INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR.378/11

WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

ON THE DIMENSION OF THE ATTRACTORS IN TWO-DIMENSIONAL TURBULENCE

R. TEMAM
Laboratoire d'Analyse Numérique
Université de Paris-Sud
91405 Orsay
France

Physica D 30 (1988) 284–296
North-Holland, Amsterdam

ON THE DIMENSION OF THE ATTRACTORS IN TWO-DIMENSIONAL TURBULENCE

P. CONSTANTIN
Department of Mathematics, University of Chicago, Chicago, IL 60637, USA

C. FOIAS
Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

R. TEMAM
Laboratoire d'Analyse Numérique, Université Paris Sud, 91405 Orsay, France

Received 6 August 1987
Revised manuscript received 20 November 1987

Using a new version of the Sobolev–Lieb–Thirring inequality, we derive an upper bound for the dimension of the universal attractor for two-dimensional space periodic Navier–Stokes equations. This estimate is optimal up to a logarithmic correction. The relevance of this estimate to turbulence and related results are also briefly discussed.

Sur la dimension des attracteurs en turbulence bidimensionnelle. En utilisant une nouvelle version des inégalités de Sobolev–Lieb–Thirring, nous établissons une borne supérieure de la dimension de l'attracteur universel des équations de Navier–Stokes bidimensionnelles avec conditions aux limites périodiques. Compte tenu de l'estimation inférieure de la dimension due à Babin et Vishik, cette estimation est alors optimale à un facteur logarithmique près. La signification de ce résultat en turbulence bidimensionnelle est discutée ainsi que quelques résultats connexes.

1. Introduction

In conventional turbulence theory a heuristic estimate of the number of degrees of freedom of a turbulent flow is given by

$$N \sim \left(\frac{l_0}{l_c}\right)^d. \tag{1.1}$$

In (1.1) l_0 denotes the linear size of the region occupied by the fluid in $\mathbb{R}^d$, $d = 2$ or 3. The length l_c is a small scale, determined by the physical properties of turbulence below which viscosity effects determine entirely the motion. The estimate (1.1) is thus simply an account of the number of parameters which is required to monitor the motion. If $d = 3$ such a scale is defined through dimensional analysis by the energy dissipation flux ϵ [14],

$$\epsilon = \nu\langle(\nabla u)^2\rangle, \tag{1.2}$$

$$l_c = \frac{\nu^{3/4}}{\epsilon^{1/4}}. \tag{1.3}$$

In (1.2) u is a velocity and $\langle\ \rangle$ denotes ensemble averaging. With an appropriate definition of "the number of degrees of freedom" and of the quantity $\langle\nabla u\rangle^2$ a rigorous upper bound of the type (1.1) was given in [6]. In the case $d = 2$ a completely different mechanism determines l_c. The role of ϵ is played by an enstrophy flux χ [1, 12],

$$\chi = \nu\langle(\Delta u)^2\rangle. \tag{1.4}$$

In two-dimensional turbulence, the amplification of vorticity gradients leads to an autonomous spectral flux of enstrophy toward larger wavenumbers, just like the amplification of enstrophy in three dimensions leads to a spectral flux of energy [26]. By dimensional analysis, the only length one can form with ν and χ is

$$l_\chi = \left(\frac{\nu^3}{\chi}\right)^{1/6}. \tag{1.5}$$

In this paper we prove a rigorous estimate of the type (1.1) for $d = 2$ and $l_c = l_\chi$ defined by (1.5). As in [7], we identify "the number of degrees of freedom" with the dimension of the universal attractor X of the d-dimensional Navier–Stokes equations. We study thus these equations in the classical functional form:

$$\frac{du}{dt} + \nu Au + B(u,u) = f, \tag{1.6}$$

$$u(0) = u_0, \tag{1.7}$$

where $\nu > 0$ is the kinematic viscosity coefficient, $f \in H$ are time independent body forces. Periodic boundary conditions are imposed. The detailed description of the functional analysis setting is given in [22, 23]. We recall that H is the L^2 space of periodic, divergence free functions in $\Omega = \prod_{i=1}^d(-L_i/2, L_i/2)$, $d = 2$, i.e.

$$H = \left\{u \mid u \in L^2(\Omega)^d,\ \ \operatorname{div} u = 0 \text{ in } \Omega,\ \ \int_\Omega u\,dx = 0,\ \ u_i|_{x_i = L_i/2} = u_i|_{x_i = -L_i/2},\ \ i = 1,2\right\}.$$

$Au = -P\Delta u$ where P is Leray's projection on divergence free vectors. Because of the periodic boundary conditions P and $-\Delta$ commute. The nonlinear term is $P((u\cdot\nabla)u) = B(u,u)$. We denote by $(\ ,\ )$ and $|\ |$ the scalar product and norm in H. We denote by $V = D(A^{1/2})$ the domain of definition of $A^{1/2}$. We denote by $((\ ,\ ))$ and $\|\ \|$ the scalar product and norm in V. The repeated eigenvalues of A are denoted by $0 < \lambda_1 \le \lambda_2 \le \cdots$. The solution $u = u(t)$ of (1.6), (1.7) is denoted $S(t)u_0$. The universal attractor X for (1.6) is defined as the largest set enjoying the properties

(i) X is bounded in H;
(ii) $S(t)X = X$ for all $t \ge 0$;
(iii) X attracts all the points of H (i.e. $\operatorname{dist}(S(t)u_0, X) \to 0$ as $t \to \infty$, $\forall\, u_0 \in H$).

