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SMR.378/15

WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

NUMERICAL BIFURCATION THEORY

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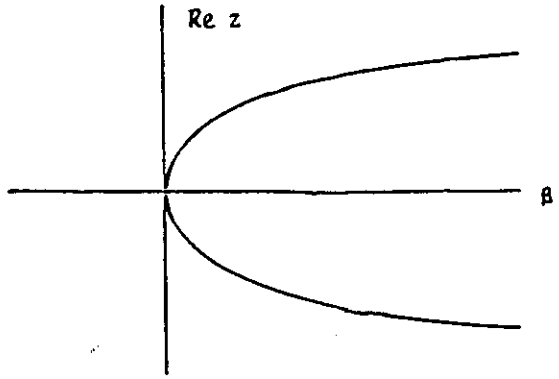
APPLICATIONS

- SPATIAL ORDERING IN CONVECTION
- PATTERN FORMATION IN REACTION-DIFFUSION
- CHEMICAL OSCILLATORS
- OSCILLATORY CONVECTION IN LIQUID METALS
- MULTIPLE SOLUTIONS IN SEMICONDUCTOR TRANSPORT
- OPTICAL PHASE CONJUGATION
- INTERFACES
- VORTEX SHEDDING
- FLOW INDUCED OSCILLATIONS IN STRUCTURES
- EXPLOSIVE CRYSTALLIZATION
- THERMAL IGNITION

SIMPLE BIFURCATION THEORY

$$z(z^2 - \beta) = 0, \ \beta \text{ real}$$

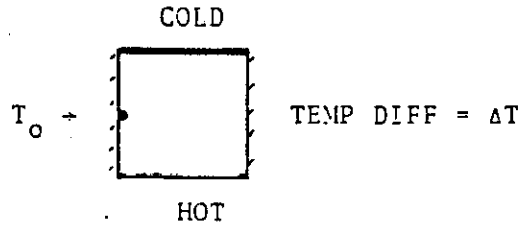
$$z = 0 \quad \text{and} \quad z = \pm \sqrt{\beta}$$

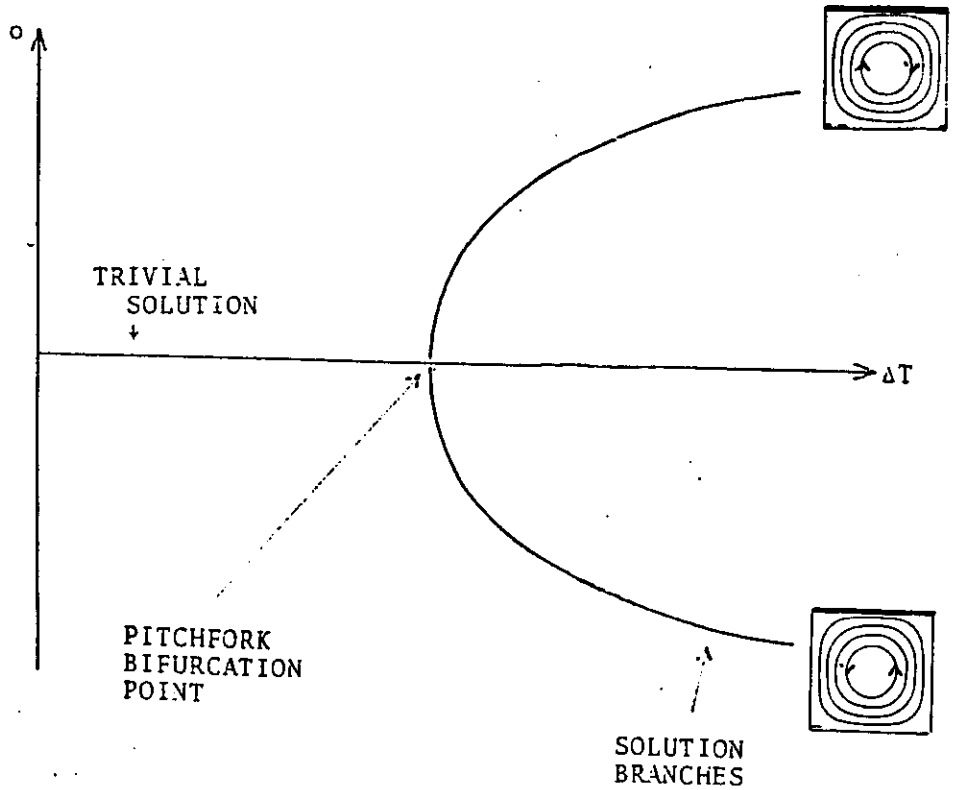


CHANGE IN NUMBER OF SOLUTIONS AT A POINT IS CALLED A BIFURCATION.

