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SMR.378/13

WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

OTHER TRIPLE-DECK FLOWS

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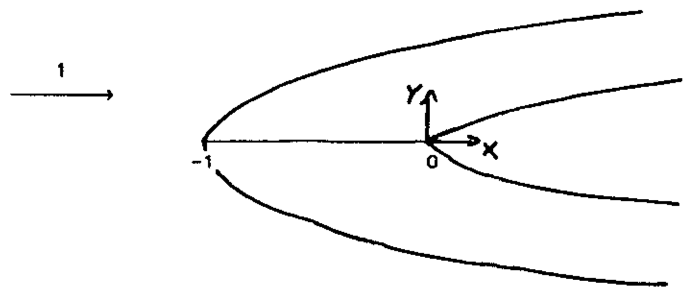
Other Triple-Deck Flows

Flow past an aligned flat plate of finite length



Classical boundary-layer solution

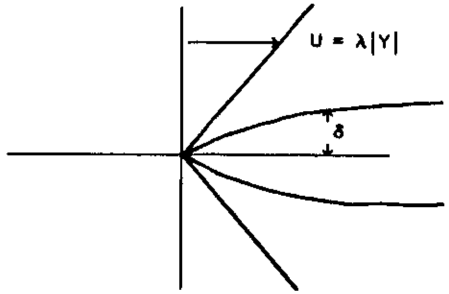
$$UU_X + VU_Y = uu_X + U_{YY}\,, \qquad U_X + V_Y = 0$$

$$V = 0 \qquad \text{on } Y = 0$$

$$\left.\begin{array}{ll} U = 0 & -1 < X < 0 \\ U_Y = 0 & 0 < X \end{array}\right\} \text{ on } Y = 0$$

$$U \to u = 1 \qquad \text{as } Y \to \infty$$

Concentrate on wake near trailing edge $X = 0+$



Due to discontinuity in boundary conditions seek similarity solution growing from $X = 0$ (Goldstein, 1930). Assume sub-viscous layer has thickness δ,

$$U = O(\lambda\delta)$$

$$UU_X = O(U_{YY}) \quad \Rightarrow \quad \lambda\delta^3 = O(X)$$

i.e. $$\delta = O\left(\frac{X}{\lambda}\right)^{1/3}, \qquad U = O(X^{1/3}\lambda^{2/3})\,.$$

Try $\quad U = X^{1/3}\lambda^{2/3}f'(\eta)\,, \qquad \eta = \left[\frac{\lambda}{X}\right]^{1/3} Y$

$$V = \frac{1}{3}\left[\frac{\lambda}{X}\right]^{1/3}(\eta f' - 2f)$$

$$3f''' + 2ff'' - f'^2 = 0$$

$$f(0) = f''(0) = 0\,, \qquad f'(\eta) \to \eta \qquad \text{as } \eta \to \infty\,.$$

Numerical solution yields

$$U = \lambda\left[Y + c\left[\frac{X}{\lambda}\right]^{1/3} + \ldots\right] \quad \text{for } X^{1/3} \ll Y \ll 1\,.$$

$(c = 1.288\ldots)$

For $Y = O(1)$,

$$U = U_0(Y) + \left[\frac{X}{\lambda}\right]^{1/3} U_1(Y) + \ldots$$

$$V = \quad -\frac{1}{3}\left[\frac{1}{\lambda X^2}\right]^{1/3}\int U_1 dY + \ldots$$

From boundary-layer equation

$$U_0 U_1 - \int U_1 dY\, U_{0Y} = 0$$

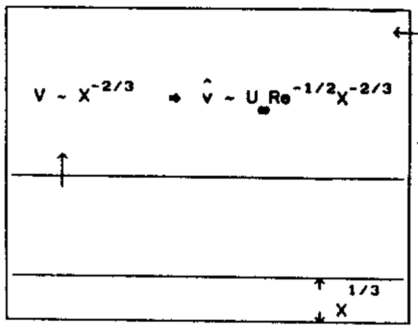
Solution which matches with the lower layer is

$$U_1 = cU_{0Y}\,.$$

Hence

$$V \to -\frac{c}{3(\lambda X^2)^{1/3}} \qquad \text{as } Y \to \infty\,.$$

Singularity at $X = 0$ $\quad\Rightarrow\quad$ boundary-layer approximation breaks down.