In many cases, and in particular for the 2D-Navier–Stokes equations, we have instead of (iii), the stronger property

(iii′) X attracts the bounded sets of H (i.e. the convergence in (iii) is uniform for u_0 in a bounded set $\mathscr{B}$ of H). In this case, X is also the largest set enjoying the properties (i), (ii).

The need for estimating the Hausdorff and fractal dimensions of the universal attractor for dissipative PDE's (and ODE's) arises from the frequent occurrence of situations in which the geometry of the unstable nifolds of some fixed points forces many trajectories to spend inordinate amounts of time far away m their ultimate destination. Thus, in numerical experiments one encounters these unstable manifolds iich are part of X, of course) as sources of apparent randomness. There are many ODE's which display h behavior. The Minea system [20, 5] is an example. One cannot expect PDE behavior to be simpler. r instance, a nonlocal Burgers equation has Minea subsystems as inertial form and thus exhibits the ne behavior [8]. Estimates of the Hausdorff dimension $d_H(X)$ and fractal dimensions $d_M(X)$ of X in case of Navier–Stokes equations have been obtained by several authors [2, 5–7, 9, 13, 15, 21, 25]. In two-dimensional case one can express these bounds in terms of the nondimensional Grashof number

$$G = \frac{|f|}{\nu^2 \lambda_1}. \tag{1.8}$$

is a generalization [11] of the classical Grashof number in Bénard convection.
The best known upper bound on $d_M(X)$ in the periodic case is given in [25] and is $d_H(X) \le d_M(X) \le cG$, iere c is an absolute constant. In this paper we prove bounds of the type

$$d_H(X) \le d_M(X) \le cG^{2/3}(\log(G+1))^{1/3}. \tag{1.9}$$

timates involving $G^{2/3}$ were predicted to us privately by O.P. Manley already in 1981. In [2] a lower und for certain choices of the body forces f and of the geometry of the problem was given in terms of e viscosity as $d_H(X) \ge a\nu^{-4/3}$. Although this estimate is not nondimensional (it assumes a certain size of e periods and of the forces) its nondimensional counterpart would be $d_H(X) \ge cG^{2/3}$, thus bringing (1.9) r particular choices of f logarithmically close to being sharp. In [10], an upper bound of the dimensions the universal attractor for the Bénard problem was given in terms of the Prandtl, Pr, and classical rashof numbers, Gr, as $d_M(X) \le c\,\mathrm{Gr}(1+\mathrm{Pr})^2$. The present work suggests that one could expect a und of the type $c(1+\mathrm{Pr})^{4/3}\mathrm{Gr}^{2/3}(\log(\mathrm{Gr}+1))^{1/3}$. In this work we denote by c, c_0, c_1, various nstants; some of them are absolute constants and some of them depend on the aspect ratio L_1/L_2. We not attempt to optimize our estimates with respect to them. Such an endeavor would be, however, eful for low Grashof numbers.
The proof of our main result is based on the techniques of [5] applied in V instead of H and relies on a finement to L^∞ [4] of some L^p estimates for Bessel potentials of orthonormal functions due to Lieb [16].

Hausdorff and fractal dimensions of bounded invariant sets

We consider a set X satisfying:
(i) X is bounded in H;
(ii) $S(t)X = X$ for all $t \ge 0$.
et us recall the definitions of Hausdorff and fractal dimensions [18, 5].

efinition 2.1. Let $X \subset H$ be a compact set. The Hausdorff dimension of X, denoted $d_H^H(X)$, is defined by

$$d_H^H(X) = \inf\{d > 0 \mid \mu_H^d(X) = 0\},$$

where

$$\mu_H^d(X) = \lim_{r\to 0} \mu_{H,r}^d(X)$$

and

$$\mu_{H,r}^d(X) = \inf\left\{ \sum_{i=1}^k r_i^d \,\middle|\, X \subset \bigcup_{i=1}^k B_i,\quad B_i \text{ open balls in } H \text{ of radius } r_i \le r \right\}.$$

Definition 2.2. Let $X \subset H$ be compact. The fractal dimension of X, denoted $d_M^H(X)$, is defined by

$$d_M^H(X) = \limsup_{r\downarrow 0} \frac{\log(n_X^H(r))}{\log(1/r)}$$

where $n_X^H(r)$ is the minimal number of open balls in H of radius r needed to cover X.

Let X be a set satisfying (i) and (ii). Because of the regularity properties of $(S(t))$, $t > 0$ it follows that $X \subset V$ and X is compact in V. The following proposition shows that the Hausdorff and fractal dimensions of X when computed in V are the same as those computed in H.

