinear terms. We need to obtain something like

$$\text{N.L.} \le c\psi(\varphi + \varphi^\beta) , \qquad \beta > 1 , \tag{6.18}$$

in such a way that (6.17) gives the boundedness of φ in R^+ if $\varphi_{|t=0}$ and $\|f\|_1^2 + \|f_t\|_{-1}^2$ are small enough.
(In fact , if we have

$$\varphi_t \le - \psi[1 - c(\varphi + \varphi^\beta)] + c(\|f\|_1^2 + \|f_t\|_{-1}^2) , \qquad \beta > 1 ,$$

φ cannot be unbounded for small data).
Estimate (6.18) heavily depends on the structure of the nonlinear terms. Hence initially we were optimist assuming that these terms <u>had</u> to be "good": for instance, quadratic terms in Dv would have been too strongly nonlinear. However, looking carefully at H

and L in (6.6), (6.7), we are able to get (6.18) for $\beta = 2$. Some sharp estimates concerning multiplication in Sobolev spaces must be repeatedly used.
Finally, it is easy to get

$$\|v\|_2^2 + \|\sigma\|_2^2 \le c(\varphi + \varphi^3 + \|f\|_0^2) ,$$

which is the a-priori estimate we need.

Let us finish with some remarks. The method now presented yields to the existence of stationary, periodic or almost-periodic solutions (for small f). Moreover these solutions are locally asymptotically stable (see [41] and Marcati-Valli [21]).
The existence of stationary solutions has been proved also by other approaches, the most interesting one due to Beirão da Veiga [3], [4] (see also Padula [26], Valli [42]).
The free boundary problem (local in time) was solved by Tani [39] and Secchi-Valli [31].

Let us mention some open problems:
(i) global existence (for large data) if $\Omega\subset R^2$, finding some new a-priori estimates on $\inf\rho$ and $\sup\rho$. As we already saw, this is related to good estimates for the L^∞-norm of $\mathrm{div}\, v$.
(ii) global existence for the free-boundary problem, even for small data. No result of this type is known in spatial dimension larger than one.

7. Compressible bulk-viscous fluids ($p = P(\rho)$, $\mu^* = 0$, $\zeta^* > 0$).

We want just to mention an interesting result due to Secchi [30], which is also valid for the complete system (1.12) (with $\mu^* = 0$), (1.13), (1.14) (with $\chi^* > 0$ or $\chi^* = 0$).

Theorem E. Suppose that the bounded domain $\Omega\subset R^3$, the initial data v_0 and ρ_0, the external force f, the functions P and ζ^* are regular enough. Assume moreover $\inf\rho_0(x) > 0$ and $\zeta^*(\xi) > 0$ for $\xi > 0$. Then there exists a (unique) local in time solution (v,ρ) to (1.12) (with $\mu^* = 0$), (1.13), (1.24), $(1.28)_{1,2}$. Moreover, $\inf\rho(t,x) > 0$.

One has to notice that equation (1.12) (with $\mu^* = 0$) is a second order equation which is <u>not</u> parabolic in the usual sense. On the other hand, $\mathrm{div}\, v$ essentially satisfies the heat equation with Neumann boundary condition, and this is a crucial remark in order to find the solution.

The existence of a global solution (even assuming small data) is an open problem, excepting in spatial dimension equal to one (in this last case there is no real distinction between viscous and bulk-viscous fluids).

8. One-dimensional compressible viscous fluids ($p = P(\rho)$, $\zeta^* + 4\mu^*/3 > 0$).

In the one-dimensional case a suitable change of variable yields to a simpler formulation which is an useful tool for showing the existence of a global in time solution for large data. We want to present a result due to Kazhikhov [17], [18] (for $f = 0$) (see also Kanel' [14]).

Theorem F. Let $\Omega \equiv]a,b[$, $v_0 \in H^1_0(\Omega)$, $\rho_0 \in H^1(\Omega)$, $\inf \rho_0(x) > 0$, $f \in L^1(R^+;L^\infty(\Omega))$, $P \in C^1(R^+)$, $P'(\xi) > 0$ for $\xi > 0$, $\nu^* \equiv (\zeta^* + 4\mu^*/3)$ a positive constant. Then there exist

$$v \in L^2(R^+;H^2(\Omega)) \cap C^0_B(R_+;H^1(\Omega)) , \quad v_t \in L^2(R^+;L^2(\Omega))$$
$$\rho \in C^0_B(R_+;H^1(\Omega)) , \quad \rho_t \in C^0_B(R_+;L^2(\Omega))$$

such that (v,ρ) is the (unique) solution of (1.12), (1.13), (1.23), $(1.28)_{1,2}$. Moreover, $\inf_{Q_\infty} \rho(t,x) > 0$.