← Linear irrotational flow.

As before:

$$\hat{p} \sim U_\infty \hat{v} \sim U_\infty^2 Re^{-1/2} X^{-2/3}$$

$$\hat{p}_{\hat{x}} \sim \frac{U_\infty^2}{D} Re^{-1/2} X^{-5/3}$$

where D is length of plate, and

$$Re = \frac{U_\infty D}{\nu}$$

Perturbation pressure gradient should be small compared with inertia and viscous terms in lower layer. But

$$\frac{\hat{p}_{\hat{x}}}{\hat{u}\hat{u}_{\hat{x}}} = O\left[\frac{U_\infty^2}{D}\frac{1}{Re^{1/2}X^{5/3}}\cdot\frac{DX}{U_\infty^2 X^{2/3}}\right] = O\left[\frac{1}{Re^{1/2}X^{4/3}}\right]\,,$$

hence require

$$X \gg Re^{-3/8} = R^{-3/4}\,.$$

When $\quad X = O(R^{-3/4}) \quad$ need to rescale.

Same triple-deck scaling as before! (Stewartson 1969, Messiter 1970)

<u>Upper Deck</u> As before.

<u>Middle Deck</u> As before.

<u>Lower Deck</u> $\quad \hat{y} = R^{-1/4}\delta y = Re^{-5/8}Dy\,, \qquad \hat{x} = Re^{-3/8}DX$

$$\hat{u} \sim Re^{-1/8}U_\infty U\,, \qquad \hat{v} \sim Re^{-3/8}U_\infty V\,, \qquad \hat{p} = Re^{-1/4}U_\infty^2 P$$

$$UU_X + VU_Y = -P_X + U_{YY}\,, \qquad U_X + V_Y = 0$$

$$0 = -P_Y$$

$U = V = 0 \quad$ on $y = 0$, $\quad X < 0$

$U_y = V = 0 \quad$ on $y = 0$, $\quad X > 0$

$U \to U_{0y}(0)(Y + A) \quad$ as $Y \to \infty$

$U \to U_{0y}(0)Y \quad$ as $X \to -\infty$

$$P = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A'(\xi)}{X - \xi}\,d\xi\,.$$

Same equations as before, but different boundary conditions. Numerical solution by Jobe and Burggraf (1974).

Define drag coefficient, C_D, by

$$C_D = \frac{\text{Drag}}{\frac{1}{2}\rho U_\infty^2 D}\,.$$

Then by integrating skin friction over surface,

$$C_D \sim 1.328Re^{-1/2} + 2.65R^{-7/8} + \ldots$$

↑ Blasius $\qquad$ ↑ Triple-deck correction

Other trailing edge problems have been examined, e.g.

(a) aerofoils with wedge shaped trailing edges

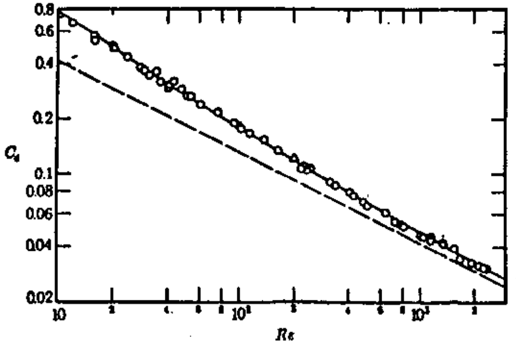


FIGURE 4. Drag against Reynolds number: a comparison of theory and experiment. ——, Equation (4.1); ○, Janour (1951); – –, Blasius (1908); △, Dennis (1973).

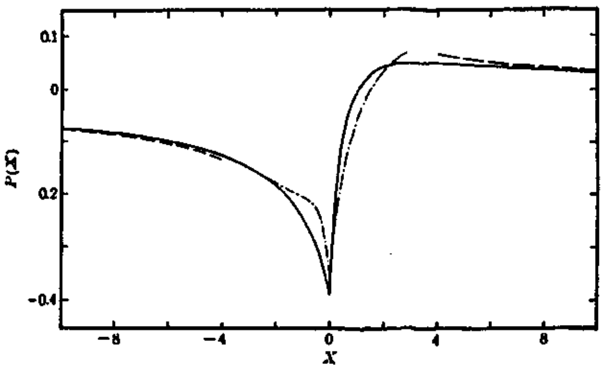


FIGURE 5. Induced pressure distribution. ——, Present results; —·—, Messiter (1970); – –, leading term of equation (2.13).

