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NONLINEAR GALERKIN METHODS

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Abstract

In this article we present a new method of integration of evolution differential equations, the nonlinear Galerkin method, that is well adapted to the long term integration of such equations.

While the usual Galerkin method can be interpreted as a projection of the considered equation on a linear space, the methods that we consider here are related to the projection of the equation on a nonlinear subspace. From the practical point of view some terms have been identified as small and we just disregard them sometime (but not always).

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Introduction

Thanks to the important increase in the computing power during the last years one can now envisage to solve numerical problems that where unthinkable in a recent past. For example, in the case of dissipative evolution partial differential equations we can now hope to solve such equations for long intervals of time and for ranges of values of the physical parameters which lead to nontrivial dynamics: by this we mean that bifurcations have occured and instead of converging to a stationary solution the system may remain constantly time dependent even if the external excitation to the system is time independent. The simplest such situation occurs after a Hopf bifurcation when the system becomes time–periodic, whereas the time and the period do not appear explicitly in the system.

These new phenomena produce new problems and new challenges to numerical analysis. The long term integration of evolution equations is not an easy problem, and little has been done in the past, due in part to the lack of motivation. A large number of existing numerical integration algorithms lead to error estimates of the form $C(h)\exp(T)$, where $C(h)$ is an appropriate constant which is small for h small and $[0,T]$ is the interval of time under consideration. Such an approximation result is irrelevant for large T and, either the error analysis ought to be refined, or the error is indeed of this order of magnitude and the algorithm is not acceptable for large T's.

Another simple way to perceive the difficulty of large time integration of differential equations is the following. Assume that we want to integrate a differential equation

(0.1) $$\frac{du}{dt}(t) = F(u(t)),\ t > 0,$$

(0.2) $$u(0) \ [illegible]$$

on an interval of time of length $0(1/\epsilon)$, ϵ small. Then it is natural to introduce the new time variable $\tau = \epsilon t$ and the new unknown function $\bar{u}(\tau) = u(\epsilon t)$ that satisfies the stiff differential equation:

(0.1') $$\frac{d\bar{u}}{d\tau}(\tau) = \frac{1}{\epsilon} F(\bar{u}(\tau)) .$$

Thus, we are faced with the difficulties of stiff equations. In practice the situation will be even more complicated; we may need to consider different physical times of order $0(1)$, $0(1/\epsilon)$, $0(1/\epsilon^2),\dots,$ and we are then lead to introduce several time scales producing several stiffness parameters of different orders.

The new integration method that we propose in this article combines time and space discretization, and we would like at this point to explain the motivations of the algorithm. A classical

Galerkin method based on functions $w_1,\dots,w_m$, is a sort of projection of the equation under consideration (say (0.1)) onto the space P_mH spanned by $w_1,\dots,w_m$. In this procedure all the terms in the orthogonal space are considered as small and are neglected. On the other hand an important and well known aspect of nonlinear dynamics is the sensitive dependence to initial data: a small variation in the initial data or more generally a slight perturbation to the system may produce after a long time very important effects and a very important change in the system. Hence it is appropriate (and necessary) for a large time integration of the evolution equations to capture the effect of some of the terms that we neglect in the Galerkin method by restricting the equation under consideration to the linear space $P_mH = \text{Span}\,[w_1,\dots,w_m]$, and this is our main motivation.

The nonlinear Galerkin method that we implement here, stems from the theory of dynamical systems and inertial manifolds [3,12,2] and proceeds as follows. For time dependent motions, the permanent regime is represented by a global attractor $\mathcal{A}$ which attracts all the orbits; when the solution of (0.1) (0.2) converges to a stationary solution u_*, and this is the case usually for very dissipative systems, then $\mathcal{A}$ is reduced to the point u_*. When more complicated dynamics occur, $\mathcal{A}$ can be a more complicated set. The usual Galerkin method produces an approximation of $\mathcal{A}$ in the space P_mH ; however the inertial manifolds and approximate inertial manifolds produce nonlinear manifolds that are closer to $\mathcal{A}$ than $P_m H$, and it is natural to look for an approximation of the equation lying in such manifolds.

For nonlinear Galerkin methods, we consider a basis consisting of $2m$ functions $w_1, \dots, w_{2m}$ instead of m functions. More generally we could consider dm functions where d is a number which is fixed and not too large. The function u is approximated by u_m with a correction term z_m:

$$(0.3) \qquad u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j,\ z_m(t) = \sum_{j=m+1}^{2m} h_{jm}(t) w_j .$$

At each given time t, $z_m(t)$ is expected to be small compared to $u_m(t)$, and nearly negligeable. However on long intervals of time the effects of z_m add up and interact on u_m. Loosely speaking one of the definitions of u_m and z_m is the following

$$(0.4) \qquad \frac{du_m}{dt}(t) = P_m F(u_m(t) + z_m(t))$$

$$(0.5) \qquad (P_{2m} - P_m)(F(u_m(t)) + \nabla F(u_m(t)) \cdot (z_m(t))) = 0 .$$

Here P_j is the orthogonal projector onto the space spanned by $w_1, \dots, w_j$, in the underlying Hilbert space H $(u(t) \in H, \forall t)$; and ∇F is the Fréchet differential of F (F is a nonlinear mapping from H into itself, $\nabla F(\varphi) \in \mathcal{L}(H)$, if $\varphi \in H$). Of course

we need appropriate hypotheses which ensure that (0.1) (0.2) is a well posed and dissipative problem and which guarantee the existence and uniqueness of u_m and z_m in (0.4) (0.5). This program is carried out in detail for a class of evolution equations that includes the Kuramoto–Sivashinsky equation, the two–dimensional Navier–Stokes equations and many other equations.

The article is organized as follows. In Section 1 we describe the evolution equation that we consider; we also describe the nonlinear Galerkin method and state the convergence result for the method. Section 2 contains the proof of convergence. Section 3 contains the statement and the proof of some improved convergence result. Another nonlinear Galerkin method, slightly different from (0.4) (0.5) is studied in Section 4. Finally in Section 5 we consider the two examples mentioned above, namely the Kuramoto–Sivashinsky equation and the two–dimensional Navier–Stokes equations and we show that our results apply to these equations.

The improvements of the nonlinear Galerkin method over the usual Galerkin method is evidenced by the theoretical results in [2] (see Remark 1.1), and by numerical computations that will be reported elsewhere [9] and that shows a significant gain in computing time. This gain is explained by the fact that we appropriately disregard some terms that have been identified as small and non effective. Although the problem is totally different, the situation is reminiscent to that of incomplete Cholesky factorisation in linear algebra, where computing time is saved and accuracy is gained, by appropriately neglecting some small terms outside the main diagonals.

CONTENTS

1. Description of the nonlinear Galerkin method and main convergence result.

1.1 The evolution equation

We are given a Hilbert space H with a scalar product $(.,.)$ and a norm $|\cdot|$. The nonlinear evolution equation that we shall study has the form

$$\frac{du}{dt} + \nu Au + R(u) = 0, \tag{1.1}$$

where

$$R(u) = B(u) + Cu - f. \tag{1.2}$$

Here, $\nu > 0$ is a viscosity parameter. The operator A is a linear unbounded self–adjoint operator in H with domain $D(A)$ dense in H. We assume that A is positive closed and that A^{-1} is compact. One can then define the powers A^s of A for $s \in \mathbb{R}$; the space $D(A^s)$ is a Hilbert space when endowed with the norm $|A^s.|$. We set $V = D(A^{\frac{1}{2}})$ and we denote by $\|.\| = |A^{\frac{1}{2}}.|$ the norm on V.