STUDY OF THE QUALITATIVE CHANGES IN THE NUMBER AND NATURE OF SOLUTIONS AS A PARAMETER VARIES IS CALLED BIFURCATION THEORY

UID FLOW





ATE DIAGRAM

NUMERICAL METHODS

- WEAK FORMULATION
- GALERKIN METHOD: TEST AND TRIAL FUNCTIONS ARE BOTH TAKEN TO BE QUADRATIC FUNCTIONS BASED ON NINE-NODED QUADRILATERAL ELEMENTS
- $$M\frac{\partial x}{\partial t} + f(x,Ra,h) = 0$$
- STEADY STATE EQUATIONS ARE SOLVED BY NEWTON ITERATION
- METHOD FAILS WHERE JACOBIAN f_x IS SINGULAR - BIFURCATION POINTS?
- USE EXTENDED SYSTEMS APPROACH TO DETERMINE EXACT LOCATION AND AND NATURE OF BIFURCATION POINT
- STABILITY DETERMINED BY (GENERALISED) EIGENVALUES OF JACOBIAN

WEAK FORMULATION

USE

$$f_\psi \nabla^2 \psi = \nabla.(f_\psi \nabla\psi) - \nabla f_\psi . \nabla\psi$$

+ GREEN'S THEOREM

$$\int_\Omega f_\psi \left\{ - \frac{Ra}{h} \frac{\partial \theta}{\partial x} \right\} + \int_\Omega \frac{\partial f_\psi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial f_\psi}{\partial y} \frac{\partial \psi}{\partial y} - \int_\Gamma f_\psi \frac{\partial \psi}{\partial n} = 0$$

$$\int_\Omega f_\theta \left\{ \frac{1}{h} \left[\frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} \right] + \frac{\partial \theta}{\partial t} \right\} + \int_\Omega \frac{\partial f_\theta}{\partial x} \frac{\partial \theta}{\partial x} + \frac{\partial f_\theta}{\partial y} \frac{\partial \theta}{\partial y} - \int_\Gamma f_\theta \frac{\partial \theta}{\partial n} = 0$$

GALERKIN FORMULATION: USE QUADRATIC FUNCTIONS BASED ON NINE-NODED QUADRILATERAL ELEMENTS AS TEST AND BASIS FUNCTIONS;

GAUSS QUADRATURE ⇒

$$M \frac{\partial x}{\partial t} + f\ (x, \lambda, \alpha) = 0.$$

TO FIND A REGULAR POINT OF

$$f(x,\lambda) = 0 \qquad f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N.$$

NEWTON-RAPHSON ITERATION:

$$f_x(x^n,\lambda)\Delta x^{n+1} = -f(x^n,\lambda)$$

WHERE: $\Delta x^{n+1} = x^{n+1} - x^n$

$f_x = \frac{\partial f}{\partial x}$ (JACOBIAN)

<u>LU DECOMPOSITION OF f_x</u>

DECOMPOSE $f_x = LU$

$\rightarrow Le = -f(x^n,\lambda)$

& $U\Delta x^{n+1} = e$.

<u>COST</u> : MAINLY LU DECOMPOSITION.

<u>FAILS</u> : when f_x IS SINGULAR.

DETECTING SINGULAR POINTS

CONSIDER $f(x,\lambda) = 0 \qquad x \in \mathbb{R}^N,\ \lambda \in \mathbb{R}^M$

$$f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$$

IMPLICIT FUNCTION THEOREM

IF $f(x_0,\lambda_0) = 0$ AT A POINT (x_0,λ_0), THEN $\exists$ A UNIQUE SOLUTION $x = x(\lambda)$ NEAR λ_0 PROVIDED THAT

$$f_x = \frac{\partial f}{\partial x} \quad \text{(JACOBIAN)}$$

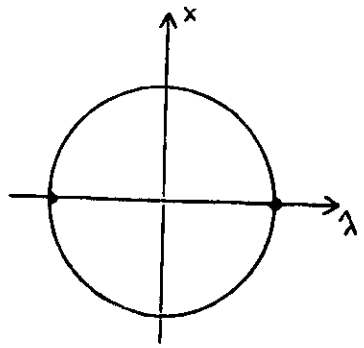
IS NON-SINGULAR (NON-ZERO).

SINGULAR POINT. A POINT (x_0,λ_0) AT WHICH f_x IS SINGULAR (I F T BREAKS DOWN).

DETECTION OF MULTIPLE SOLUTIONS	≡	DETECTION OF SINGULAR POINTS

(SINGULAR f_x IS NECESSARY BUT NOT SUFFICIENT)

EXAMPLE



$f(x,\lambda) = x^2 + \lambda^2 - 1 = 0$

UNIQUE SOLUTION NEAR (x_0,λ_0) EXCEPT AT $\lambda = \pm 1$ WHERE $\frac{\partial f}{\partial x}$ $(= 2x)$ IS ZERO.