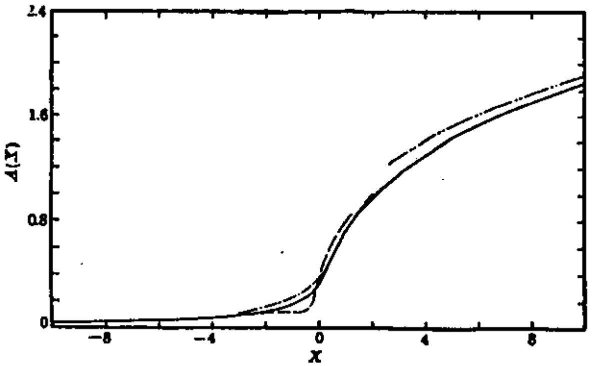
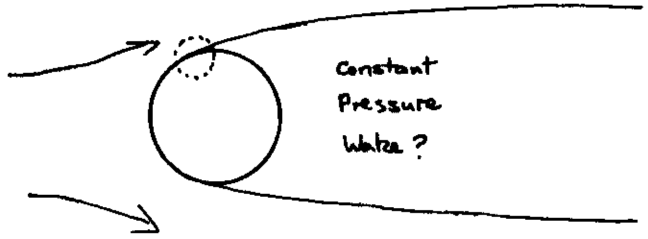


FIGURE 6. Lower deck displacement function, $A(X)$. ——, Present results; – –, Messiter (1970) computed; —, Messiter (1970) assumed; –·–, leading term of equation (2.12).

(b) aerofoils with cusped trailing edges at incidence

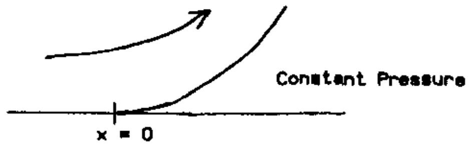
(c) oscillating aerofoils.

Gross Separation



Neighbourhood of separation point.

$$y = S(x) = S_0 x^{n+1} + \dots \qquad n > 0$$



Free streamline must come off tangentially, otherwise deceleration associated with stagnation point will lead to separation upstream.

Assume small velocity perturbations for $|x| \ll 1$, and that slip velocity normalised to 1 at separation point. Then

$$u \sim 1 + \tilde{u}\ , \text{ etc.}$$

where

$$\tilde{u}_x = -\tilde{p}_x\ , \qquad \tilde{v} = -\tilde{p}_y\ , \qquad \tilde{u}_x + \tilde{v}_y = 0$$

$$\nabla^2 \tilde{p} = 0 \qquad \Rightarrow \qquad \tilde{p} = -nS_0 r^n \sin n\theta$$

Boundary conditions on $y = 0,\ x < 0$ and $y = 0,\ x > 0$ yield

$$\tilde{p} = -nS_0 r^n \sin n\theta \qquad \text{with } n = \tfrac{1}{2} + m, \qquad m = 0,1,2,\dots \quad \text{and } S_0 > 0\ .$$

With $n = 1/2$

$$\frac{d\tilde{p}}{dx} = \frac{S_0}{4(-x)^{1/2}} > 0 \qquad \text{on } y = 0,\ x < 0\ .$$

Adverse, infinite pressure gradient at $x = 0$. Usual argument is that this leads to separation, hence require $S_0 = 0$: Brillouin-Villat condition.
Sychev (1972) argues that

$$S_0 = O(\varepsilon) \qquad \text{where} \qquad \varepsilon \propto Re^{-a}$$

Blowing velocity	$\Rightarrow v \sim Re^{-1/2}\varepsilon^{1/2}(-x)^{3/4}$
	$p \sim Re^{-1/2}\varepsilon^{1/2}(-x)^{3/4}$
No slip	$\Rightarrow u \sim U_0 + \varepsilon^{1/2}(-x)^{1/4}$
Continuity	$\Rightarrow v \sim Re^{-1/2}\varepsilon^{1/2}(-x)^{-3/4}$
$p_x \sim \varepsilon(-x)^{-1/2}$,	
nonlinear	$\Rightarrow u \sim \varepsilon^{1/2}(-x)^{1/4}$
shear	$\Rightarrow y \sim \varepsilon^{1/2}(-x)^{-1/4}Re^{-1/2}$
$p_x \sim \frac{1}{Re}u_{yy}$	$\Rightarrow (-x) \sim \varepsilon^{6}$