Since A^{-1} is compact and self–adjoint, there exists an orthonormal basis $\{w_j\}$ of H consisting of eigenvectors of A

$$\begin{cases} Aw_j = \lambda_j w_j, \\ 0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_j \to \infty \text{ as } j \to +\infty. \end{cases} \tag{1.3}$$

The nonlinear term $R(u)$ satisfies (1.2) where $B(u) = B(u,u)$, $B(.,.)$ is a bilinear operator from $V \times V$ into V', C is a linear operator from V into H and $f \in H$. Let us denote by b the trilinear form on V given by

$$b(u,v,w) = \langle B(u,v),w \rangle_{V',V}, \ \forall u,v,w \in V.$$

We assume that

$$b(u,v,w) = -b(u,w,v), \ \forall u,v,w \in V, \tag{1.4}$$

$$|b(u,v,w)| \leq c_1 |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\| |w|^{\frac{1}{2}} \|w\|^{\frac{1}{2}}, \ \forall u,v,w \in V, \tag{1.5}$$

$$|Cu| \leq c_2 \|u\|, \ \forall u \in V, \tag{1.6}$$

where c_1, c_2 like the quantities c_i which will appear subsequently are positive constants. In addition, we require that B maps $V \times D(A)$ into H and

$$|B(u,v)| \leq c_3 |u|^{\frac{1}{2}} \|u\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}} |Av|^{\frac{1}{2}}, \ \forall u \in V, \forall v \in D(A), \tag{1.7}$$

$$|B(u,v)| \leq c_4 |u|^{\frac{1}{2}} |Au|^{\frac{1}{2}} \|v\|, \ \forall u,v \in D(A). \tag{1.8}$$

Finally, we require that $\nu A + C$ is positive, i.e.

$$((\nu A + C)u,u) \geq \alpha \|u\|^2, \ \forall u \in D(A), \tag{1.9}$$

where $\alpha > 0$.

The above abstract setting applies in particular to the Navier–Stokes equations in a bounded domain of $\mathbb{R}^2$ associated to the non slip boundary condition or to the space–periodic boundary condition. Using the same methods as for these equations (see Lions [4], Temam [10]), one can check that the initial value problem for (1.1) with initial condition

$$(1.10) \qquad u(0) = u_0, \ u_0 \in H,$$

has a unique solution $u = u(t)$ defined for all $t > 0$ and such that

$$u \in \mathscr{C}(\mathbb{R}^+;H) \cap L^2(0,T;V), \forall T > 0 .$$

Moreover, if $u_0 \in V$, then

$$u \in \mathscr{C}(\mathbb{R}^+;V) \cap L^2(0,T;D(A)), \forall T > 0 .$$

1.2 The nonlinear Galerkin method.

The method is implemented using as a basis of H the eigenvectors w_j, $j \in \mathbb{N}$, of the operator A. For every integer m, we are looking for an approximate solution of Problem (1.1)(1.10) of the form

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j,$$

$$u_m : \mathbb{R}^+ \to W_m \ = \ \text{the space spanned by } w_1,\dots,w_m .$$

The function u_m is determined by the resolution of a system involving another unknown function z_m where

$$z_m(t) = \sum_{j=m+1}^{2m} h_{jm}(t) w_j,$$

$$z_m : \mathbb{R}^+ \to \tilde{W}_m = \text{the space spanned by } w_{m+1},\dots,w_{2m} .$$

The pair (u_m, z_m) satisfies

$$(1.11) \qquad \frac{d}{dt}(u_m,v) + \nu((u_m,v)) + (Cu_m,v) + b(u_m,u_m,v) + \\ + b(z_m,u_m,v) + b(u_m,z_m,v) = (f,v), \forall v \in W_m,$$

$$(1.12) \qquad \nu((z_m,\tilde{v})) + (Cz_m,\tilde{v}) + b(u_m,u_m,\tilde{v}) = (f,\tilde{v}), \forall \tilde{v} \in \tilde{W}_m,$$

together with

$$(1.13) \qquad u_m(0) = P_m u_0,$$

where P_m is the orthogonal projector in H onto W_m.

The system (1.11) (1.12) is equivalent to an ordinary differential equation for u_m. Indeed, the equation (1.12) for z_m is linear and can be rewritten

$$(1.14) \qquad \nu A z_m + (P_{2m}-P_m)Cz_m = (P_{2m}-P_m)(f-B(u_m)) .$$

The assumption (1.9) guarantees the coerciveness and invertibility of the operator $\nu A + (P_{2m}-P_m)C$ on $\dot{W}_m$, so that z_m is explicitly given in terms of u_m by

(1.15) $$z_m = (\nu A+(P_{2m}-P_m)C)^{-1}(P_{2m}-P_m)(f-B(u_m)) .$$

Therefore, the system (1.11)(1.12) is equivalent to the following ordinary differential system

(1.16) $$\frac{du_m}{dt} + \nu Au_m + P_m(Cu_m+B(u_m) + B(z_m,u_m) + B(u_m,z_m)) = P_mf, \quad z_m \text{ given by (1.15).}$$

Note that (1.16) with $z_m = 0$ is the system obtained by the classical Galerkin method.

The existence and uniqueness of a solution u_m of (1.16)(1.13) defined on a maximal interval $[0,T_m)$ follows from standard theorems on ordinary differential equations. The a priori estimates that we derive in Section 2 below guarantee that $T_m = +\infty$. Also, they will allow us to study the limit $m \to +\infty$ and to obtain the following convergence result.

THEOREM 1.1

The hypotheses are (1.4) to (1.9). For u_0 given in H, the solution u_m of (1.16)(1.13) converges, as $m \to \infty$, to the solution u of Problem (1.1) (1.10) in the following sense:

(1.17) $u_m \to u$ *in* $L^2(0,T;V)$ *and* $L^p(0,T;H)$ *strongly, for all* $T > 0$, *and all* $1 \leq p < +\infty$,
$u_m \to u$ *in* $L^\infty(\mathbb{R}^+;H)$ *weak-star.*

REMARK 1.2

(i) We can prove that z_m is small compared to u_m. Thus at each given time $z_m(t)$ is a minor correction to $u_m(t)$. However, on long interval of times, z_m modifies u_m in a non negligible way.

(ii) It is shown in Foias, Manley and Temam [2] (where $C = 0$), that the global attractor $\mathcal{A}$ to (1.1) lies at a distance in H of P_mH bounded by $C(\lambda_1/\lambda_{m+1})$. In [2] we constructed an approximate inertial manifold $\mathcal{M}_0$ for (1.1) of equation

$$(I-P_m)\varphi = (\nu A)^{-1} (I-P_m)(f-B(P_m\varphi))$$

such that $\mathcal{A}$ lies at a distance $\leq C(\lambda_1/\lambda_{m+1})^{3/2}$ of $\mathcal{M}_0$. Thus, for large m, $\mathcal{M}_0$ is a better approximation of $\mathcal{A}$ than P_mH. By replacing I by P_{2m}, we construct with z_m a Galerkin approximation (based on $w_{m+1},\dots,w_{2m}$) of

$$(\nu A)^{-1} (I-P_m)(f-B(P_mu_m)) .$$

The algorithm considered in Section 4 corresponds to another (better) approximate inertial manifold $\mathcal{M}_1$ for (1.1), introduced in [13]; $\mathcal{A}$ lies at a distance $\leq c(\lambda_1/\lambda_{m+1})^2$ of $\mathcal{M}_1$.

Other aspects of approximate inertial manifolds appear in Marion [5] [6].

2. PROOF OF THEOREM 1.1.

We start by deriving various a priori estimates on the norms of u_m, z_m (Section 2.1). As already mentioned, these estimates yield in particular that u_m, z_m are defined for all $t > 0$. We then investigate in Section 2.2 the limit $m \to +\infty$.