Proposition 2.3. Let $d_H^V(X)$ and $d_M^V(X)$ denote respectively the Hausdorff and fractal dimensions of X computed in V. Assume X satisfies (i) and (ii). Then

$$d_H^H(X) = d_H^V(X), \tag{2.1}$$

$$d_M^H(X) = d_M^V(X). \tag{2.2}$$

Proof. In view of the Poincaré inequality, if $X \subset \bigcup_{i=1}^k B^V(x_i, r_i)$ then $X \subset \bigcup_{i=1}^k B^H(x_i, r_i/\sqrt{\lambda_1})$. (We denoted $B^V(x,r) = \{u \in V \mid \|u - x\| < r\}$ and $B^H(x,r) = \{u \in H \mid |u - x| < r\}$.) On the other hand, it is well known that for any $t > 0$, there exists a constant $k(t) > 0$ such that

$$\|S(t)u - S(t)v\| \le k(t)|u - v|,$$

see [22]. Taking, for instance $t = 1$ and denoting $k = k(1)$ we infer that if $X \subset \bigcup_{i=1}^m B^H(x_i, r_i)$ then $X \subset \bigcup_{i=1}^m B^V(S(1)x_i, kr_i)$.
After these observations, the proof of proposition 2.3 is straightforward.

Let us recall the estimates of $d_H^H(X)$ and $d_M^H(X)$ given in [5] (see also [6]). For each $u \in X$ we define the linearized operator

$$\mathscr{A}(u) = \nu A + B(u, \cdot) + B(\cdot, u). \tag{2.3}$$

The evolution of an infinitesimal displacement v_0 along the trajectory $S(t)u_0$ obeys

$$\frac{dv}{dt} + \mathscr{A}(S(t)u_0)v = 0, \tag{2.4}$$

$$v(0) = v_0. \tag{2.5}$$

If $|v_1(0)\wedge\cdots\wedge v_N(0)|$ is an arbitrary N-dimensional volume element at u_0, $(v_1(0)\wedge\cdots\wedge v_N(0)\in\Lambda^N H)$, its evolution in time along $S(t)u_0$ is given by

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v_1(t)\wedge\cdots\wedge v_n(t)|^2+\operatorname{Tr}(\mathscr{A}(S(t)u_0)Q(t))|v_1(t)\wedge\cdots\wedge v_N(t)|^2=0, \tag{2.6}$$

where $Q(t)$ is the orthogonal projection on the linear space spanned by the solutions $v_1(t),\dots,v_N(t)$ of (2.4) starting, at $t=0$, from $v_1(0),\dots,v_N(0)$.

It is proven by a geometrical argument (see [5, 6]) that if $(1/T)\int_0^T\operatorname{Tr}(\mathscr{A}(S(t)u_0)Q(t))\,\mathrm{d}t$ becomes and remains positive for large T and all choices of $u_0\in X$, $v_1(0),\dots,v_N(0)$, then N (resp. $\frac{3}{2}N$) is an upper bound for $d_{\mathrm{H}}^{H}(X)$ (resp. $d_{\mathrm{M}}^{H}(X)$). All the geometrical arguments can be carried over to V without change. The volume elements must, of course, be taken in $\Lambda^N V$. We obtain

Theorem 2.4. Let X satisfy (i) and (ii). Suppose $N_0\geq 1$ is an integer satisfying, for all $N\geq N_0$,

$$q_N=\liminf_{t\to\infty}\inf_{\substack{u_0\in X\\ v_1(0),\dots,v_N(0)\in V}}\frac{1}{t}\int_0^t\operatorname{Tr}(\mathscr{A}(S(s)u_0)Q(s))\,\mathrm{d}s>0, \tag{2.7}$$

where the infimum is taken over all $u_0\in X$ and linearly independent $v_1(0),\dots,v_N(0)$ unit vectors in V; $Q(s)$ depends on $v_1(0),\dots,v_N(0)$ and is the orthonormal projector in V on the span of $v_1(s),\dots,v_N(s)$, solutions of (2.4) with initial data $v_1(0),\dots,v_N(0)$. Then

$$d_{\mathrm{H}}^{V}(X)\leq N_0, \tag{2.8}$$

$$d_{\mathrm{M}}^{V}(X)\leq N\max_{1\leq j<N_0}\left(\frac{-q_j}{q_N}+1\right), \tag{2.9}$$

where N is any integer $\geq N_0$.

3. The main estimate

The eigenvalues of A can be computed explicitly and it is well known that

$$\lambda_1+\cdots+\lambda_N\geq c_1\lambda_1N^2\quad\text{for all } N\geq 1. \tag{3.1}$$

From the Rayleigh–Ritz principle follows that

$$\operatorname{Tr}AQ\geq\lambda_1+\cdots+\lambda_N \tag{3.2}$$

for any orthogonal projector Q in V of N-dimensional range.

