INTERNATIONAL ATOMIC ENERGY AGENCY
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SMR.378/16

WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

AN INTRODUCTION TO BIFURCATION PHENOMENA IN FLUID MECHANICS

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AN INTRODUCTION TO BIFURCATION IN FLUID MECHANICS

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TRIESTE JAN. '87

CONTENTS

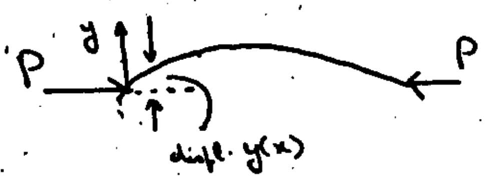
BIFURCATION

Bifurcation – change in number of solutions as a parameter is varied — or qualitative change in equilibrium solutions of equations.
– more generally – multiplicity

Singularity theory – steady state phenomena only – gives equivalence of types of bifurcation ⇒ "universal unfolding".
Schaeffer (1980) SLN 505
Golubitsky + Shaeffer (1985)

Time-dependent bifurcation – no complete theory available but, see Guckenheimer and Holmes (1983)
"Nonlinear Oscillations, Dynamial Systems and Bifurcation of Vector Fields"
"Universality in Chaos" – Citanovic (1984/5)
"The Lorenz Equations: Bifurcations, Chaos and Strange Attractors" C. Sparrow (1983) Vol 41 Allied Math Serie

EULER STRUT (EULER (1744))



Equilibrium equation

$$B\frac{d^2y}{dx^2} + Py = 0 \quad \ldots (1)$$

(lin. y ≈ sin y)

$B = EI$ is the elastic bending stiffness

B.C. $y(0) = y(L) = 0$

Rewrite (1) as

$$(D^2 + \omega^2)y = 0 \quad \ldots (2) \quad \text{where } \omega^2 = P/B$$

gen. soln.

$$y = A\sin\omega x + B\cos\omega x \quad \ldots (3)$$

$y(0) = 0 \rightarrow B = 0$

$y(L) = 0 \rightarrow A\sin\omega L = 0$

$\rightarrow A = 0$ — trivial soln.

or $\omega l = n\pi$

$$P_c = B\left(\frac{n\pi}{L}\right)^2 \quad (n = 1, 2, \ldots)$$

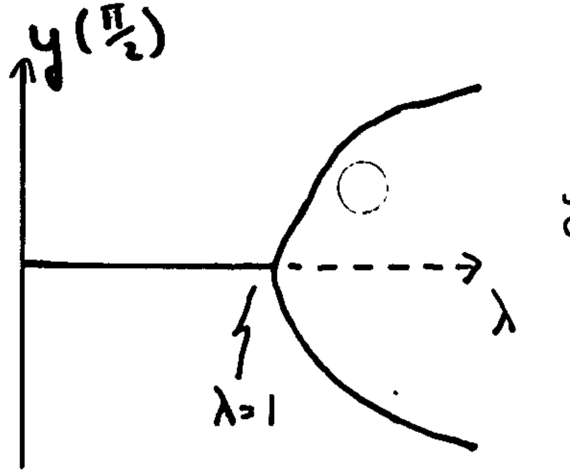
$$y = A\sin(\pi x/L) \quad \ldots (4)$$

<u>Euler</u> (Zeeman (1975)) S.L.N. in Maths 525) pp 3/5

Smallest λ for non-trivial solutions is $\lambda = 1$ $\{\lambda \equiv P/B$ at $L = \pi\}$

$\lambda < 1$ .... 1 soln.

$\lambda > 1$ .... 3 solns.



—— stable

- - - unstable

$y = e^{-\mu_i t} v_j$

Re. for $\mu_i > 0$ ⇒ stable

Exchange of stability at $\lambda = 1$

Using Lyapunov-Schmidt reduction the bifurcation may be shown to have the algebraic form $G(x, \lambda) = x^3 - \lambda x$

CRITICISM - imperfection sensitivity.

Golubitsky + Schaeffer (1985) Show a classification scheme for bifurcations

eg.

If at a bif.ⁿ pt. a set of eqn's g satisfy the following conditions

x - variable
λ - parameter

$$g = g_x = g_{xx} = g_\lambda = 0 \; ; \; g_{xxx} > 0 \quad g_{\lambda x} < 0$$

then the bifurcation is of the type;

$$G(x, \lambda) = x^3 - \lambda x = 0$$

This may be implemented into numerical schemes to identify types of bifurcation.