IN THIS SIMPLE CASE $x = x(\lambda)$ CAN BE WRITTEN DOWN EXPLICITLY.

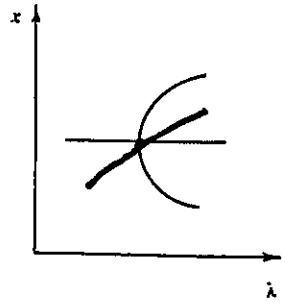
CONSIDER

$$f(x,\lambda) = e^x \sin \lambda + x^2\lambda^2 - \lambda \ln x + 1 = 0.$$

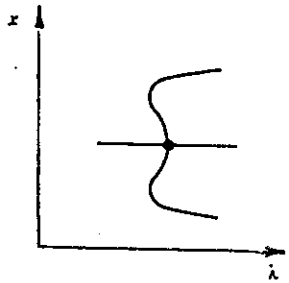
HERE

$x = x(\lambda)$ IS GIVEN IMPLICITLY.

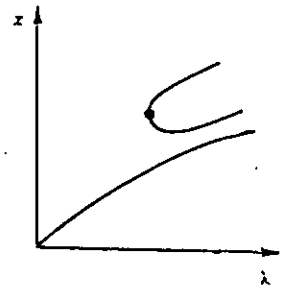
EXAMPLES OF CRITICAL POINTS



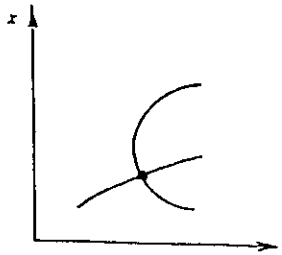
SUPERCRITICAL SYMMETRY-BREAKING BIFURCATION POINT



SUBCRITICAL SYMMETRY-BREAKING BIFURCATION POINT



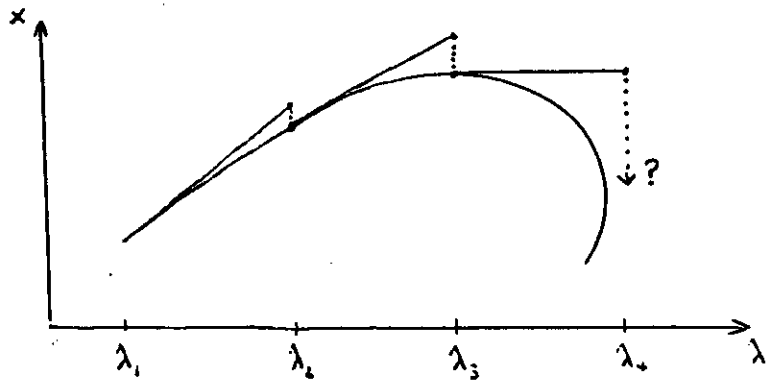
ONE-SIDED BIFURCATION POINT OR LIMIT POINT



TRANSCRITICAL BIFURCATION POINT

FOLLOW SOLUTION BRANCH WHILE CHECKING FOR f_x SINGULAR

PARAMETER CONTINUATION



AT $\lambda = \lambda_{n+1}$ USE INITIAL GUESS

$$x(\lambda_{n+1}) = x(\lambda_n) + (\lambda_{n+1} - \lambda_n) \left.\frac{\partial x}{\partial \lambda}\right|_{\lambda=\lambda_n}$$

OBTAIN $\frac{\partial x}{\partial \lambda}$ VIA $f_x \frac{\partial x}{\partial \lambda} = -f_\lambda$.

COST : SMALL USE LU DECOMPOSITION OF f_x ON LAST NEWTON-RAPHSON ITERATION AT $\lambda = \lambda_n$.

WHEN IS f_x SINGULAR?

(i) DET $(f_x) = 0$

(ii) $f_x \varphi = 0 \qquad \varphi \in \mathbb{R}^N, \ \varphi \neq 0$

i.e. $f_x \varphi = \sigma \varphi$ WITH $\sigma = 0$ (ZERO EIGENVALUE MULTIPLICITY 1)

(i) $\mathrm{DET}(f_x) = \mathrm{DET}(L)\ \mathrm{DET}(U)$

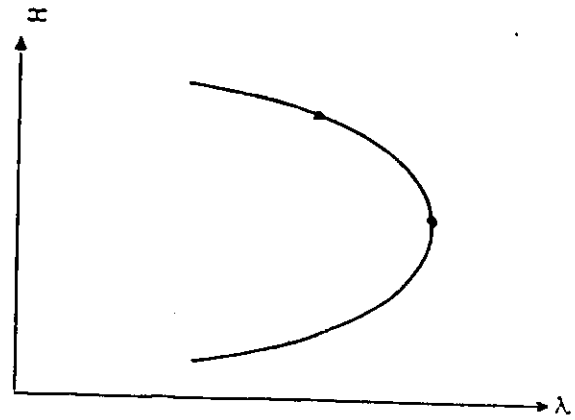
$$= \prod_{i=1}^{N} L_{ii} \prod_{j=1}^{N} U_{jj}$$

DET (f_x) GENERALLY CHANGES SIGN AT SINGULAR POINT

(ii) REQUIRES EIGENVALUE CALCULATION ALONG THE BRANCH.