Define

$$\rho^* \equiv \frac{1}{b-a} \int_a^b \rho(t,x) , \quad M \equiv (b-a)\rho^* ,$$

and set

$$y = y(t,x) \equiv \int_a^{U(0,t,x)} \rho_0(\lambda) d\lambda = \int_a^x \rho(t,\lambda) d\lambda , \qquad x = x(t,y) = a + \int_0^y \eta(t,\xi) d\xi ,$$
$$u(t,y) \equiv v(t,x) , \quad \eta(t,y) \equiv \rho^{-1}(t,x) ,$$

where $U(t,s,x)$ is defined in (2.2), (2.3). Since $\inf \rho_0(x) > 0$, the map $x \to y(0,x)$ has an inverse $\Lambda : y \to x$.
It is easy to see that equations (1.12), (1.13), (1.23), $(1.28)_{1,2}$ are transformed into

$$\text{(8.1)} \quad u_t + P(\eta^{-1})_y - (\nu^* \eta^{-1} u_y)_y = f(t, a + \int_0^y \eta(t,\xi) d\xi) \qquad \text{in } M_\infty \equiv R^+ \times]0,M[,$$

$$\text{(8.2)} \quad \eta_t = u_y \qquad \text{in } M_\infty ,$$

$$\text{(8.3)} \quad u(t,0) = u(t,M) = 0 \qquad \text{in } R^+ ,$$

$$\text{(8.4)} \quad u_{|t=0} = u_0(y) , \qquad \eta_{|t=0} = \eta_0(y) > 0 \qquad \text{in }]0,M[,$$

where $u_0(y) \equiv v_0(\Lambda(y))$, $\eta_0(y) \equiv [1/\rho_0(\Lambda(y))]$.
For simplicity choose $a = 0$, $b = \rho^* = 1$ (hence $M = 1$), $\nu^* = 1$ .We want just to present some a-priori estimates for u and η in $L^\infty(R^+;H^1(\Omega))$, which permit to find a global in time solution for large data.

Step 1.
Set

$$R(\gamma) \equiv \int_1^\gamma [P(1) - P(\xi^{-1})] d\xi$$

(which satisfies $R(\gamma) \geq 0$, since $P'(\xi) > 0$), multiply (7.1) by u and integrate over $]0,1[$. Recalling that

$$\int_0^1 P(\eta^{-1})_y u = - \int_0^1 P(\eta^{-1}) u_y = - \int_0^1 P(\eta^{-1}) \eta_t , \qquad \int_0^1 \eta_t = \int_0^1 u_y = 0 ,$$

one easily obtains

$$\text{(8.5)} \quad \frac{d}{dt} \int_0^1 [\frac{u^2}{2} + R(\eta)] + \int_0^1 \eta^{-1} u_y^2 = \int_0^1 u \, f(t, \int_0^y \eta) \leq (\sup_y |f(t, \int_0^y \eta)|) (\int_0^1 u^2)^{1/2} .$$

Hence by Gronwall's lemma

$$\text{(8.6)} \qquad \int_0^1 u^2 \leq c(u_0,\eta_0) \exp(\int_0^{+\infty} \|f(\tau)\|_\infty d\tau) ,$$

and

$$\text{(8.7)} \qquad \int_0^t \int_0^1 \eta^{-1} u_y^2 \leq c(u_0,\eta_0,f) ,$$

where $\|\cdot\|_\infty$ is the norm in $L^\infty(0,1)$.

Step 2.
We need now to find an estimate for $\inf_{M_\infty} \eta(t,y)$ and $\sup_{M_\infty} \eta(t,y)$. Since

$$\text{(8.8)} \qquad (\eta^{-1} u_y)_y = (\eta^{-1} \eta_t)_y = (\log \eta)_{yt} ,$$

multiplying (8.1) by $(\log\eta)_y$ we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\log\eta)_y^2 = \int_0^1 u_t(\log\eta)_y + \int_0^1 P(\eta^{-1})_y(\log\eta)_y - \int_0^1 f(t,\int_0^\xi \eta)(\log\eta)_y .$$

Moreover

$$\int_0^1 u_t(\log\eta)_y = \frac{d}{dt}\int_0^1 u(\log\eta)_y - \int_0^1 u(\log\eta)_{yt} = \frac{d}{dt}\int_0^1 u(\log\eta)_y + \int_0^1 \eta^{-1}u_y^2 ,$$

$$\int_0^1 P(\eta^{-1})_y(\log\eta)_y = -\int_0^1 P'(\eta^{-1})\eta^{-3}\eta_y^2 \le 0 ,$$

hence integrating over $]0,t[$

$$\frac{1}{2}\int_0^1 (\log\eta)_y^2 + \int_0^t\int_0^1 P'(\eta^{-1})\eta^{-3}\eta_y^2 = \frac{1}{2}\int_0^1 (\log\eta_0)_y^2 + \int_0^1 u(\log\eta)_y - \int_0^1 u_0(\log\eta_0)_y +$$

$$+ \int_0^t\int_0^1 \eta^{-1}u_y^2 - \int_0^t\int_0^1 f(\tau,\int_0^\xi \eta)(\log\eta)_y .$$