Interaction when

$$\varepsilon(-x)^{1/2} \sim \frac{Re^{-1/2}\varepsilon^{1/2}}{(-x)^{3/4}}$$

i.e. $\qquad \varepsilon \sim Re^{-1/16}$

$$(-x) \sim Re^{-3/8}.$$

Standard triple-deck scaling, but expansion now proceeds in powers of $Re^{-1/16}$ rather than $Re^{-1/8}$.

Lower Deck Problem

$$UU_X + VU_Y = -P_X + U_{YY}, \qquad U_X + V_Y = 0$$

$$U = V = 0 \qquad \text{on } Y = 0$$

$$U \to Y + D_1Y^{1/2} + D_2\ln Y + A \qquad \text{as } Y \to \infty$$

$$\left.\begin{aligned} &U \to Y, \\ &P \to -\alpha_0|X|^{1/2} - \alpha_1|X|^{-1/2} \\ &A \to 6^{1/3}\Gamma\left(\frac{2}{3}\right)\alpha_0|X|^{1/6} + E_1\ln|X| + E_2 \end{aligned}\right\} \quad \text{as } X \to -\infty$$

$$A \to -\frac{1}{2}\alpha_0X^{3/2}, \qquad P \to -\frac{9C_0^{\,2}}{8\alpha^2}X^{-6/3} \qquad \text{as } X \to +\infty$$

$$A' = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{P(\xi)}{X-\xi}\,d\xi$$

where $\quad \alpha_1$, C_0, D_1, D_2, E_1 and E_2 are known constants.

Numerical solution by Smith (1977), Korolev (1980)

$$\alpha_0 \approx 0.44.$$

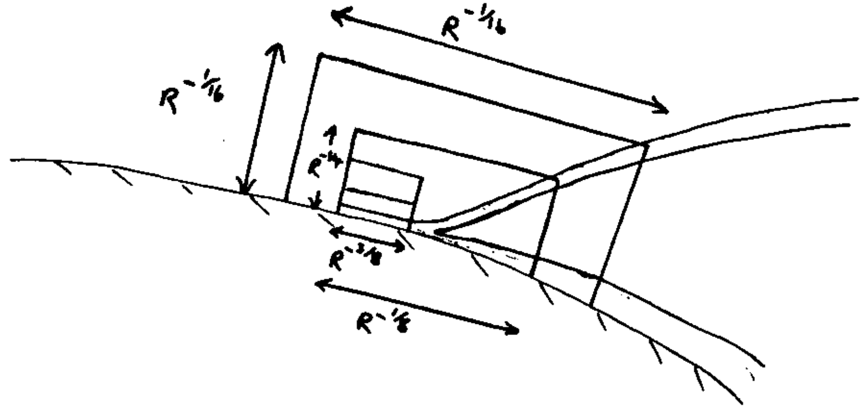
On global scale

$$\hat{y} \sim Re^{-1/16}\,\frac{2}{3}\alpha_0\lambda^{9/8}x^{3/2} \qquad \text{as } x \to 0+$$

where λ is the boundary-layer skin friction.

Gross separation past a bluff body

(Schematic)



Free streamline

$$\hat{y} \sim \underset{\substack{\uparrow \\ \text{curvature}}}{k_0 x^2} + \underset{\substack{\uparrow \\ \text{leading order} \\ \text{inviscid solution}}}{\tfrac{2}{5}k_1 x^{5/2}} + \underset{\text{triple-deck correction}}{R^{-1/16}\left[\tfrac{2}{3}\alpha_0\lambda^{9/8}\right]x^{3/2}}$$

How is the position of separation fixed?

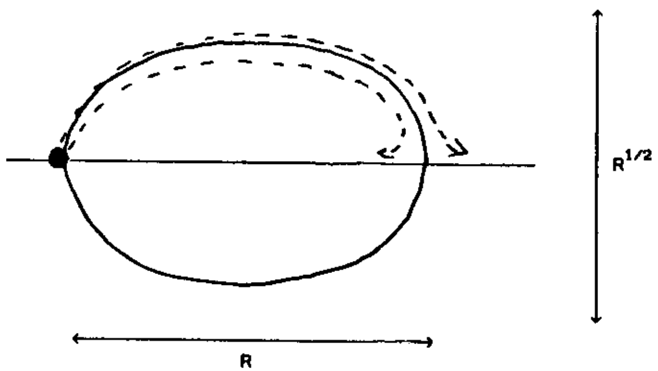
(a) Solve inviscid free-streamline problem to fix slip velocity

(b) Integrate boundary layer equations from stagnation point to separation point to find λ.