LOCATING SINGULAR POINTS EXACTLY

A. TURNING (OR LIMIT) POINTS



CODIMENSION : 0

CANONICAL FORM : $f = x^2 + \lambda$

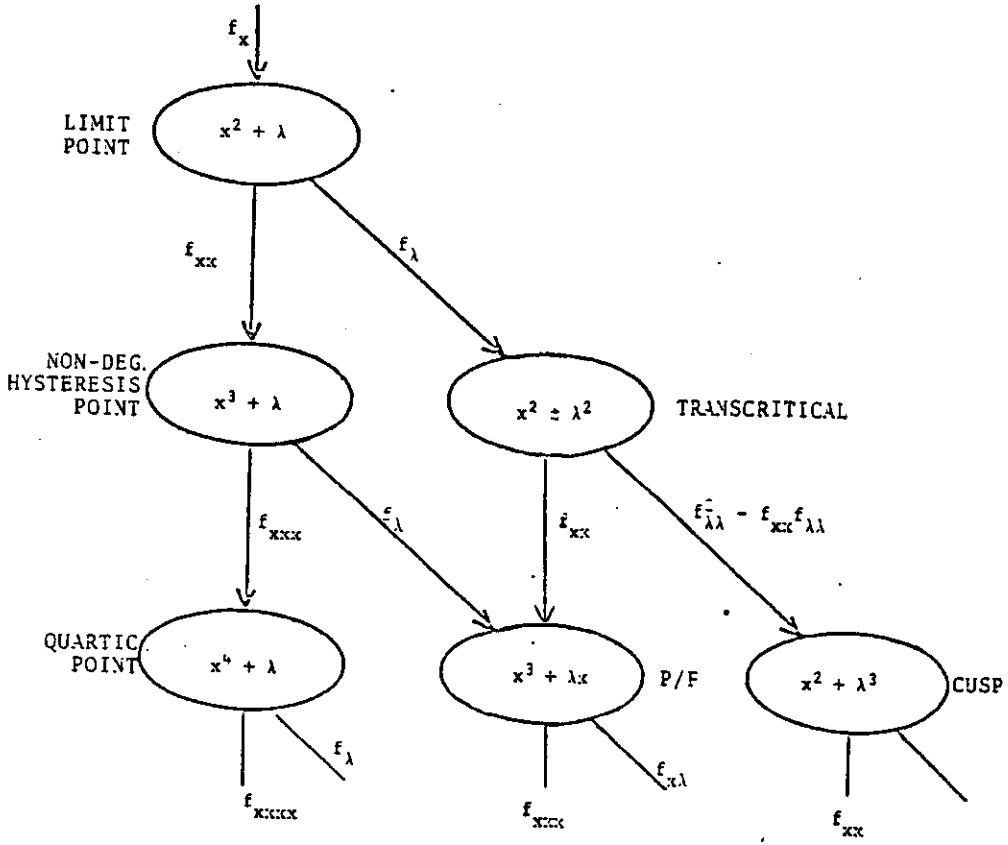
DEFINING CONDITIONS : $f_x = 0$

SIDE CONSTRAINTS : $f_{xx} \neq 0$

$f_\lambda \neq 0.$

FOR SCALAR PROBLEM $f(x,\lambda) = 0 \qquad x \in \mathbb{R}$

JEPSON - SPENCE HIERARCHY

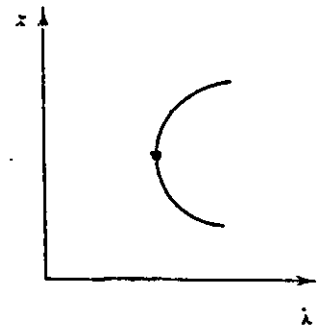


DEFINING CONDITION LABELS ON PATH TO CANONICAL FORM VANISH

SIDE CONSTRAINTS LABELS ON PATH LEAVING NODE ARE NON-ZERO.

EXTENDED SYSTEMS

E.G. FOR A LIMIT POINT



$f(x,\lambda) = x^2 - \lambda = 0.$

AT CRITICAL POINT $x = 0, \quad \lambda = 0$

$$\frac{\partial f}{\partial x} = 0; \quad \frac{\partial^2 f}{\partial x^2} \neq 0, \quad \frac{\partial f}{\partial \lambda} \neq 0.$$

EXTENDED SYSTEM

$$f(x,\lambda,\alpha) = 0$$

$$f_x(x,\lambda,\alpha)\varphi = 0$$

$$\ell\varphi = 1$$

$$\psi f_\lambda = 0$$

$$\psi f_{xx}\varphi\varphi = 0$$

MOORE-SPENCE EXTENDED SYSTEM.