In order to proceed, let us recall the identity

$$(B(v,v),Av)=0 \tag{3.3}$$

valid for $v\in D(A)$ [23]. Differentiating, we obtain

$$(B(u,v),Av)+(B(v,u),Av)+(B(v,v),Au)=0. \tag{3.4}$$

Thus we have

$$((B(u,v)+B(v,u),v))=-(B(v,v),Au). \tag{3.5}$$

Suppose now that Q is an orthogonal projector on V whose range is spanned by the orthonormal (in V) vectors $\phi_1,\dots,\phi_N$. Let $u\in X$. Let $L(u)$ be the linear operator

$$L(u)v=B(u,v)+B(v,u). \tag{3.6}$$

Then $\operatorname{Tr}L(u)Q$ can be computed using (3.5) as

$$\operatorname{Tr}L(u)Q=\sum_{i=1}^{N}((L(u)\phi_i,\phi_i))=-\sum_{i=1}^{N}(B(\phi_i,\phi_i),Au).$$

Now since ϕ_i, u are divergence free and since $Au=-\Delta u$, it follows that

$$\operatorname{Tr}L(u)Q=\sum_{i=1}^{N}\int_\Omega(\phi_i\cdot\nabla\phi_i)\,\Delta u\,\mathrm{d}x. \tag{3.7}$$

Let us denote by $\rho(x)$ the function

$$\rho(x)=\sum_{i=1}^{N}|\phi_i(x)|^2 \tag{3.8}$$

and by $\sigma(x)$ the function

$$\sigma(x)=\sum_{i=1}^{N}|\nabla\phi_i(x)|^2. \tag{3.9}$$

Here we used the notation $|M|^2$ for the sum of the squares of the entries of the matrix M.

It follows from (3.7) and Hölder's inequality that

$$|\operatorname{Tr}L(u)Q|\leq|\rho|^{1/2}_{L^2(\Omega)}|\sigma|^{1/2}_{L^1(\Omega)}|\Delta u|_{L^4(\Omega)}. \tag{3.10}$$

Let us notice that, since ϕ_i are orthonormal in V and $Au=-\Delta u$

$$\int_\Omega\nabla\phi_i\cdot\nabla\phi_j\,\mathrm{d}x=\delta_{ij}\ \text{(the Kronecker symbol)}. \tag{3.11}$$

Therefore, using a vectorial version of an estimate of Lieb and Thirring [17, 24] it follows that

$$|\sigma|^{1/2}_{L^1(\Omega)}\leq c_2\left(\sum_{i=1}^{N}|A\phi_i|^2\right)^{1/4}=c_2(\operatorname{Tr}AQ)^{1/4}. \tag{3.12}$$

The constant c_2 is independent of N. This fact, which is due to the orthogonality in L^2 of the vector valued functions $\nabla\phi_i$, is the main improvement achieved by the use of the Lieb–Thirring inequalities instead of standard Sobolev inequalities.

In order to estimate $|\rho|_{L^\infty(\Omega)}$ we use a L^∞ refinement of some L^p inequalities for Bessel potentials due to Lieb [16]. In this case $\rho(x)=\sum_{i=1}^{N}|\phi_i(x)|^2$ and $\psi_i = A^{1/2}\phi_i$ are orthogonal in L^2 vector valued functions. One has [4]

$$|\rho|_{L^\infty} \le c_3\left(1+\log\lambda_1^{-1}\sum_{i=1}^{N}|A\phi_i|^2\right). \tag{3.13}$$

For the convenience of the reader, we give an alternate proof of (3.13) at the end of this paper.

Now we estimate

$$\left(\int_\Omega|\Delta u(x)|^{4/3}\,dx\right)^{3/4} \le c_4\lambda_1^{-1/4}|Au|. \tag{3.14}$$

From (3.10), (3.12), (3.13) and (3.14) we obtain

$$|\mathrm{Tr}\,L(u)Q| \le c_5\left(1+\log\lambda_1^{-1}\mathrm{Tr}\,AQ\right)^{1/2}\left(\lambda_1^{-1}\mathrm{Tr}\,AQ\right)^{1/4}|Au|. \tag{3.15}$$

Now we shall apply (3.15) for $u=S(s)u_0$ with $u_0\in X$ and $Q(s)$ described in theorem 2.4. Let $t>0$. Then

$$\begin{aligned}
&\frac{1}{t}\int_0^t \mathrm{Tr}(\mathscr{A}(S(s)u_0)Q(s))\,ds\\
&\quad=\frac{\nu}{t}\int_0^t \mathrm{Tr}\,AQ(s)\,ds+\frac{1}{t}\int_0^t \mathrm{Tr}\,L(S(s)u_0)Q(s)\,ds\\
&\quad\ge\frac{\nu}{t}\int_0^t \mathrm{Tr}\,AQ(s)\,ds-c_5\frac{1}{t}\int_0^t\left(1+\log\lambda_1^{-1}\mathrm{Tr}\,AQ(s)\right)^{1/2}\left(\lambda_1^{-1}\mathrm{Tr}\,AQ(s)\right)^{1/4}|A(S(s)u_0)|\,ds\\
&\quad\ge\frac{\nu}{t}\int_0^t \mathrm{Tr}\,AQ(s)\,ds-c_5\left(\frac{1}{t}\int_0^t\left(1+\log\lambda_1^{-1}\mathrm{Tr}\,AQ(s)\right)\left(\lambda_1^{-1}\mathrm{Tr}\,AQ(s)\right)^{1/2}ds\right)^{1/2}\\
&\qquad\times\left(\frac{1}{t}\int_0^t|A(S(s)u_0)|^2\,ds\right)^{1/2}.
\end{aligned}$$