(c) Given λ find change in free streamline solution.

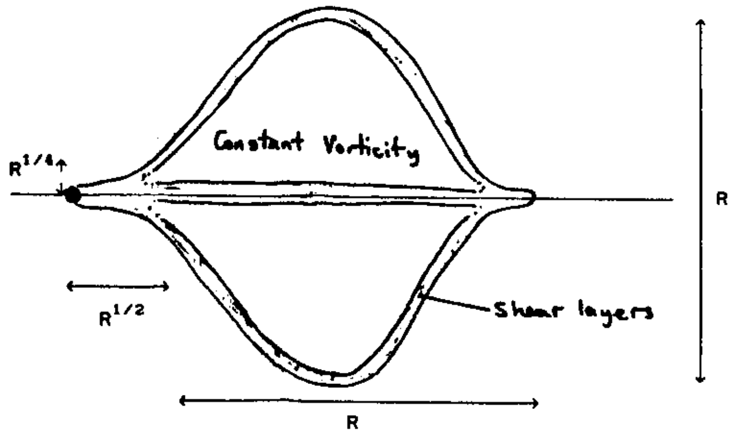
Far downstream

Smith (1979a)



Wake has an elliptic shape. Difficulty with wake reattachment.

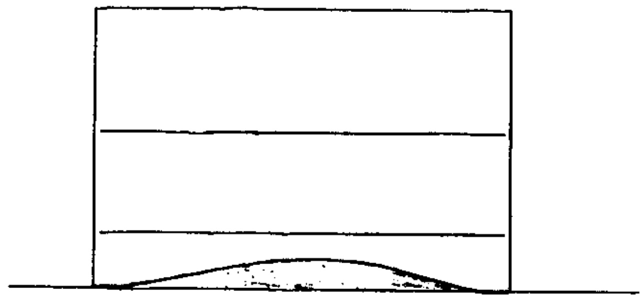
Smith (1985)



Sadovskii Vortex

Unsteady Triple-Deck

Example



Oscillating rivet/ribbon $\quad y = HF(X,T)$,

where $\quad t = Re^{-1/4}T$: unsteady triple-deck scaling

$$U_T + UU_X + VU_Y = -P_X + U_{YY}\,, \qquad U_X + V_Y = 0$$
$$U = 0\,, \quad V = HF_T \qquad \text{on } Y = HF(X, T)$$
$$u \to \lambda(Y + A(X, T)) \qquad \text{as } Y \to \infty$$
$$P = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A_\xi(\xi, T)}{X - \xi}\, d\xi\ .$$

Suppose $F \equiv 0$, i.e. no forcing. Then

$$U = \lambda Y$$

is a solution. What happens if seek linear solution

$$U \sim \lambda Y + \tilde{U}\,, \quad \text{etc.}$$

Seek normal mode solution for $\tilde{U}$, with

$$\tilde{U} = \bar{U} \exp\,(iKX - i\Omega T), \quad \text{etc.}$$

Then

$$iK\bar{U} + \bar{V}_Y = 0\,, \qquad \bar{P}_Y = 0$$
$$i(\lambda YK - \Omega)\bar{U} + \lambda\bar{V} + iK\bar{P} - \bar{U}_{YY} = 0$$
$$\bar{U} = \bar{V} = 0 \qquad \text{on } Y = 0\,,$$
$$\bar{U} \to \lambda\bar{A} \qquad \text{as } Y \to \infty\,.$$
$$\bar{P} = K\,\bar{A}$$