2.1 A priori estimates.

Let us take $v = u_m$ in (1.11), $\tilde{v} = z_m$ in (1.12) and add the corresponding equalities. Thanks to (1.4), we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m|^2 + \nu\|u_m\|^2 + (Cu_m, u_m) + \nu\|z_m\|^2 + (Cz_m, z_m) = (f, u_m + z_m). \tag{2.1}$$

Hence, by virtue of (1.9),

$$\frac{1}{2}\frac{d}{dt}|u_m|^2 + \alpha(\|u_m\|^2 + \|z_m\|^2) \le |f|\,|u_m + z_m|. \tag{2.2}$$

Since we have

$$\|v\| = |A^{\frac{1}{2}}v| \ge \lambda_1^{\frac{1}{2}}|v|, \ \forall v \in V, \tag{2.3}$$

it follows from (2.2) that

$$\frac{1}{2}\frac{d}{dt}|u_m|^2 + \alpha(\|u_m\|^2 + \|z_m\|^2) \le \lambda_1^{-\frac{1}{2}}|f|\,\|u_m + z_m\|$$
$$\le \frac{\alpha}{2}(\|u_m\|^2) + \|z_m\|^2) + \frac{1}{\alpha\lambda_1}|f|^2$$

$$\frac{d}{dt}|u_m|^2 + \alpha(\|u_m\|^2 + \|z_m\|^2) \le \frac{2|f|^2}{\alpha\lambda_1}. \tag{2.4}$$

Dropping momentarily the term $\alpha\|z_m\|^2$ and using again (2.3), we obtain

$$\frac{d}{dt}|u_m|^2 + \alpha\lambda_1|u_m|^2 \le \frac{2}{\alpha\lambda_1}|f|^2.$$

Hence, by integrating

$$|u_m(t)|^2 \le |u_m(0)|^2\exp(-\alpha\lambda_1 t) + \frac{2|f|^2}{\lambda_1^2\alpha^2}(1 - \exp(-\alpha\lambda_1 t)), \ \forall t \ge 0.$$

Therefore

(2.5) The sequence u_m remains in a bounded set of $L^\infty(\mathbb{R}^+; H)$, as $m \to +\infty$.

We come back to (2.4) that we now integrate between 0 and T. This gives that

(2.6) For all $T > 0$, u_m and z_m remain bounded in $L^2(0,T;V)$ as $m \to \infty$.

We aim now to derive an estimate similar to (2.5) for the sequence z_m. Taking $\tilde{v} = z_m$ in (1.12), we have

$$\nu\|z_m\|^2 + (Cz_m,z_m) = -b(u_m,u_m,z_m) + (f,z_m)\,.$$

Thus, by use of (1.9) (1.7),

$$\alpha\|z_m\|^2 \le |B(u_m)|\,|z_m| + |f|\,|z_m|, \qquad (2.7)$$
$$\le c_3|u_m|^{1/2}\|u_m\|\,|Au_m|^{1/2}|z_m| + |f|\,|z_m|\,.$$

Since $u_m \epsilon W_m$ and $z_m \epsilon \hat{W}_m$, we have

$$|Au_m| \le \lambda_m^{1/2}\|u_m\|,\ \|u_m\| \le \lambda_m^{1/2}|u_m|, \qquad (2.8)$$

$$|Az_m| \ge \lambda_{m+1}^{1/2}\|z_m\|,\ \|z_m\| \ge \lambda_{m+1}^{1/2}|z_m|\,. \qquad (2.9)$$

Combining these inequalities with (2.7) we obtain

$$\alpha\lambda_{m+1}|z_m|^2 \le c_3\lambda_m|u_m|^2|z_m + |f|\,|z_m|,$$
$$\alpha\lambda_{m+1}|z_m| \le c_3\lambda_m|u_m|^2 + |f|\,. \qquad (2.10)$$

This gives, thanks to (2.5), that

(2.11) z_m remains bounded in $L^\infty(\mathbb{R}^+;H)$, as $m \to +\infty$.

Using again (2.8) (2.9), we also infer from (2.7)

$$\alpha\lambda_{m+1}|z_m| \le c_3\lambda_m^{1/2}|u_m|\,\|u_m\| + |f|$$

Hence, since $\lambda_1 \le \lambda_m \le \lambda_{m+1}$,

$$\alpha\lambda_{m+1}^{1/2}\,|z_m| \le c_3|u_m|\ \|u_m\| + \lambda_1^{-1/2}\,|f|\,.$$

This inequality along with (2.5) (2.6) shows that

(2.12) For all $T > 0$, $\lambda_{m+1}^{1/2} z_m$ remains bounded in $L^2(0,T;H)$, as $m \to +\infty$.

We conclude this section by deriving an estimate on $\frac{du_m}{dt}$. Due to (1.4) (1.5), we have

$$\|B(u,v)\|_{V'} \le c_1|u|^{1/2}\|u\|^{1/2}|v|^{1/2}\|v\|^{1/2},\ \forall u, v \epsilon V\,.$$

Therefore, the estimates (2.5), (2.6) and (2.11) imply that $B(u_m)$, $B(z_m,u_m)$ and $B(u_m,z_m)$ remain bounded in $L^2(0,T;V')$. Also by virtue of (1.6) and (2.6) Cu_m remains bounded in $L^2(0,T;H)$. Hence, the differential equation (1.14) gives that

(2.13) For all $T > 0$, $\frac{du_m}{dt}$ remains bounded in $L^2(0,T,V')$, as $m \to +\infty$

(recall that $\|P_m\|_{\mathcal{L}(V',V')} \leq 1$).

2.2 Passage to the limit.

We first note that, since $\lambda_m \to +\infty$ as $m \to +\infty$, (2.12) implies that

(2.14) For all $T > 0$, $z_m \to 0$ in $L^2(0,T;H)$ strongly, as $m \to +\infty$

Thus, using the estimates (2.6) and (2.11), we have also

(2.15) For all $T > 0$, $z_m \rightharpoonup 0$ in $L^2(0,T;V)$ weakly, and $L^\infty(\mathbb{R}^+,H)$ weak-star, as $m \to +\infty$.

We now study the convergence of the sequence u_m. The estimates (2.5), (2.6) and (2.13) insure the existence of an element u^* and a subsequence $m' \to +\infty$ such that

(2.16) $u_{m'} \rightharpoonup u^*$ in $L^2(0,T,V)$ weakly, for all $T > 0$, and $L^\infty(\mathbb{R}^+;H)$ weak-star, as $m' \to +\infty$,

$\frac{du_{m'}}{dt} \rightharpoonup \frac{du^*}{dt}$ in $L^2(0,T;V')$ weakly, for all $T > 0$, as $m' \to +\infty$.

Due to a classical compactness theorem ([4,10]), it follows from (2.16) that

(2.17) For all $T > 0$, $u_{m'} \to u^*$ in $L^2(0,T;H)$ strongly, as $m' \to +\infty$.

Thanks to (2.14) – (2.17) we can now pass to the limit in (1.11). The only difficulty concerns the bilinear terms. Let $v \in W_m$ be fixed and let $m' \geq m$. Thanks to (1.4), we have

$$b(u_{m'},u_{m'},v) = -\,b(u_{m'},v,u_{m'})\,.$$

In view of (1.7), $b(\cdot,v,\cdot)$ is bilinear continuous from $V \times H$ into $\mathbb{R}$. Therefore, (2.16) and (2.17) imply that

$$b(u_{m'},v,u_{m'}) \to b(u^*,v,u^*) \text{ in } L^1(0,T) \text{ strongly,}$$
for all $T > 0$, as $m' \to +\infty$.

Hence,

$$b(u_{m'},u_{m'},v) \to b(u^*,u^*,v) \text{ in } L^1(0,T) \text{ strongly,}$$
for all $T > 0$ as $m' \to +\infty$.

Similarly, we have

$$b(z_{m'},u_{m'},v) \to b(0,u^*,v) = 0, \text{ as } m' \to +\infty,$$
$$b(u_{m'},z_{m'},v) \to b(u^*,0,v) = 0, \text{ as } m' \to +\infty,$$

where the convergences hold in $L^1(0,T)$ strongly, for all $T > 0$.
Therefore, we find at the limit that u^* satisfies

$$\frac{d}{dt}(u^*,v) + \nu((u^*,v)) + (Cu^*,v) + b(u^*,u^*,v) = (f,v), \tag{2.18}$$

for all $v \in W_m$ and by continuity for all $v \in V$. Furthermore, (2.16) yields that

$$u_{m'}(0) \rightharpoonup u^*(0) \text{ weakly in } H. \tag{2.19}$$

Recalling that $u_{m'}(0) = P_{m'}u_0$, we conclude from (2.19) that

$$u^*(0) = u_0. \tag{2.20}$$

In view of (2.18) and (2.20), u^* is a solution of Problem (1.1) (1.10). Hence, $u^* = u$ and the whole sequence u_m converges to u in the sense (2.16).