STABILITY OF SOLUTIONS

$$M\frac{\partial x}{\partial t} + f(x,\lambda,\alpha) = 0$$

LINEARISE ABOUT STEADY SOLUTION $x = x_0$: $x = x_0 + x_1$

$$M\frac{\partial x_1}{\partial t} + f_x(x_0,\lambda_0,\alpha)x_1 = 0$$

$$\rightarrow x_1(t) = e\, e^{-\sigma t}\varphi$$

WHERE $f_x\varphi = \sigma M\varphi$.

FOR σ REAL :

$\sigma > 0$	STABILITY
$\sigma < 0$	INSTABILITY
$\sigma = 0$	POSSIBLE MULTIPLE SOLUTIONS

SIGN(DET(f_x)) = $(-1)^n$, n = NUMBER OF NEGATIVE EIGENVALUES.

HOPF BIFURCATIONS

HOPF POINTS ARE NOT SINGULAR POINTS

EIGENVALUES :

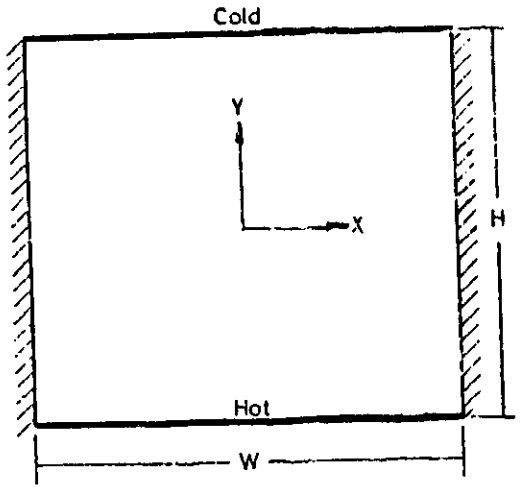
$$\sigma = i\sigma_1, -i\sigma_1, \sigma_2, \sigma_3, \ldots \qquad (\sigma_i > 0)$$