Using now (8.6) and (8.7) we get

$$\int_0^1 (\log\eta)_y^2 + \int_0^t\int_0^1 P'(\eta^{-1})\eta^{-3}\eta_y^2 \le c(u_0,\eta_0,f) + c\int_0^t \|f(\tau)\|_\infty \Big(\int_0^1 (\log\eta)_y^2\Big)^{1/2} ,$$

and from Gronwall's lemma

(8.9) $$\int_0^1 (\log\eta)_y^2 + \int_0^t\int_0^1 P'(\eta^{-1})\eta^{-3}\eta_y^2 \le c(u_0,\eta_0,f) .$$

On the other hand, from (8.2) and (8.3) we know that

$$\int_0^1 \eta = \int_0^1 \eta_0 = 1 ,$$

hence for each $t\in\mathbb{R}_+$ there exists a point $y_1(t)\in\,]0,1[$ such that

$$\eta(t,y_1(t)) = 1 .$$

As a consequence, Poincaré's inequality shows that for each $t\in\mathbb{R}_+$

(8.10) $$\|\log\eta(t)\|_1 \le c(u_0,\eta_0,f) .$$

By using Sobolev's embedding theorem $H^1(0,1)\subset C^0([0,1])$ we finally get

(8.11) $$0 < \inf_{M_\infty}\eta(t,y) \le \sup_{M_\infty}\eta(t,y) < +\infty .$$

<u>Step 3.</u>

From (8.7), (8.9)-(8.11) one obtains at once that for each $t\in\mathbb{R}_+$

(8.12) $$\|\eta(t)\|_1 + \int_0^t\int_0^1 \eta_y^2 + \int_0^t\int_0^1 u_y^2 \le c(u_0,\eta_0,f) ,$$

which is one of the global estimates we need.

Moreover, multiplying (8.1) by u_{yy} and integrating by parts we find

(8.13) $$\frac{1}{2}\frac{d}{dt}\int_0^1 u_y^2 + \int_0^1 \eta^{-1}u_{yy}^2 = \int_0^1 P(\eta^{-1})_y u_{yy} + \int_0^1 \eta^{-2}\eta_y u_y u_{yy} - \int_0^1 f(t,\int_0^y \eta)u_{yy} .$$

By (8.12) and interpolation

$$\int_0^1 \eta_y^2 u_y^2 \le \sup_y u_y^2 \int_0^1 \eta_y^2 \le c(u_0,\eta_0,f)\,\|u_y\|_0\,\|u_{yy}\|_0 ,$$

hence a standard argument gives

(8.14) $$\int_0^1 u_y^2 + \int_0^t\int_0^1 u_{yy}^2 \le c(u_0,\eta_0,f) ,$$

which is the last estimate we need.

Some interesting questions arise if we assume that $f = f(x)\in L^\infty(\Omega)$. We find again a global in time solution, since we can repeat the arguments we used in the proof of Theorem F considering the problem in the set Q_T for each $T\in\mathbb{R}^+$. But in general we are not able to show that (8.11) holds, since the estimate on $\log\eta$ depends on T. Can we say anything about the asymptotic behaviour of η (i.e. of the density ρ)? Is it possible to obtain that (8.11) is valid also in this case?

It is not difficult to realize that an answer to these questions has to be related to the existence of a *stationary* solution (u,η) satisfying (8.11). A result of Beirão da Veiga [4] shows that such a stationary solution does exist if and only if the pressure P and the external force field f satisfy a suitable compatibility condition. For instance, if $P(\rho) = A\rho^\gamma$ and $f(x) = \lambda$ ($A>0$, $\gamma>1$ and λ given constants), then the stationary solution satisfying (8.11) exists if and only if

$$|\lambda| < A(\gamma/\gamma\text{-}1)^{\gamma}.$$

Starting from this result, a first answer concerning the asymptotic behaviour of the density is given by Straškraba-Valli [36], where it is proved that a (regular enough) solution to (8.1)-(8.4) satisfying (8.11) exists only if the same compatibility condition on P and f discovered in [4] for the stationary solution holds. Hence, when such a condition does not hold, the global solution must asymptotically develop either vacuum or infinite density.

As a consequence, it is not possible to extend Theorem F to the case $f = f(x)$ if this compatibility condition is not satisfied. An interesting open problem is to prove this existence result when such a compatibility condition holds (for instance, choose $P(\rho) = A\rho^{\gamma}$ and f small enough in $L^{\infty}(\Omega)$). A particular case was solved by Shelukhin [33], assuming that

$$\exists\, c \geq 1: \qquad c^{-1}\xi^{-1} \leq P'(\xi) \leq c\,\xi^{-1}, \qquad \forall\, \xi > 0.$$

Under this hypothesis the compatibility condition is satisfied for each $f = f(x)$.

Another open problem is concerned with the existence of an attractor for this system of equations. The presence of the (hyperbolic) equation (8.2) makes difficult to apply the standard methods, since they are usually employed for dissipative systems. A first partial result proved in [36] shows that if a (regular enough) solution to (8.1)-(8.4) satisfying (8.11) exists, it has to converge to the stationary solution (remark that under these assumptions the compatibility condition between P and f must be satisfied).

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