In order to complete the proof of Theorem 1.1, it remains to check the strong convergence results in (1.17). Let us introduce the expression

$$X_m = \frac{1}{2}|u_m(T) - u(T)|^2 + \int_0^T \{\nu\|u_m - u\|^2 + (C(u_m - u), u_m - u) +$$
$$+ \nu\|z_m\|^2 + (Cz_m,z_m)\}dt$$

Then, we note that it suffices to show that

$$\lim_{m\to+\infty} X_m = 0 \tag{2.21}$$

Indeed, thanks to (1.9), (2.21) gives the strong convergence of u_m towards u in $L^2(0,T,V)$, $\forall T > 0$. Also, (2.21) yields that

$$u_m(t) \to u(t) \text{ in } H \text{ strongly, for all } t \geq 0. \tag{2.22}$$

Combining (2.22) with (2.5), we can apply the Lebesgue dominated convergence Theorem and therefore obtain the strong convergence of u_m towards u in $L^p(0,T;H)$ for all $p \in [1,+\infty)$. We can also note that, besides the strong convergence results (1.17) for u_m, (2.21) yields

$$z_m \to 0 \text{ in } L^2(0,T;V) \text{ strongly, for all } T > 0, \text{ as } m \to +\infty \tag{2.23}$$

This remark will be useful in next section.

We now prove (2.21). Integrating (2.1) between 0 and T, we obtain

$$\frac{1}{2}|u_m(T)|^2 + \int_0^T\{\nu\|u_m\|^2 + (Cu_m,u_m) + \nu\|z_m\|^2 + (Cz_m,z_m)\}dt =$$
$$= \frac{1}{2}|u_m(0)|^2 + \int_0^T(f,u_m+z_m)dt,$$

so that X_m can be rewritten

(2.24) $$X_m = -(u_m(T), u(T)) + \frac{1}{2}|u(T)|^2 + \frac{1}{2}|u_m(0)|^2 + \int_0^T\{-2\nu((u_m,u)) + \nu\|u\|^2 - (Cu,u_m-u) - (Cu_m,u) + (f,u_m+z_m)\}dt .$$

By use of (2.14) (2.16), we can pass to the limit in (2.24) (which is a linear expression with respect to u_m and z_m) and we obtain

$$\lim_{m\to+\infty} X_m = -\frac{1}{2}|u(T)|^2 + \frac{1}{2}|u_0|^2 + \int_0^T\{-\nu\|u\|^2 - (Cu,u) + (f,u)\}dt,$$

and using equation (2.18) with $u^*,v,$ replaced by $u,u,$ we find that this limit is 0; hence (2.21).

Theorem 1.1 is proved. □

3. Improved convergence results

Our aim in this Section is to prove the convergence of the nonlinear Galerkin method in stronger topologies. We shall derive the

THEOREM 3.1

The hypotheses are (1.4) to (1.9). For u_0 given in V, the solution u_m of (1.16) (1.13) converges to the solution u of Problem (1.1) (1.10) as $m\to+\infty$, in the following sense

(3.1) $u_m \to u$ in $L^2(0,T;D(A))$ and $L^p(0,T;V)$ strongly, for all $T>0$ and all $1\le p<+\infty$,

$u_m \to u$ in $L^\infty(\mathbb{R}^+;V)$ weak-star.

PROOF. The proof relies on further a priori estimates on u_m and z_m.

We start by deriving an estimate of u_m in $L^\infty(\mathbb{R}^+;V)$. Let us take $v = Au_m$ in (1.11), $\tilde{v} = Az_m$ in (1.12) and add the corresponding relations. We find

(3.2) $$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 + \nu|Az_m|^2 = (f,A(u_m+z_m)) - (Cu_m,Au_m) - (Cz_m,Az_m) - b(u_m,u_m,Au_m) - b(z_m,u_m,Au_m) - b(u_m,z_m,Au_m) - b(u_m,u_m,Az_m) .$$

The different terms in the right-hand side of (3.2) are majorized as follows. We have

(3.3) $$|(f,A(u_m+z_m))| \le \frac{\nu}{12}(|Au_m|^2 + |Az_m|^2) + \frac{6}{\nu}|f|^2 .$$

Also, by using (1.6),

$$(3.4)\qquad |(Cu_m,Au_m)| \le c_2\|u_m\||Au_m|, \le \frac{\nu}{12}|Au_m|^2 + \frac{3c_2^2}{\nu}\|u_m\|^2,$$

and

$$(3.5)\qquad |(Cz_m,Az_m)| \le c_2\|z_m\||Az_m|, \le \frac{\nu}{12}|Az_m|^2 + \frac{3c_2^2}{\nu}\|z_m\|^2 .$$

We then bound the trilinear terms by using (1.7). We have

$$(3.6)\qquad \begin{aligned} |b(u_m,u_m,Au_m)| &\le |B(u_m,u_m)||Au_m|,\\ &\le c_3|u_m|^{1/2}\|u_m\||Au_m|^{3/2},\\ &\le \text{(with Young inequality)},\\ &\le \frac{\nu}{12}|Au_m|^2 + \frac{c_5}{\nu}|u_m|^2\|u_m\|^4, \end{aligned}$$

where c_5 is an absolute constant. Similarly,

$$|b(z_m,u_m,Au_m)| \le \frac{\nu}{12}|Au_m|^2 + \frac{c_5}{\nu}|z_m|^2\|z_m\|^2\|u_m\|^2 .$$

Also

$$(3.7)\qquad \begin{aligned} |b(u_m,z_m,Au_m)| &\le c_3|u_m|^{1/2}\|u_m\|^{1/2}\|z_m\|^{1/2}|Az_m|^{1/2}|Au_m|,\\ &\le \frac{\nu}{12}|Au_m|^2 + \frac{c_6}{\nu}|u_m|\|u_m\|\|z_m\||Az_m|,\\ &\le \frac{\nu}{12}|Au_m|^2 + \frac{\nu}{12}|Az_m|^2 + \frac{c_7}{\nu^2}|u_m|^2\|u_m\|^2\|z_m\|^2 . \end{aligned}$$

Finally, for the last term in the right-hand side of (3.2)

$$(3.8)\qquad \begin{aligned} |b(u_m,u_m,Az_m)| &\le c_3|u_m|^{1/2}\|u_m\||Au_m|^{1/2}|Az_m|,\\ &\le \frac{\nu}{12}|Au_m|^2 + \frac{\nu}{12}|Az_m|^2 + \frac{c_7}{\nu^2}|u_m|^2\|u_m\|^4 . \end{aligned}$$

Combining the above majorizations, we conclude from (3.2) that

$$(3.9)\qquad \begin{aligned} \frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 + \nu|Az_m|^2 &\le \frac{12}{\nu}|f|^2 + \frac{6c_2^2}{\nu}\|z_m\|^2 +\\ &+ c_8\|u_m\|^2(1+|u_m|^2\|u_m\|^2 + |z_m|^2\|z_m\|^2 + |u_m|^2\|z_m\|^2), \end{aligned}$$

where $c_8 = c_8(\nu)$ depends on ν.

This gives in particular the differential inequality.

$$(3.10)\qquad \frac{dy_m}{dt} \le g_m y_m + h_m,$$

where we have set

$$(3.11)\qquad \begin{aligned} &y_m(t) = \|u_m(t)\|^2,\ h_m(t) = \frac{12}{\nu}|f|^2 + \frac{6c_2^2}{\nu}\|z_m\|^2,\\ &g_m(t) = c_8(1+|u_m(t)|^2\|u_m(t)\|^2 + |z_m(t)|^2\|z_m(t)\|^2\\ &\qquad\qquad + |u_m(t)|^2\|z_m(t)\|^2 \end{aligned}$$

By integrating (3.10), we find that

(3.12) $$y_m(t) \le y_m(0)\exp\left(\int_0^t g_m(s)ds\right) + \int_0^t h_m(s)\exp\left(\int_s^t g_m(\sigma)d\sigma\right),$$
$\forall t \ge 0$.