Now for each s, $x(s)=\lambda_1^{-1}\mathrm{Tr}\,AQ(s)$ satisfies $x(s)\ge c_1N^2$ (see (3.1), (3.2)) and also $x(s)\ge 1+\lambda_2/\lambda_1+\cdots+\lambda_m/\lambda_1\ge 1$. The function $g(x)=x^{1/2}(1+\log x)$ is concave on $x\ge 1/e$. Since our functions $x(s)$ verify $x(s)\ge 1$, we can apply Jensen's inequality and infer that

$$\begin{aligned}
&\frac{1}{t}\int_0^t\left(1+\log\lambda_1^{-1}\mathrm{Tr}\,AQ(s)\right)\left(\lambda_1^{-1}\mathrm{Tr}\,AQ(s)\right)^{1/2}ds\\
&\quad\le\left(1+\log\lambda_1^{-1}\frac{1}{t}\int_0^t\mathrm{Tr}\,AQ(s)\,ds\right)\left(\lambda_1^{-1}\frac{1}{t}\int_0^t\mathrm{Tr}\,AQ(s)\,ds\right)^{1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{t}\int_0^t\mathrm{Tr}\,\mathscr{A}(S(s)u_0)Q(s)\,ds\ge{}&\frac{\nu}{t}\int_0^t\mathrm{Tr}\,AQ(s)\,ds-c_5\left(1+\log\lambda_1^{-1}\frac{1}{t}\int_0^t\mathrm{Tr}\,AQ(s)\,ds\right)^{1/2}\\
&\times\left(\lambda_1^{-1}\frac{1}{t}\int_0^t\mathrm{Tr}\,AQ(s)\,ds\right)^{1/4}\left(\frac{1}{t}\int_0^t|A(S(s)u_0)|^2\,ds\right)^{1/2}.
\end{aligned}$$

Let us denote

$$m(t)=\frac{1}{t}\int_0^t x(s)\,ds=\frac{1}{\lambda_1}\frac{1}{t}\int_0^t\mathrm{Tr}\,AQ(s)\,ds$$

and let us consider also

$$C=C(t,u_0)=\frac{c_5}{\nu\lambda_1}\left(\frac{1}{t}\int_0^t|AS(s)u_0|^2\,ds\right)^{1/2}.$$

Then

$$\frac{1}{t}\int_0^t\mathrm{Tr}\,\mathscr{A}(S(s)u_0)Q(s)\,ds\ge\nu\lambda_1 m-C\nu\lambda_1 m^{1/4}(1+\log m)^{1/2}. \tag{3.16}$$

Let us define

$$\bar{C}=(\nu\lambda_1)^{-1}\limsup_{t\to\infty}\left(\sup_{u_0\in X}\frac{1}{t}\int_0^t|AS(s)u_0|^2\,ds\right)^{1/2}. \tag{3.17}$$

Then there exists a t_1 depending only on X such that

$$C(t,u_0)\le 2\bar{C},\quad\forall\, t\ge t_1,\quad\forall\, u_0\in X,$$

and for $t\ge t_1$, (3.16) yields

$$\frac{1}{t}\int_0^t\mathrm{Tr}\,\mathscr{A}(S(s)u_0)Q(s)\,ds>\nu\lambda_1 m-2\bar{C}\nu\lambda_1 m^{1/4}(1+\log m)^{1/2}.$$

By elementary calculations we find that

$$m-2\bar{C}\nu\lambda_1 m^{1/4}(1+\log m)^{1/2}\ge\frac{m}{2}-c_1\bar{C}^{4/3}(1+\log\bar{C})^{2/3},\quad\forall\, m\ge 1, \tag{3.18}$$

and hence

$$\begin{aligned}
\frac{1}{t}\int_0^t\mathrm{Tr}\,\mathscr{A}(S(s)u_0)Q(s)\,ds&\ge\frac{\nu\lambda_1}{2}m-\nu\lambda_1c_1\bar{C}^{4/3}(1+\log\bar{C})^{2/3}\\
&\ge\frac{\nu\lambda_1c_1N^2}{2}-\nu\lambda_1c_1'\bar{C}^{4/3}(1+\log\bar{C})^{2/3},\quad\forall\, u_0\in X,\quad\forall\, t\ge t_1.
\end{aligned} \tag{3.19}$$

We conclude that (see (2.7))

$$q_N\ge\frac{\nu\lambda_1c_1N^2}{2}-\nu\lambda_1c_1'\bar{C}^{4/3}(1+\log\bar{C})^{2/3},\quad\forall\, N\in\mathbb{N}.$$