$\therefore$ product $\neq 0$

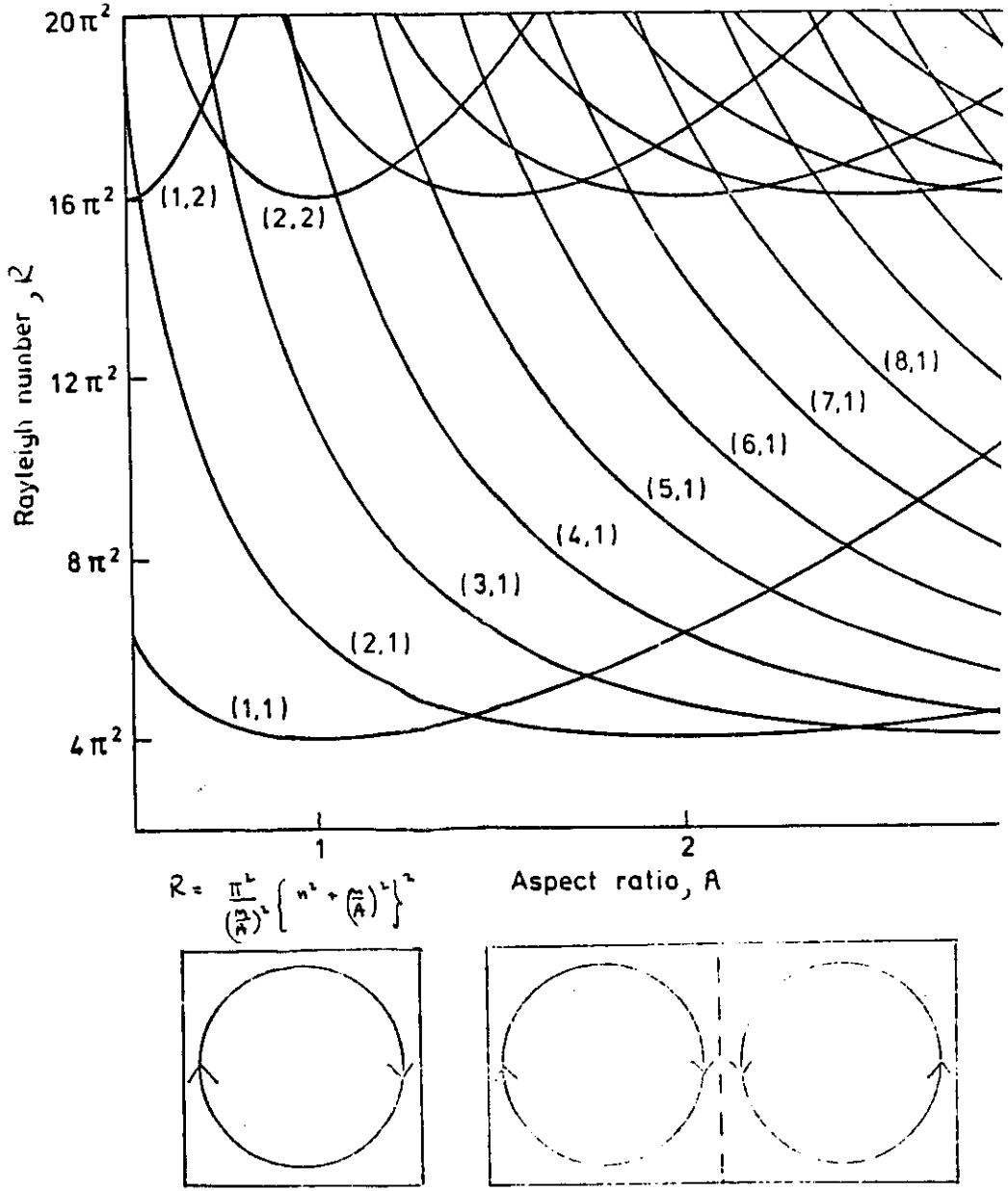
$\therefore$ JACOBIAN NONSINGULAR

CRIEWANK AND REDDIEN:

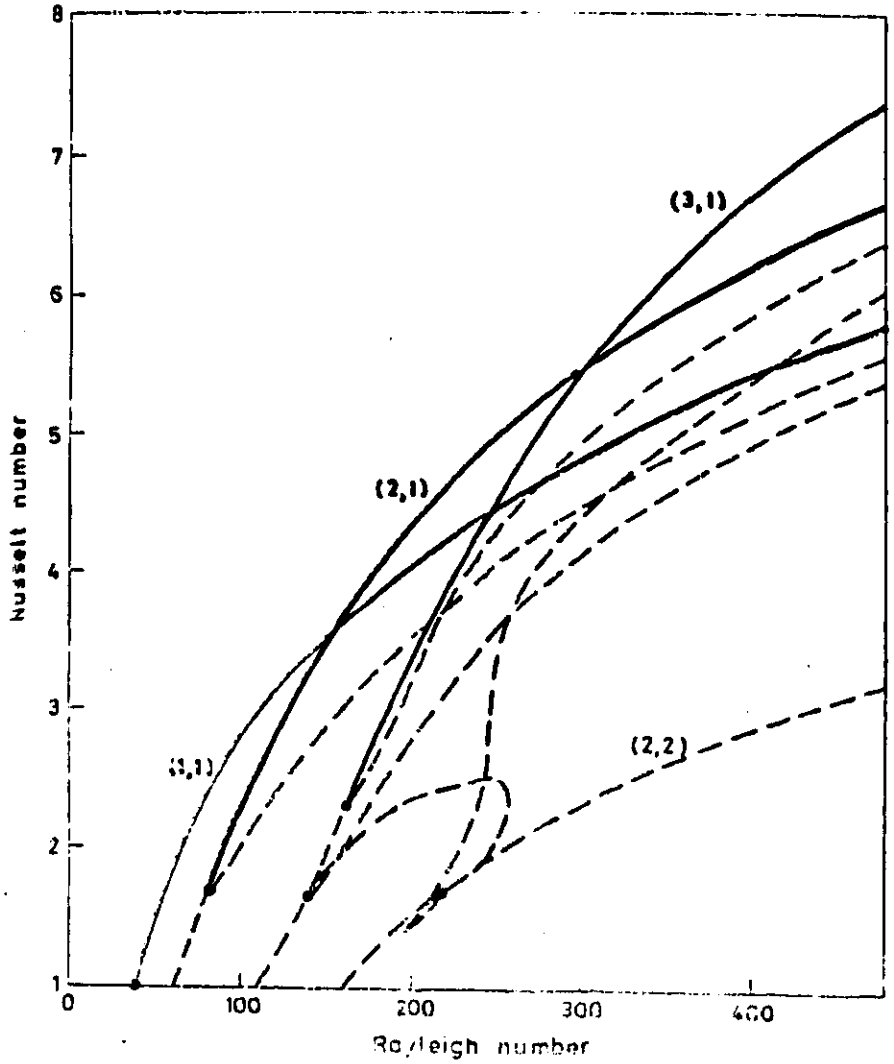
$$f = 0$$
$$f_x\xi_r + \sigma_1 M\xi_i = 0$$
$$f_x\xi_i - \sigma_1 M\xi_r = 0$$
$$l\xi_r = 0$$
$$l\xi_i - 1 = 0$$