This inequality combined with the a priori estimates (2.5), (2.6), and (2.11) provides a bound of u_m in $L^\infty(0,T;V)$, for all $T > 0$.

A bound valid on $\mathbb{R}^+$ is obtained by application of the uniform Gronwall Lemma that we first recall (see for instance [1] [12]).

LEMMA 3.2

Let g,h,y be three locally integrable positive functions on $]t_0,+\infty[$ which satisfy

$$\frac{dy}{dt} \in L^1_{loc}(]t_0,+\infty[) \text{ and } \frac{dy}{dt} \le gy + h \text{ for } t \ge t_0,$$

(3.13) $$\int_t^{t+1} g(s)ds \le a_1, \int_t^{t+1} h(s)ds \le a_2, \int_t^{t+1} y(s)ds \le a_3, \text{ for } t \ge t_0,$$

where a_1,a_2,a_3 are positive constants. Then

(3.14) $$y(t) \le (a_3+a_2)\exp(a_1), \forall t \ge t_0 + 1 .$$

Returning to (3.10), the assumption (3.13) is checked thanks to the a priori estimates of Section 2.1. We know that u_m,z_m are bounded in $L^\infty(\mathbb{R}^+;H)$. Moreover, by integrating (2.4) between t and $t+1$, we find that

$$\int_t^{t+1} \|u_m\|^2 ds, \int_t^{t+1} \|z_m\|^2 ds \text{ are bounded for all } t \ge 0$$
by a constant independent of m.

Consequently, the functions y_m, h_m, g_m given by (3.11) satisfy (3.13) with constants a_i's independent of m and we infer from (3.14) that

(3.15) $$y_m(t) = \|u_m(t)\|^2 \le c_9, \ \forall t \ge 1,$$

where $c_9 = c_9(\nu)$ is independent of m.

Hence, (3.15) provides a uniform bound for $\|u_m(t)\|$, $t \ge 1$, while (3.12) gives a uniform bound for $0 \le t \le 1$. Therefore,

(3.16) u_m remains bounded in $L^\infty(\mathbb{R}^+;V)$ as $m \to +\infty$.

Then, integrating (3.9), we obtain that

(3.17) For all $T > 0$, u_m and z_m remain bounded in $L^2(0,T;D(A))$, as $m \to +\infty$.

Our goal now is to derive an estimate analogous to (3.16) for the sequence z_m. By taking $\tilde{v} = Az_m$ in (1.12), we find

$$\nu|Az_m|^2 = -(Cz_m,Az_m) - b(u_m,u_m,Az_m) + (f,Az_m),$$
$$\le \text{(thanks to (1.6) (1.7))},$$
$$\le c_2\|z_m\||Az_m| + c_3|u_m|^{1/2}\|u_m\||Au_m|^{1/2}|Az_m| + |f||Az_m|.$$
$$\nu|Az_m| \le c_2\|z_m\| + c_3|u_m|^{1/2}\|u_m\||Au_m|^{1/2} + |f|.$$

Then, using (2.8) (2.9), we obtain

$$\nu\lambda_{m+1}^{1/2}\|z_m\| \le c_2\|z_m\| + c_3\lambda_m^{1/4}|u_m|^{1/2}\|u_m\|^{3/2} + |f|,$$

which gives for large m

$$\|z_m\| \le \frac{1}{\nu\lambda_{m+1}^{1/2} - c_2}(c_3\lambda_m^{1/4}|u_m|^{1/2}\|u_m\|^{3/2} + |f|).$$

This inequality combined with the estimate (3.16) enables us to say that

(3.18) $\quad z_m \to 0$ in $L^\infty(\mathbb{R}^+;V)$ strongly, as $m \to +\infty$.

Finally, the last a priori estimate we shall need here concerns $\frac{du_m}{dt}$. From (1.7) and the estimates (3.16) – (3.18), we have that $B(u_m)$, $B(z_m,u_m)$ and $B(u_m,z_m)$ are bounded

independently of m in $L^4(0,T;H)$ for all $T > 0$, while Au_m is bounded in $L^2(0,T;H)$. Therefore, the equation (1.16) for u_m gives that

(3.19) $\quad$ For all $T > 0$, $\frac{du_m}{dt}$ remains bounded in $L^2(0,T;H)$, as $m \to +\infty$.

The convergence results (3.1) will follow now from the estimates (3.16) – (3.19). First, combining these estimates with our previous convergence results ((1.17) (2.14) (2.15)), we obtain that, as $m \to +\infty$,

(3.20) $\quad u_m \rightharpoonup u$ in $L^2(0,T;D(A))$ weakly, for all $T > 0$,

(3.21) $\quad u_m \rightharpoonup u$ in $L^\infty(\mathbb{R}^+;V)$ weak–star,

(3.22) $\quad \frac{du_m}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2(0,T;H)$ weakly, for all $T > 0$,

(3.23) $\quad z_m \rightharpoonup 0$ in $L^2(0,T;D(A))$ weakly, for all $T > 0$.

This gives, in particular, the weak convergence result in (3.1). Next, to check the strong convergence results in (3.1), we introduce the expression

$$Y_m = \frac{1}{2}\|u_m(T) - u(T)\|^2 + \nu\int_0^T(|Au_m - Au|^2 + |Az_m|^2)ds,$$

and we note that it suffices to show that

$$\lim_{m\to+\infty} Y_m = 0,$$

(indeed, the strong convergence in $L^p(0,T;V)$ follows then from the estimate (3.16), thanks to the Lebesgue dominated convergence Theorem). Integrating (3.2) between 0 and T, we obtain

$$\frac{1}{2}\|u_m(T)\|^2 + \nu\int_0^T(|Au_m|^2 + |Az_m|)^2 = Z_m + \frac{1}{2}\|u_{0m}\|^2,$$

(3.24)
$$Z_m = \int_0^T \{(f,A(u_m+z_m)) - (Cu_m,Au_m) - (Cz_m,Az_m) - b(u_m,u_m,Au_m) - b(z_m,u_m,Au_m) - b(u_m,z_m,Au_m) - b(u_m,u_m,Az_m)\}ds,$$

so that Y_m can be rewritten

$$Y_m = -((u_m(T),u(T))) + \frac{1}{2}\|u(T)\|^2 + \frac{1}{2}\|u_{0m}\|^2 + \nu\int_0^T(-2(Au_m,Au) + |Au|^2)ds + Z_m .$$

It follows easily from (3.20) and (3.22) that

$$\lim_{m\to+\infty} \{-((u_m(T),u(T))) - 2\nu\int_0^T(Au_m,Au)ds\} = = -\|u(T)\|^2 - 2\nu\int_0^T |Au|^2 ds .$$

Next, we can pass to the limit $m\to+\infty$ in Z_m by using that u_m and z_m converge weakly in $L^2(0,T;D(A))$ ((3.20), (3.23)), strongly in $L^2(0,T;V)$ ((1.17),(2.23)) and are bounded in $L^\infty(0,T;V)$ ((3.16), (3.18)). In particular, since the passages to the limit in the different trilinear terms in (3.24) are similar, we will only consider here the first one. We have

(3.25)
$$\int_0^T b(u_m,u_m,Au_m)ds - \int_0^T b(u,u,Au)ds = \int_0^T b(u_m-u, u_m,Au_m)ds + \int_0^T b(u,u_m-u,Au_m)ds + \int_0^T b(u,u,A(u_m-u))ds$$