Set

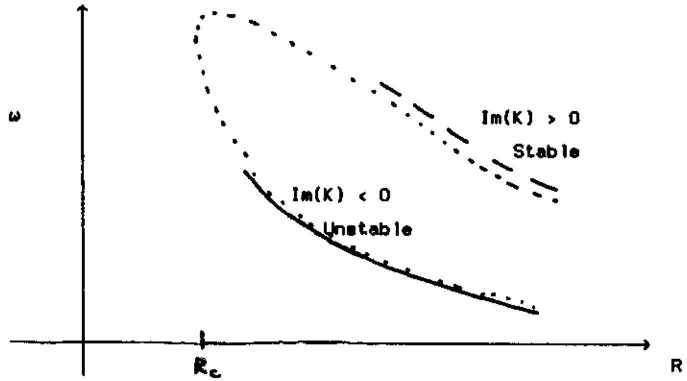
$$\xi = s(Y - \frac{\Omega}{\lambda K})\,, \qquad s = (i\lambda K)^{1/3}$$
$$\xi\bar{U}_\xi - \bar{U}_{\xi\xi\xi} = 0$$
$$\bar{U}_\xi = \frac{iK_1P_1}{s^2 Ai'(\xi_0)}\, Ai(\xi)\,, \qquad \xi_0 = -\frac{s\Omega}{\lambda K}$$
$$iK^2 \int_{\xi_0}^{\infty} Ai(q)\, dq = \lambda s^{2/3} Ai'(\xi_0) \qquad : \quad \text{Eigenrelation.}$$

Hence given frequency, Ω, and wall shear λ, can solve for (complex) wavenumber K. For $\Omega \geq \Omega_c$ unstable waves can be found. $\Omega = \Omega_c$ specifies frequency of neutral curve in terms of T variable. Conventionally stability analysis performed in terms of scaled variable

$$\hat{t} = \frac{\delta}{\bar{U}}\,\tau \qquad \rightarrow \qquad \tau = Re^{1/2}t = Re^{1/4}T\ .$$

In terms of τ variable neutral frequency, ω_c, given by

$$\omega_c = Re^{-1/4}\Omega_c = R^{-1/2}\Omega_c\ .$$



—— : Triple-deck lower branch asymptote

- — — : Multiple-deck upper branch asymptote

.... : Orr-Sommerfeld approximation

<u>Orr-Sommerfeld theory</u> yields an accurate dispersion relation between frequency, wavenumber and Reynolds number for parallel flows:

$$\underline{u} = (u_0(y),\ 0\ ),$$

where u_0 satisfies Navier-Stokes equations. Write

$$\psi = \psi_0(y) + \tilde{\psi}\exp(ikX - ict), \qquad \text{etc.}$$

Substitute into Navier-Stokes equation, then

$$(i\alpha R)^{-1}(D^2 - k^2)^2\tilde{\psi} = (u_0 - c)(D^2 - k^2)\tilde{\psi} - u_0''\tilde{\psi}$$

where $D = \frac{d}{dy}$.

Solution to this equation with no slip boundary conditions yields eigenrelation

$$F(\alpha,\ c,\ R) = 0\ .$$

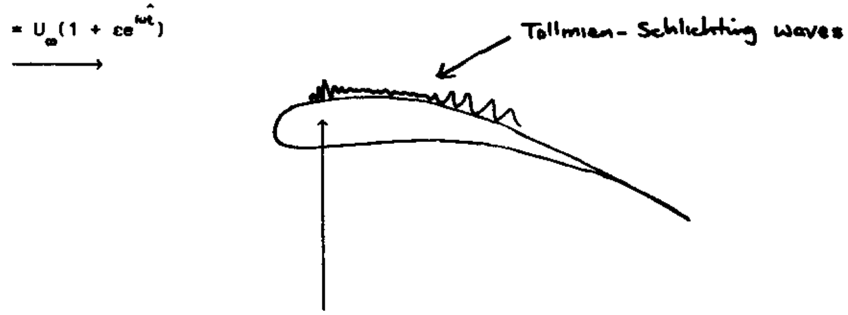
This theory is often applied to quasi-parallel flows where the velocity profile $u_0(y)$ actually varies *slowly* with x, e.g. the Blasius boundary layer. If the slow dependence in x is neglected then the frozen velocity profile $u_0(y)$ does not satisfy the Navier-stokes equation - as a result the Orr-Sommerfeld equation is a *heuristic* approximation. Note that R is assumed both order one in deriving the Orr-Sommerfeld equation, and asymptotically large so that the basic flow can be assumed to be quasi-parallel - this is inconsistent. However, Orr-Sommerfeld theory does yield a critical Reynolds number for instability.