If N_0 is defined by

$$N_0-1\le\left[\frac{2c_1'}{c_1}\right]^{1/2}\bar{C}^{2/3}(1+\log\bar{C})^{1/3}<N_0, \tag{3.20}$$

then $q_{N_0} > 0$ and

$$\max_{1<j<N_0} \frac{-q_j}{q_N} \le \max_{1\le j<N} \frac{-\nu\lambda_1\left|(j^2-N_0^2)\right|2}{q_N} \le \max_{1\le j\le N} \frac{-j^2+N_0^2}{N^2-N_0^2} \le \frac{N_0^2}{N^2-N_0^2}, \quad \forall N > N_0.$$

Minimizing in $N \ge N_0$ we deduce from (2.9) the following:

Theorem 3.1. Let X be a set satisfying (i) and (ii). Let $\bar{C}$ be defined in (3.19). Assume N_0 is an integer defined by (3.20).
Then

$$d_H(X) \le N_0, \tag{3.21}$$

$$d_M(X) \le 2.6 N_0. \tag{3.22}$$

Let us define the average $\langle \Delta u\rangle^2$ by

$$\langle \Delta u\rangle^2 = \lambda_1\left(\limsup_{t\to\infty}\left(\sup_{u_0\in X}\frac{1}{t}\int_0^t\left|AS(s)u_0\right|^2 \mathrm{d}s\right)^{1/2}\right)^2. \tag{3.23}$$

Via $\langle \Delta u\rangle^2$ we define the spectral enstrophy flux χ and the smallest scale of motion l_χ associated to it by

$$\chi = \nu\langle \Delta u\rangle^2 = \nu\lambda_1\left(\limsup_{t\to\infty}\left(\sup_{u_0\in X}\frac{1}{t}\int_0^t\left|AS(s)u_0\right|^2 \mathrm{d}s\right)^{1/2}\right)^2, \tag{3.24}$$

$$l_\chi = \left(\frac{\nu^3}{\chi}\right)^{1/6} \quad \text{(see [1], [12])}. \tag{3.25}$$

From (3.19) we observe that $\langle \Delta u\rangle^2 = \nu^2\lambda_1^2\bar{C}^2$, $\chi = \nu^3\lambda_1^3\bar{C}^2$, $l_\chi = \lambda_1^{-1/2}(\bar{C})^{-1/3}$. Therefore, taking $l_0 = \lambda_1^{-1/2}$

$$\left(\frac{l_0}{l_\chi}\right)^2 = \bar{C}^{2/3}. \tag{3.26}$$

With these definitions we can reformulate theorem 3.1:

Theorem 3.2. Let X satisfy (i) and (ii). Let the average $\langle \Delta u\rangle^2$ on X be defined in (3.23). Let χ and l_χ be the associated spectral enstrophy flux (3.24) and corresponding microscale (3.25). Let $l_0 = \lambda_1^{-1/2}$ be taken to represent the macroscale. Then, there exist nondimensional constants c_6, c_7 such that

$$d_H(X) \le c_6\left(\left(\left(\frac{l_0}{l_\chi}\right)^2+1\right)\left(1+\log\left(1+\frac{l_0}{l_\chi}\right)\right)\right)^{1/3}, \tag{3.27}$$

$$d_M(X) \le 2.6 c_6\left(\left(\frac{l_0}{l_\chi}\right)^2\left(1+\log\left(\frac{l_0}{l_\chi}\right)\right)\right)^{1/3}. \tag{3.28}$$

Let us give now an upper bound for $\bar{C}$. Taking the scalar product of (1.6) with Au and using (3.3) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \nu|Au|^2 \le |f||Au| \le \frac{|f|^2}{2\nu} + \nu\frac{|Au|^2}{2}.$$

Integrating we get

$$\nu\int_0^t\left|Au(s)\right|^2 \mathrm{d}s \le \|u(0)\|^2 + t\frac{|f|^2}{\nu}.$$

Now, if $u(0) = u_0$ belongs to X it follows, since X is bounded in V, say $X \subset B^V(0,\zeta)$, that

$$\frac{1}{\nu^2\lambda_1^2}\sup_{u_0\in X}\frac{1}{t}\int_0^t\left|AS(s)u_0\right|^2 \mathrm{d}s \le \frac{|f|^2}{\nu^4\lambda_1^2} + \frac{\zeta^2}{\nu^3\lambda_1^2}\frac{1}{t}.$$

Denoting by G the quantity

$$G = \frac{|f|}{\nu^2\lambda_1} \tag{3.29}$$

we obtain, taking the square roots and then $\limsup_{t\to\infty}$ in the last inequality, that

$$\bar{C} \le G. \tag{3.30}$$

In conclusion we have proved the following:

Theorem 3.3. Let X satisfy (i) and (ii). There exists a constant c_7 such that

$$d_H(X) \le c_7 G^{2/3}(1+\log G)^{1/3}, \tag{3.31}$$

$$d_M(X) \le 2.6 c_7 G^{2/3}(1+\log G)^{1/3}. \tag{3.32}$$

Remark 3.1

(i) As indicated in the introduction, A.V. Babin and M.I. Vishik provide in [2] a lower bound for the dimension of the universal attractor X of the two-dimensional Navier–Stokes equations for certain choices of f the body forces f and of the geometry (L_1/L_2 sufficiently small). This lower bound was given in terms of the viscosity as $d_H(X) \ge a\nu^{-4/3}$; although this estimate is not nondimensional, its nondimensional counterpart would be $d_H(X) \ge cG^{2/3}$, showing thus that for particular choices of f (3.31), (3.32) are logarithmically close to being optimal (for G large).