For the first term in the right–hand side of (3.25) we obtain, thanks to (1.8) (3.16), and Hölder inequality

$$|\int_0^T b(u_m-u,u_m,Au_m)ds| \le c_4\int_0^T |u_m-u|^{1/2}\|u_m\||A(u_m-u)|^{1/2}|Au_m|ds,$$
$$\le c\int_0^T |u_m-u|^{1/2}|A(u_m-u)|^{1/2}|Au_m|ds,$$
$$\le c(\int_0^T |u_m-u|^2ds)^{1/4}(\int_0^T|A(u_m-u)|^2ds)^{1/4}(\int_0^T|Au_m|^2ds)^{1/2},$$

which, along with (1.17) and (3.17), shows that this term goes to zero as $m\to+\infty$. Then, for the second one, by using (1.7),

$$|\int_0^T b(u,u_m-u,Au_m)ds|$$
$$\le c_3\int_0^T |u|^{1/2}\|u\|^{1/2}\|u_m-u\|^{1/2}|A(u_m-u)|^{1/2}|Au_m|ds$$
$$\le c\int_0^T \|u_m-u\|^{1/2}|A(u_m-u)|^{1/2}|Au_m|ds,$$
$$\le c(\int_0^T \|u_m-u\|^2 ds)^{1/4}(\int_0^T |A(u_m-u)|^2 ds)^{1/4}(\int_0^T |Au_m|^2 ds$$

and this goes to 0, as $m \to +\infty$, by virtue of (1.17), (3.17).
Finally, the last term in the right–hand side of (3.25) is linear with respect to u_m and one checks easily that this term goes to zero, as $m \to +\infty$, thanks to (3.20). We have thus shown that

$$\int_0^T b(u_m,u_m,Au_m)ds \to \int_0^T b(u,u,Au)ds, \text{ as } m \to +\infty .$$

The limit for the different terms in (3.24) can be studied in a similar manner and one obtains finally

$$\lim_{m\to+\infty} Y_m = -\frac{1}{2}\|u(T)\|^2 + \frac{1}{2}\|u_0\|^2 - \nu\int_0^T |Au|^2 ds + \int_0^T \{(f,Au) -$$
$$- (Cu,Au) - b(u,u,Au)\}ds,$$
$$= (\text{by integrating the equation (2.18) with } u^*,v \text{ replaced by } u,Au)$$
$$= 0 .$$

This shows the strong convergence results in (3.1) and concludes the proof of Theorem 3.1. □

4. Another nonlinear Galerkin method

We present in this section a second nonlinear Galerkin method for approximating Problem (1.1) (1.10). Its motivation and its relation with the first one were explained in Remark 1.1.

We assume that u_0 is given in V. The method is implemented by using as a basis of V the eigenvectors of the operator A and we are looking for an approximate solution u_m of the form

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j,$$
$$u_m:\mathbb{R}^+ \to W_m = \text{Span}\{w_1,\dots,w_m\} .$$

We again introduce the unknown function z_m

$$z_m(t) = \sum_{j=m+1}^{2m} h_{jm}(t)w_j$$
$$z_m:\mathbb{R}^+ \to \tilde{W}_m = \text{Span}\{w_{m+1},\dots,w_{2m}\} .$$

In this second method, the couple (u_m,z_m) will be determined by the resolution of the following system

$$(4.1)\qquad \frac{d}{dt}(u_m,v) + \nu((u_m,v)) + (Cu_m,v) + b(u_m,u_m,v) + b(z_m,u_m,v) + b(u_m,z_m,v) + b(z_m,z_m,v) = (f,v),\ \forall v\in W_m,$$

$$(4.2)\qquad \nu((z_m,\tilde{v})) + (Cz_m,\tilde{v}) + b(u_m,u_m,\tilde{v}) + b(z_m,u_m,\tilde{v}) + b(u_m,z_m,\tilde{v}) = (f,\tilde{v}),\ \forall\tilde{v}\in\tilde{W}_m,$$

(4.3) $u_m(0) = P_m u_0$.

Note that the equation (4.2) can be rewritten

(4.4) $$\nu A z_m + (P_{2m}-P_m)Cz_m + (P_{2m}-P_m)(B(z_m,u_m) + B(u_m,z_m)) = \\ = (P_{2m}-P_m)(f-B(u_m)).$$

We denote by $D(u_m)$ the linear operator (operating on z_m) in the left–hand side of (4.4).

The existence of a solution $\{u_m,z_m\}$ of (4.1) and (4.2) defined even on a small interval of time is not straightforward and we must prove that the operator $D(u_m)$ is invertible on $\tilde{W}_m$; furthermore the proof of the invertibility of $D(u_m)$ depends itself on the derivation of suitable a priori estimates. Indeed, for $\tilde{v}\in\tilde{W}_m$, we have

(4.5) $$\begin{aligned}(D(u_m)\tilde{v},\tilde{v}) &= \nu\|\tilde{v}\|^2 + (C\tilde{v},\tilde{v}) + b(\tilde{v},u_m,\tilde{v}),\\ &\geq \text{(thanks to (1.5) and (1.9))},\\ &\geq \alpha\|\tilde{v}\|^2 - c_1|\tilde{v}|\,\|\tilde{v}\|\|u_m\|,\\ &\geq \text{(since } \tilde{v}\in\tilde{W}_m, \text{ see (2.9))},\\ &\geq \|\tilde{v}\|^2(\alpha - c_1\lambda_{m+1}^{-1/2}\|u_m\|).\end{aligned}$$

However, if m is chosen such that

(4.6) $$\alpha - c_1\lambda_{m+1}^{-1/2}\|u_0\| \geq \frac{\alpha}{2},$$

then due to the general theorems on ordinary differential equations, (4.1) (4.2) possesses a maximal solution $\{u_m,z_m\}$ defined on some interval $[0,T_m)$; on this interval the system (4.1) (4.2) is equivalent to the ordinary differential system for u_m

(4.7) $$\begin{cases}\frac{d}{dt}u_m + \nu A u_m + P_m(Cu_m+B(u_m+z_m)) = P_m f,\\ z_m = D(u_m)^{-1}\{(P_{2m}-P_m)(f-B(u_m))\}.\end{cases}$$

The condition (4.6) means that m is large enough (recall that $\lambda_m \to +\infty$, as $m\to+\infty$). Our aim in the sequel is to show that (at least for m sufficiently large), $T_m = +\infty$ i.e. that (4.7) has a solution u_m on $\mathbb{R}^+$. Furthermore we prove that this solution converges towards the solution u of Problem (1.1) as $m\to+\infty$.

Our main result is the following

THEOREM 4.1

The hypotheses are (1.4) to (1.9). Assume that u_0 is given in V. Then

i) There exists a constant $K = K(u_0)$ depending on u_0 only through $\|u_0\|$ such that, if m satisfies

(4.8) $$\alpha - c_1 K\lambda_{m+1}^{-1/2} \geq \frac{\alpha}{2},$$

the system (4.7) (4.3) possesses a solution u_m defined on $\mathbb{R}^+$.

ii) <u>The solution</u> u_m <u>of</u> (4.7) (4.3) <u>converges to the solution</u> u <u>of</u> (1.1) (1.10) <u>as</u> $m \to +\infty$, <u>in the following sense</u>

(4.9) $u_m \to u$ <u>in</u> $L^2(0,T;D(A))$ <u>and</u> $L^p(0,T;V)$ <u>stronly for all</u> $T > 0$ <u>and all</u> $1 \le p < +\infty$, <u>and in</u> $L^\infty(\mathbb{R}^+,V)$ <u>weak-star.</u>

PROOF.

The constant K in (4.8) will be determined later (see (4.26)). We start by assuming that (4.6) holds so that (4.1), (4.2) possesses a solution $\{u_m,z_m\}$ on some interval $(0,T_m)$ and we derive some a priori estimates for $\{u_m,z_m\}$ on $(0,T_m)$. The derivation of these estimates has some common points with those of Theorems 1.1 and 3.1 and certain details will be omitted.