<u>Triple-Deck theory</u> is only valid at large Reynolds numbers - hence it does not yield a critical Reynolds number. However it provides a consistent way of obtaining higher order corrections, in particular the effects of non-parallelism can be included (Smith, 1979b).

Above approach can be extended in many ways.

I. Receptivity problems:

$u = U_\infty(1 + \varepsilon e^{i\hat{\omega}t})$



almost separated flow region

Interaction of modulated freestream in region of almost separated flow can lead to the generation of Tollmien-Schlichting waves (Goldstein, Leib & Cowley, 1987).

II. Nonlinear Effects can be examined:

(a) in the neighbourhood of the neutral curve (analytical) - Smith (1979c).

(b) for order one triple-deck frequencies (numerical) - Tutty & Cowley (1986). Tollmien-Schlichting waves are found to grow until a *singularity* forms with an associated rapid shortening of scales (cf. transition and unsteady classical boundary-layer separation).

(c) for high frequency T-S waves (analytical) - Smith (1986).

In addition, the analysis can be extended to three dimensions, and a systematic description of effects important in transition to turbulence can be identified, e.g.

(a) resonant triad interactions - Smith & Stewart (1987).

(b) Tollmien-Schlichting/Görtler wave interactions - Hall & Smith (1988).

III. Similarly asymptotic analysis can be performed to look at linear and nonlinear instability in the neighbourhood of the upper neutral curve. For example Smith & Bodonyi (1982) have shown that Hagan-Poiseuille pipe flow is unstable to *nonlinear* disturbances.

IV. The stability of Görtler vortices has been examined by Hall & Lakin (1988) using a high Re asymptotic approach. Goldstein & Leib (1988) have examined the stability of shear layers using the idea of non equilibrium critical layers.

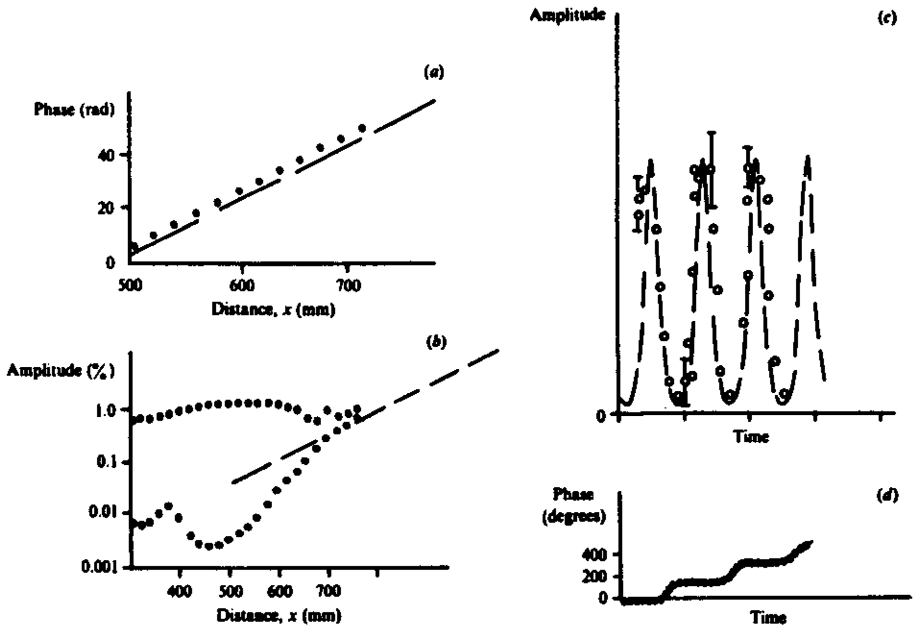


FIGURE 7. Comparisons with the (nonlinear) experiments of Kachanov & Levchenko (1984). (*a*), (*b*) The growths [versus distance x] of phase and amplitude for the fundamental and subharmonic components: ○○○○ (fundamental), ●●●● (subharmonic) experiments; ---, theory, from (4.14) combined with the triple-deck scalings. (*c*), (*d*) compare representative experimental values (○○○○, with typical scatter/reading error shown I) of the subharmonic amplitude and phase with the theory (——), versus normalized/slow time, using figure 17(*b*) of Kachanov & Levchenko (1984) and figures 4(*c*), 4(*f*), 5 of the present paper.