(ii) Although the example of [2] shows that X can be indeed of large dimension (as $cG^{2/3}$), it is interesting to recall here the example of C. Marchioro [19] who shows that for particular choices of f corresponding to arbitrarily large values of G, the attractor of the two-dimensional space periodic Navier–Stokes equations can be trivial, i.e. reduced to one single stationary solution which attracts all the trajectories. For the sake of completeness, we give here a very simple and short proof of the result in [19].

We take $L_1 = L_2 = L > 0$, $\bar{u} = \nu\rho\binom{0}{1}\sin(2\pi x_1/L)$, $\rho > 0$, $f = \nu\lambda_1\bar{u}$. Now $\bar{u}$ is an eigenvector of A corresponding to the first eigenvalue λ_1 ($= \lambda_2 = \lambda_3 = \lambda_4$), and since $B(\bar{u},\bar{u}) = 0$, we have

$$\nu A\bar{u} + B(\bar{u},\bar{u}) = f. \tag{3.33}$$

at $\bar{u}$ is also a stationary solution of (1.5) for this particular choice of f. It is easy to check that $\rho(\pi\sqrt{2}\,\nu^2/L^2)$, so that $G = |f|/\nu^2\lambda_1 = (1/\sqrt{2}\,2\pi)\rho$ and G proportional to ρ can be chosen arbitrarily

us show now that any solutions of (1.6) and (1.7) (for this choice of f) converges to $\bar{u}$ as $t \to \infty$. We $(t) = u(t) - \bar{u}$ and it follows readily from (1.6) and (3.33) that

$$\frac{dw}{dt} + \nu Aw + B(w, \bar{u}) + B(u, w) = 0. \tag{3.34}$$

uccessively take the scalar product in H of (3.34) with w and Aw. Recalling (3.3), (3.4) and the ity $(B(v, v), Av) = 0$, $\forall\, v \in D(A)$, we find

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 + (B(w, \bar{u}), w) = 0, \tag{3.35}$$

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu|Aw|^2 + (B(w, A\bar{u}), w) = 0. \tag{3.36}$$

nultiply (3.35) by λ_1, and subtract (3.36) to find (by using also $A\bar{u} = \lambda_1\bar{u}$)

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2 - \lambda_1|w|^2\right) + \nu\left(|Aw|^2 - \lambda_1\|w\|^2\right) = 0. \tag{3.37}$$

iave

$$|Aw|^2 - \lambda_1\|w\|^2 = \sum_{j=5}^{\infty}\lambda_j(\lambda_j - \lambda_1)(w, w_j)^2 \geq \lambda_5\sum_{j=5}^{\infty}(\lambda_j - \lambda_1)(w, w_j)^2 = \lambda_5\left(\|w\|^2 - \lambda_1|w|^2\right),$$

e $\{w_j\}$ is the sequence of orthonormal eigenvectors of A. We deduce from (3.37) that $w(t) - (w(t), w_j)w_j \to 0$ in V as $t \to \infty$. Then using also the fact that $(B(w_i, w_j), w_k) = 0$ for $1 \leq i, j, k \leq 4$, we : in (3.35)

$$(B(w, \bar{u}), w) = \left(B(w, \bar{u}), w - \sum_{j=1}^{4}(w, w_j)w_j\right) + \sum_{j=1}^{4}(w, w_j)\left(B\left(w - \sum_{k=1}^{4}(w, w_k)w_k, \bar{u}\right), w_j\right)$$

we obtain that $w(t)$ itself converges to 0 in H as $t \to \infty$.

i) Both results in (i) and (ii) above leave totally open the question of the structure of the universal ictor of the 2D-Navier–Stokes equations in the space periodic case: We do not know if it is reduced to onary solutions and their unstable manifolds or if more complex components can appear.

nally let us give, as promised, the following:

of of (3.13). For $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{R}$ and $\phi = \sum_{i=1}^{N}\xi_i\phi_i$ we have (see [3])

$$|\phi|_{L^\infty} \leq c_{10}\|\phi\|\left(\log\frac{|A\phi|^2}{\lambda_1\|\phi\|^2} + 1\right)^{1/2}.$$

Therefore for $x \in \Omega$

$$\left|\sum_{i=1}^{N}\xi_i\phi_i(x)\right|^2 \leq |\phi|_{L^\infty}^2 \leq c_{10}^2\left(\sum_{i=1}^{N}|\xi_i|^2\right)\left(\log\frac{\sum_{i=1}^{N}|\xi_i A\phi_i|^2}{\lambda_1\sum_{i=1}^{N}\xi_i^2} + 1\right)$$
$$\leq c_{10}^2\left(\sum_{i=1}^{N}|\xi_i|^2\right)\left(\log\frac{\sum_{i=1}^{N}|A\phi_i|^2}{\lambda_1} + 1\right)$$

and so, since the coefficients $\xi_1, \ldots, \xi_N$ are arbitrary,

$$\sum_{i=1}^{N}|\phi_i(x)|^2 \leq 2c_{10}^2\left(\log\frac{\sum_{i=1}^{N}|A\phi_i|^2}{\lambda_1} + 1\right),$$

which yields (3.13).