(i) <u>A priori estimates (I)</u>. We take $v = u_m$ in (4.1), $\tilde{v} = z_m$ in (4.2) and add the corresponding equalities. Thanks to (1.4), we obtain

$$\frac{1}{2}\frac{d}{dt}|u_m|^2 + \nu\|u_m\|^2 + (Cu_m,u_m) + \nu\|z_m\|^2 + (Cz_m,z_m) = (f,u_m+z_m) \tag{4.10}$$

Therefore, the analog of (2.1) is satisfied and, as for (2.5) and (2.6), we obtain

(4.11) u_m is bounded independently of m in $L^\infty(0,T_m;H)$

(4.12) u_m,z_m are bounded independently of m in $L^2(0,T;H)$ for all $T, 0 < T < T_m$ (and $T = T_m$ if $T_m < +\infty$).

ii) <u>A priori estimates (II)</u>. We now take $v = Au_m$ in (4.1), $\tilde{v} = Az_m$ in (4.2) and add the corresponding equalities. We find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 + \nu|Az_m|^2 &= (f,A(u_m+z_m)) - (Cu_m,Au_m) \\ &- (Cz_m,Az_m) - b(u_m,u_m,Au_m) - b(z_m,u_m,Au_m) - \\ &- b(u_m,z_m,Au_m) - b(z_m,z_m,Au_m) - b(u_m,u_m,Az_m) - \\ &- b(z_m,u_m,Az_m) - b(u_m,z_m,Az_m)\,.\end{aligned} \tag{4.13}$$

Some terms in the right-hand side of (4.13) are bounded as in the proof of Theorem 3.1. The majorizations (3.3)–(3.8) are still used here.

Then thanks to (1.7), (2.8) and (2.9), we have

$$\begin{aligned}|b(z_m,u_m,Au_m)| &\le c_3|z_m|^{1/2}\|z_m\|^{1/2}\|u_m\|^{1/2}|Au_m|^{3/2}, \\ &\le c_3\left(\frac{\lambda_m}{\lambda_{m+1}}\right)^{3/4}\|u_m\|^2|Az_m|, \\ &\le c_3\|u_m\|^2|Az_m|, \\ &\le \frac{\nu}{24}|Az_m|^2 + \frac{c_{10}}{\nu}\|u_m\|^4\,.\end{aligned} \tag{4.14}$$

Also

$$(4.15)\qquad |b(z_m,z_m,Au_m)| \le c_3|z_m|^{1/2}\|z_m\||Az_m|^{1/2}|Au_m|,$$
$$\le c_3\left(\frac{\lambda_m}{\lambda_{m+1}}\right)^{1/2}\|u_m\|\|z_m\||Az_m|,$$
$$\le \frac{\nu}{24}|Az_m|^2 + \frac{c_{10}}{\nu}\|u_m\|^2\|z_m\|^2.$$

$$(4.16)\qquad |b(z_m,u_m,Az_m)| \le c_3|z_m|^{1/2}\|z_m\|^{1/2}\|u_m\|^{1/2}|Au_m|^{1/2}|Az_m|,$$
$$\le c_3\left(\frac{\lambda m}{\lambda_{m+1}}\right)^{1/4}\|u_m\|\|z_m\||Az_m|$$
$$\le \frac{\nu}{24}|Az_m|^2 + \frac{c_{10}}{\nu}\|u_m\|^2\|z_m\|^2.$$

Finally, for the last term in the right–hand side of (4.13)

$$(4.17)\qquad |b(u_m,z_m,Az_m)| \le c_3|u_m|^{1/2}\|u_m\|^{1/2}\|z_m\|^{1/2}|Az_m|^{3/2},$$
$$\le \frac{\nu}{24}|Az_m|^2 + \frac{c_{11}}{\nu}|u_m|^2\|u_m\|^2\|z_m\|^2.$$

Combining (3.3)–(3.8), (4.14)–(4.17) and (4.13), we obtain the differential inequality

$$(4.18)\qquad \frac{d}{dt}\|u_m\|^2 + \nu|Au_m|^2 + \nu|Az_m|^2 \le \frac{12}{\nu}|f|^2 + \frac{6c_2^2}{\nu}\|z_m\|^2 +$$
$$+ c_{12}\|u_m\|^2(1+\|u_m\|^2 + |u_m|^2\|u_m\|^2 + \|z_m\|^2 + |u_m|^2\|z_m\|^2)$$

where $c_{12} = c_{12}(\nu)$ depends on ν.

This gives in particular

$$(4.19)\qquad \frac{d}{dt}\|u_m\|^2 \le \frac{12}{\nu}|f|^2 + \frac{6c_2^2}{\nu}\|z_m\|^2 + c_{12}\|u_m\|^2(1+\|u_m\|^2+$$
$$+ |u_m|^2\|u_m\|^2 + \|z_m\|^2 + |u_m|^2\|z_m\|^2).$$

Integrating (4.19) between 0 and T and using the estimates (4.11), (4.12), we obtain that

(4.20) u_m is bounded independently of m in $L^\infty(0,T;V)$, for all $0 < T < T_m$ (and $T = T_m$ if $T_m < +\infty$)

Thus, if $T_m < +\infty$, (4.20) provides a bound on $\|u_m(t)\|$ for $0 \le t \le T_m$. Let us check that an analogous bound exists if $T_m = +\infty$, by applying the uniform Gronwall Lemma. It is easy to infer from (4.10) that

(4.21) $\int_t^{t+1}\|u_m\|^2 ds, \int_t^{t+1}\|z_m\|^2 ds$, are bounded for $t \ge 0$ independently of m.

By combination of (4.21) and (4.11), we see that (4.19) satisfies the assumptions of the uniform Gronwall Lemma, and, as in the proof of Theorem 3.1, (3.14) provides a bound independent of m for $\|u_m(t)\|$, $t \ge 1$. Since (4.20) gives a bound for $\|u_m(t)\|$, $0 \le t \le 1$, we have that

(4.22) u_m is bounded independently of m in $L^\infty(0,T_m;V)$

Then, integrating (4.18) between 0 and T, we find that

(4.23) u_m and z_m are bounded independently of m in $L^2(0,T;D(A))$ for all $0 < T < T_m$ (and $T = T_m$ if $T_m < +\infty$).

(iii) A priori estimates (III). By taking $\tilde{v} = Az_m$ in (4.2), we find

$$\begin{aligned}\nu|Az_m|^2 &= -(Cz_m,Az_m) - b(u_m,u_m,Az_m) - b(z_m,u_m,Az_m) - \\ &\quad - b(u_m,z_m,Az_m) + (f,Az_m), \\ &\le \text{(thanks to (1.6), (1.7), (1.8))}, \\ &\le c_2\|z_m\||Az_m| + c_3|u_m|^{1/2}\|u_m\||Au_m|^{1/2}|Az_m| + \\ &\quad + c_4|z_m|^{1/2}|Az_m|^{3/2}\|u_m\| + \\ &\quad + c_3|u_m|^{1/2}\|u_m\|^{1/2}\|z_m\|^{1/2}|Az_m|^{3/2} + |f||Az_m| .\end{aligned}$$

Hence, dividing by $|Az_m|$, and using (2.8), (2.9), along with the estimate (4.22), we obtain

$$\nu|Az_m| \le c_2\lambda_{m+1}^{-1/2}|Az_m| + c_3\lambda_m^{1/4} + c_4\lambda_{m+1}^{-1/2}|Az_m| + c_3\lambda_{m+1}^{-1/4}|Az_m| + |f|, \text{ on } [0,T_m).$$

Hence

$$\{\nu - c_2\lambda_{m+1}^{-1/2} - c_4\lambda_{m+1}^{-1/2} - c_3\lambda_{m+1}^{-1/4}\}|Az_m| \le c_3\lambda_m^{1/4} + |f|$$

which implies for m sufficiently large

(4.24) $\frac{\nu}{2}|Az_m| \le c_3\lambda_m^{1/4} + |f|$.

Since $|Az_m| \ge \lambda_{m+1}^{1/2}\|z_m\|$, (4.24) yields that

(4.25) $z_m \to 0$ in $L^\infty(0,T_m;V)$ as $m \to +\infty$.

(iv) Passage to the limit. Let us first check that the solution u_m of (4.7) is defined on $\mathbb{R}^+$, for sufficiently large m. From (4.22), we know the existence of a constant K independent of m such that

(4.26) $\|u_m(t)\| \le K$, for $0 \le t < T_m$.