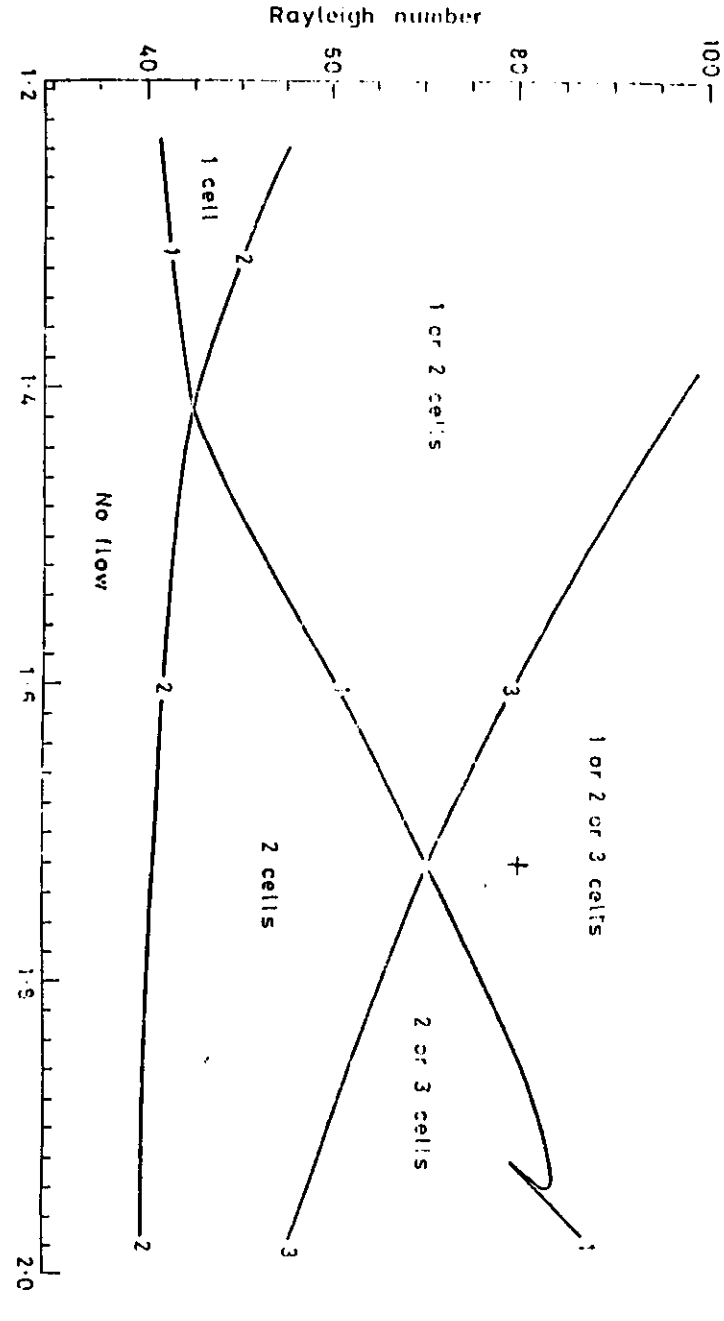


≡ insulated boundaries



Square container





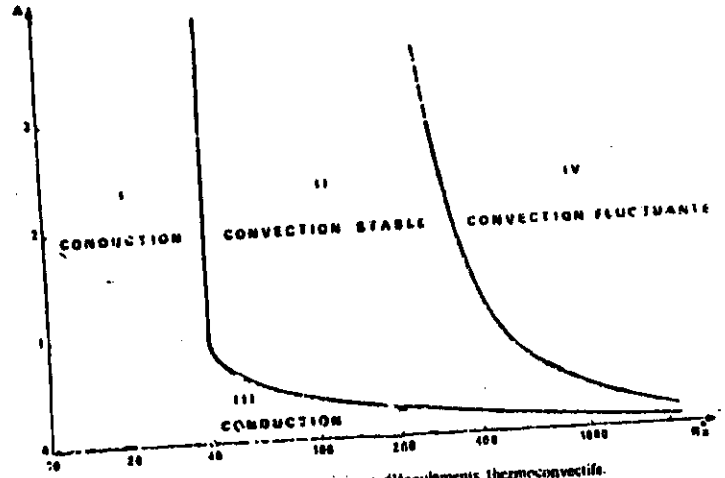


Fig. 1. — Différents régimes d'écoulements thermoconvectifs.