Acknowledgements

This research was partially supported by the Applied Mathematical Sciences Program of the U.S. Department of Energy, Contract DE-AC02-82ER-12049 and Grant DE-FG02-86ER-25020; by the National Science Foundation, Grant DMS-860-2031; and the Research Fund of Indiana University. Also the first author is an A.P. Sloan Research Fellow.

References

[1] G.K. Batchelor, Computation of the energy spectrum in homogeneous, two dimensional turbulence, Phys. Fluids, 12, suppl. 2 (1969) 233–239.

[2] A.V. Babin and M.I. Vishik, Les attracteurs des équations d'évolution et les estimations de leurs dimensions, Usp. Math. Nauk 38, 4 (232) (1983) 133–187.

[3] H. Brézis and T. Gallouet, Nonlinear Schrödinger evolution equations, Nonlinear Analysis Theory, Methods and Applications 4 (1980) 677–681.

[4] P. Constantin, Collective L^∞ estimates for families of functions with orthonormal derivatives, Indiana University Math. J. 36 (1987) 603–615.

[5] P. Constantin and C. Foias, Global Lyapunov exponents, Kaplan–Yorke formulas and the dimension of the attractors for 2D Navier–Stokes equations, Comm. Pure Appl. Math. XXXVIII (1985) 1–27.

[6] P. Constantin, C. Foias and R. Temam, Attractors representing turbulent flows, Memoirs of A.M.S. 53 (January, 1985) No. 314.

[7] P. Constantin, C. Foias, O. Manley and R. Temam, Determining modes and fractal dimensions of turbulent flows, J. Fluid Mech. 150 (1985) 427–440.

[8] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, Integral and inertial manifolds for dissipative PDE's, preprint.

[9] C. Foias and R. Temam, Some analytic and geometric properties of the solutions of the Navier–Stokes equations, J. Math. Pures Appl. 58 (1979) 339–368.

[10] C. Foias, O. Manley and R. Temam, Attractors for the Bénard problem: Existence and physical bounds on their fractal dimension, to appear.

[11] C. Foias, O. Manley, R. Temam and Y. Treve, Asymptotic analysis of the Navier–Stokes equations, Physica 9D (1983) 157–188.

[12] R.H. Kraichnan, Inertial ranges in two-dimensional turbulence, Phys. Fluids 10 (1967) 1417–1423.

[13] O.A. Ladyzhenskaia, On the finiteness of the dimension of bounded invariant sets for the Navier–Stokes equations and other related dissipative systems, in: The Boundary Values of Problems of Mathematical Physics and Related Questions in Functional Analysis, Seminar of the Steklov Institute, Leningrad, 14.

[14] L. Landau and I.M. Lifschitz, Fluid Mechanics (Addison–Wesley, New York, 1953).

[15] E. Lieb, On characteristic exponents in turbulence, Comm. Math. Phys. 92 (1984) 473–480.
[16] E. Lieb, An L^p bound for the Riesz and Bessel potentials of orthonormal functions, J. Funct. Analysis 51, No. 2 (1983) 159–165.
[17] E. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger equation and their relation to Sobolev inequalities, in: Studies in Mathematical Physics: Essays in Honor of Valentine Bargman, E. Lieb, B. Simon and A. Wightman, eds. (Princeton Univ. Press, Princeton, NJ, 1976).
[18] B. Mandelbrot, Fractals: Form, Chance and Dimension (Freeman, San Francisco, 1977).
[19] C. Marchioro, An example of absence of turbulence for any Reynolds number, Comm. Math. Phys. 105 (1986) 99–106.
[20] Gh. Minea, Remarques sur l'unicité de la solution stationnaire d'une équation de type Navier–Stokes, Revue Roumaine Math. Pures et Appl. 21 (1976) 1071–1075.
[21] D. Ruelle, Large volume limit distribution of characteristic exponents in turbulence, Comm. Math. Phys. 87 (1982) 287–301.
[22] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, 2nd ed. (North-Holland, Amsterdam, 1979).
[23] R. Temam, Navier–Stokes equations and nonlinear functional analysis, NSF/CBMS Regional Conference Series in Appl. Math. (SIAM, Philadelphia, 1983).
[24] R. Temam, Remarks on Lieb–Thirring–Sobolev inequalities and some generalizations, preprint.
[25] R. Temam, Infinite dimensional dynamical systems in fluid mechanics, in: Nonlinear Functional Analysis and Applications, Proc. AMS Summer Research Institute, Berkeley, CA, 1983, F. Browder, ed.
[26] H. Tennekes, A comparative pathology of atmospheric turbulence in two and three dimensions, in: Turbulence and Predictability in Geophysical Fluid Dynamics and in Climate Dynamics, Proc. Int. School of Physics, Varenna, 1983, M. Ghil, ed.