Therefore, if m satisfies

$$\alpha - c_1K\lambda_{m+1}^{-1/2} \ge \frac{\alpha}{2},$$

we have by (4.5)

$$(Du_m(t)\tilde{v},\tilde{v}) \ge \frac{\alpha}{2}\|\tilde{v}\|^2, \text{ for } 0 \le t < T_m, \tilde{v} \in \dot{W}_m,$$

so that the operator $D(u_m)$ is uniformly coercive on $(0,T_m)$. This implies immediately $T_m = +\infty$ and shows Theorem 4.1–i).

We now assume that (4.8) holds, K given by (4.26). Then the estimates (4.22), (4.23) and (4.25) hold with $T_m = +\infty$ and these estimates are similar to (3.16), (3.17), (3.18). Therefore we can now pass to the limit $m \to +\infty$, by using arguments similar to those in the proofs of Theorems 1.1 and 3.1. The details are omitted; one obtains successively that u_m converges towards the solution u of (1.1) (1.10) in the sense (1.17) and (3.1). This shows (4.9).

Theorem 4.1 is proved. □

5. Examples

In this section we intend to show that Theorems 1.1 and 4.1 apply to the two–dimensional Navier–Stokes equations and to the Kuramoto–Sivashinsky equation.

5.1 The two–dimensional Navier–Stokes equations

We restrict ourselves to the space–periodic case where the w_j's are proportional to sines and cosines; for other boundary conditions and in particular the no–slip boundary condition (Dirichlet problem) the eigenfunctions are not known but we intend in a forthcoming article to extend our results to more general bases.

The Navier–Stokes equations for an incompressible fluid are written

(5.1) $$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f,$$

(5.2) $$\operatorname{div} u = 0,$$

where $u = \{u_1,u_2\}$ is the velocity vector and p the pressure; $\nu > 0$ is given (the kinematic viscosity) and f represents volume forces. Equations (5.1), (5.2) must be supplemented by a boundary condition. We shall assume that the flow is periodic with period L_1, L_2, in directions x_1, x_2 and we denote by $\Omega = (0,L_1) \times (0,L_2)$ the period.

As usual this boundary value problem can be reduced to an evolution equation for u only in an appropriate Hilbert space H; H is a closed subspace of $L^2(\Omega)^2$, its definition and the details can be found in [11]. The equations have the form (1.1) (1.2) with $C = 0$, $f \in H$ and $Au = -\Delta u$, $B(u) = B(u,u)$,

$$(B(\varphi,\psi),\theta) = \sum_{i,j=1}^{2} \int_\Omega \varphi_i \frac{\partial \psi_j}{\partial x_i} \theta_j dx, \quad \forall \varphi,\psi,\theta \in D(A).$$

The eigenfunctions w_m are simply the functions (see [10]):

$$\frac{\bar{j}}{|j|} \sin\left(2\pi \frac{jx}{L}\right), \quad \frac{\bar{j}}{|j|} \cos\left(2\pi \frac{jx}{L}\right),$$

where $j = (j_1,j_2) \in \mathbb{N}^2$, $\bar{j} = (j_2,-j_1)$, $|j| = (j_1^2+j_2^2)^{1/2}$, and

$\frac{x}{L} = \frac{j_1 x_1}{L_1} + \frac{j_2 x_2}{L_2}$. The hypotheses (1.4)–(1.9) reduce to standard properties concerning B that can be found for instance in [11]. Theorems 1.1 and 4.1 apply.

5.2. The Kuramoto–Sivashinsky equation

This is an evolution equation in space dimension 1, with a Burgers nonlinearity, a fourth order dissipation term and a second order antidissipative term. It reads

$$\frac{\partial v}{\partial t} + \frac{\partial^4 v}{\partial x^4} + \frac{\partial^2 v}{\partial x^2} + v\,\frac{\partial v}{\partial x} = 0\,. \tag{5.3}$$

We consider the solutions of (5.3) that are periodic in space of period L. The existence and uniqueness of solution of (5.3) is easy; however the stability of solutions for large t has been proved only for odd solutions and we therefore restrict ourselves to this case [7, 8]. Hence

$$H = \{u \in L^2(-\tfrac{L}{2}, \tfrac{L}{2}),\ u \text{ is odd}\} \tag{5.4}$$
$$D(A) = H^4_{per}(-\tfrac{L}{2}, \tfrac{L}{2}) \cap H,\ A = \frac{d^4}{dx^4}$$

In order to satisfy (1.9) we transform the equation with the translation method used in [7, 8] for the study of the long time stability. This consists in setting

$$v = u + \varphi, \tag{5.5}$$

where φ is an appropriate function in D(A) constructed in [7, 8]. The new equation for u reads

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u\,\frac{\partial \varphi}{\partial x} + \varphi\frac{\partial u}{\partial x} + u\,\frac{\partial u}{\partial x} = g(\varphi), \tag{5.6}$$
$$g(\varphi) = -\frac{d^4\varphi}{dx^4} - \frac{d^2\varphi}{dx^2} - \varphi\frac{d\varphi}{dx}\,.$$

This equation is of the form (1.1), (1.2) with $\nu = 1$, A as before, $f = g(\varphi)$, while $B(u) = u\,\frac{du}{dx}$ and

$$Cu = \frac{d^2 u}{dx^2} + u\,\frac{d\varphi}{dx} + \varphi\,\frac{du}{dx}\,. \tag{5.7}$$

The choice of φ such that (1.9) holds is one of the main tasks in [7, 8]. The relation (1.5) to (1.8) are easily proved using Sobolev's imbeddings. Finally we note that (1.4) is not satisfied but instead, we have

$$b(u,u,u) = 0,\ \forall u \in V = D(A^{1/2})\,. \tag{5.8}$$

The replacement of (1.4) by (5.8) induces some slight changes in the proofs of Theorems 1.1 and 4.1, which are left as an exercise to the reader[3].

[3]The main difference is that some terms which canceled each other do not disappear anymore and some estimates such as the estimate on B before (2.13) do not follow anymore from (1.4) – (1.9). However better estimates on B are valid and all the new terms and all the desired estimates are easily obtained.

REFERENCES

[1] C. Foias, O. Manley and R. Temam, Attractors for the Bénard problem: existence and physical bounds on their fractal dimension, Nonlinear Analysis, TMA, 11, 1987, p. 939–967.

[2] C. Foias, O. Manley and R. Temam, Modelization of the interaction of small and large eddies in turbulent flows, Math. Mod. and Num. Anal., 22, 1988, p. 93–114.

[3] C. Foias, G. Sell and R. Temam, Inertial manifolds for nonlinear evolutionary equations, J. Diff. Eq., 73, 1988, p. 309–353.

[4] J. L. Lions, Quelques methodes de résolution des problémes aux limites non linéaires, Dunod, Paris, 1969.

[5] M. Marion, Approximate inertial manifolds for reaction–diffusion equations in high space dimension, to appear.

[6] M. Marion, Approximate inertial manifolds for the Kuramoto–Sivashinsky equation and the Cahn–Hilliard equation, to appear.

[7] B. Nicolaenko, B. Scheurer and R. Temam, Some global dynamical properties of the Kuramoto–Sivashinsky equations: Nonlinear stability and attractors, Physica 16D, 1985, p. 155–183.

[8] B. Nicolaenko, B. Scheurer and R. Temam, Some global dynamical properties of a class of pattern formation equations, IMA Preprint Series #381, 1988.

[9] C. Rosier, Thesis in preparation, Université Paris–Sud, Orsay, 1989.

[10] R. Temam, Navier–Stokes equations, 3rd ed., North–Holland, Amsterdam, 1984.

[11] R. Temam, Navier–Stokes equations and Nonlinear functional analysis, CBMS–NSF Regional Conference Series in Applied Mathematics, SIAM Philadelphia, 1983.

[12] R. Temam, Infinite dimensional dynamical systems in mechanics and physics, Applied Mathematics Series, Vol. 68, Springer–Verlag, New York, 1988.

[13] R. Temam, Variétés inertielles approximatives pour les équations de Navier–Stokes bidimensionnelles, C.R. Acad. Sc. Paris, 306, Serie II, 1988, p. 399–402.

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