Les limites du domaine d'investigation sont comprises entre $Ra^* = 10$ et $Ra^* = 2\,000$ pour une variation de l'allongement de la cellule $A = L/d$ de 0,02 à 4. Le programme numérique permet de résoudre le système avec des conditions aux limites du type Neumann sur les plans verticaux, du type Dirichlet ou Fourier sur les plans horizontaux; dans le cas présent où ces plans sont isothermes, ce sont donc des conditions de Dirichlet et on a ainsi, pour les quatre côtés du rectangle :

$$T = 1 \quad \text{pour} \quad z = 0; \quad \forall x, \quad \frac{\partial T}{\partial x} = 0 \quad \text{pour} \quad x = 0,\ 1; \quad \forall z;$$

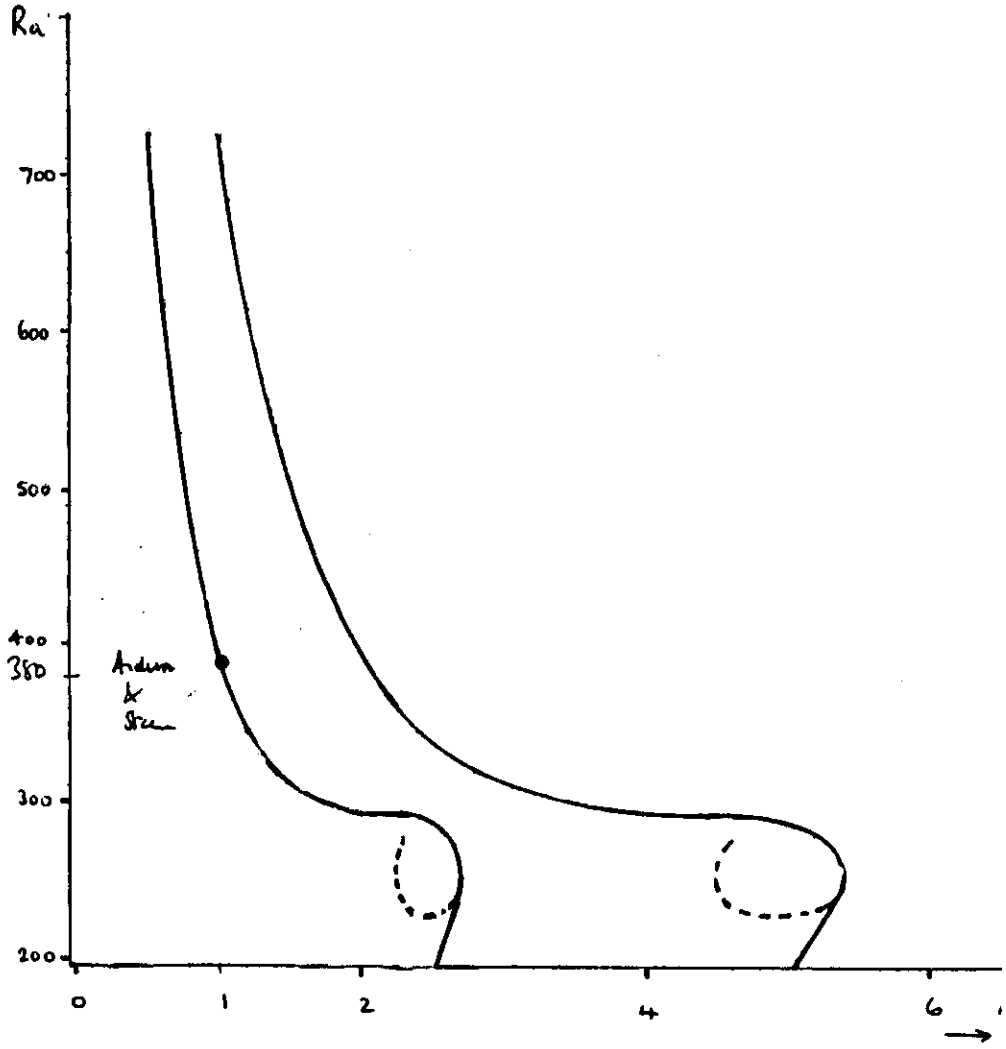
$$T = 0 \quad \text{pour} \quad z = 1; \quad \forall x.$$

3. Les résultats de cette étude numérique font ressortir plusieurs types de convection suivant les valeurs du nombre de Rayleigh Ra^* et de l'allongement A de la cellule (*fig.* 1) :

Région I. — Pour un nombre de Rayleigh inférieur à $4\pi^2$, les perturbations introduites dans le calcul s'amortissent quel que soit l'allongement de la cellule; ce résultat est en parfait accord avec la théorie.

Région II. — Les perturbations initiales s'amplifient pour donner une solution convergente stable; le champ de température est d'autant plus affecté par la présence du rouleau convectif que le nombre de Rayleigh est grand. La variation du transfert de chaleur, représenté par le nombre de Nusselt, en fonction de l'allongement présente un maximum qui se déplace vers les petits allongements lorsque le nombre de Rayleigh Ra^* augmente, tout en restant dans la région II, résultat analogue à (9).

Pour des nombres de Rayleigh inférieurs à 240 et des allongements de l'ordre de 3, le calcul montre qu'un passage a lieu sans transition du régime stable avec un tourbillon



Paths of Hopf bifurcations