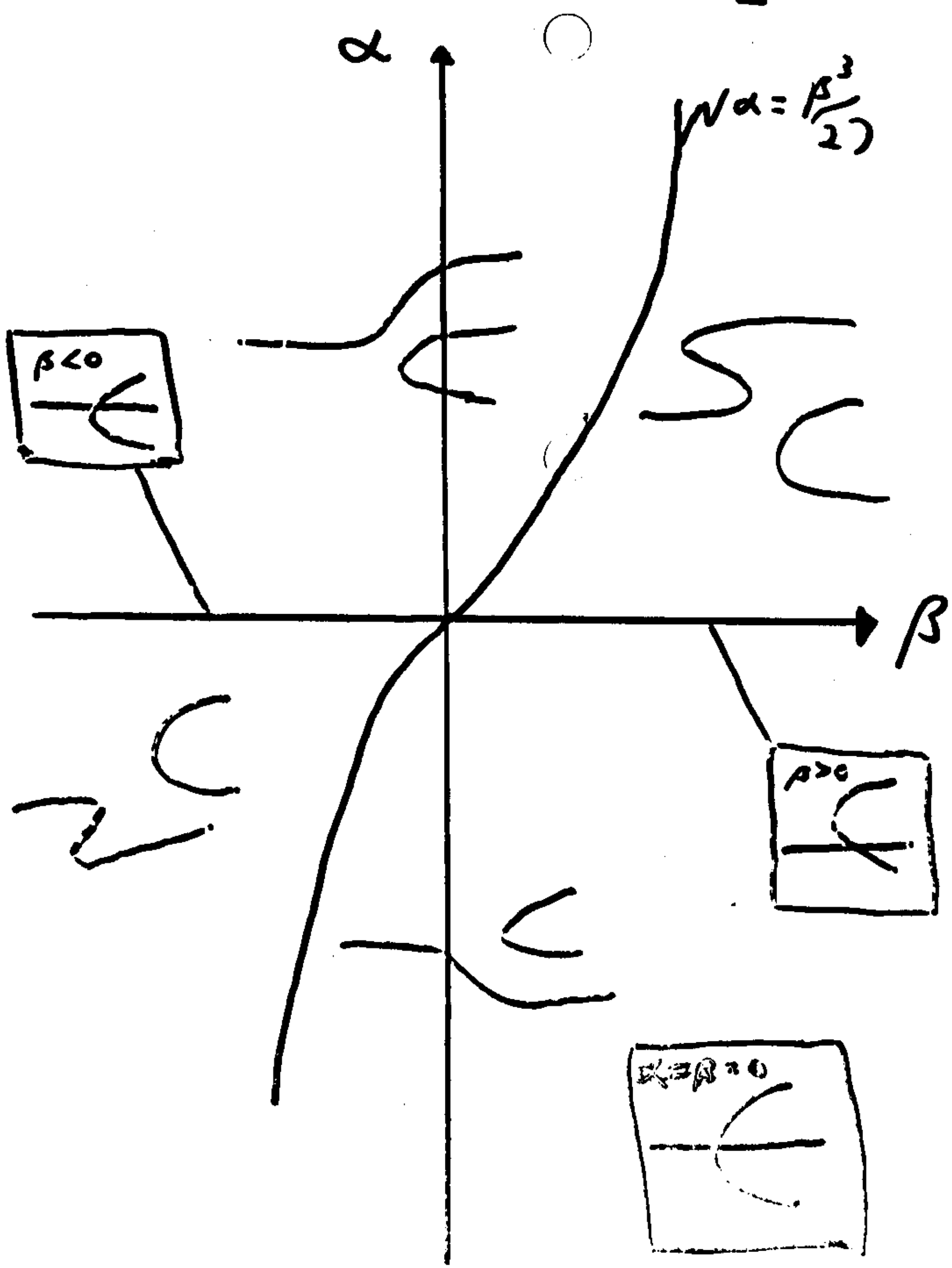
$F(x,\lambda,\alpha,\beta) = x^3 - \lambda x + \underline{\underline{\alpha}} + \underline{\underline{\beta}} x^2 = 0$ (8)

<u>CoDimension</u> GOLUBITSKY + SCHAEFFER (1985)(+)

NUMBER OF ADDITIONAL PARAMETERS REQUIRED TO GIVE UNIVERSAL UNFOLDING OF A BIFURCATION.

eg. $G(x,\lambda) = x^3 - \lambda x = 0$

Univ. unf. $F(x,\lambda,\alpha,\beta) = x^3 - \lambda x + \alpha + \beta x^2 = 0$



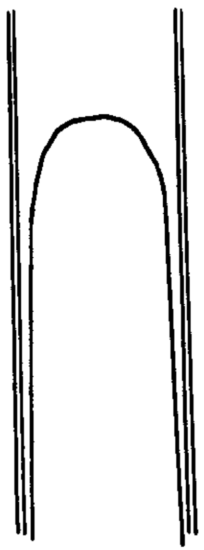
C-D $\Rightarrow$ TYPICALITY

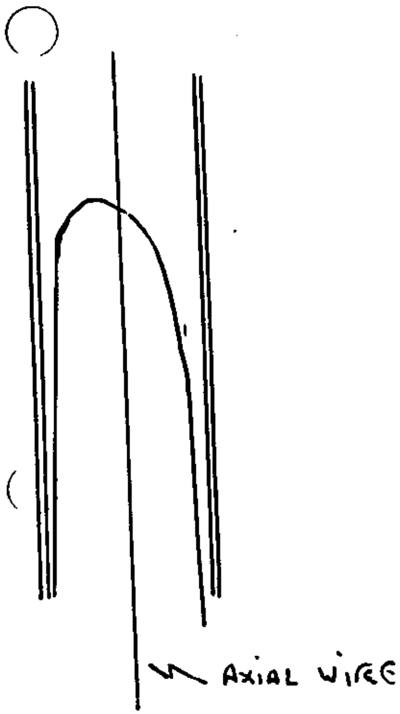
C-D 0 - FOLDS

C-D 1 - CUSPS, DOUBLE SINGULAR P

C-D 2 - PITCHFORK, SUBCRITICAL

GAS BUBBLE RISING IN A VERTICAL TUBE





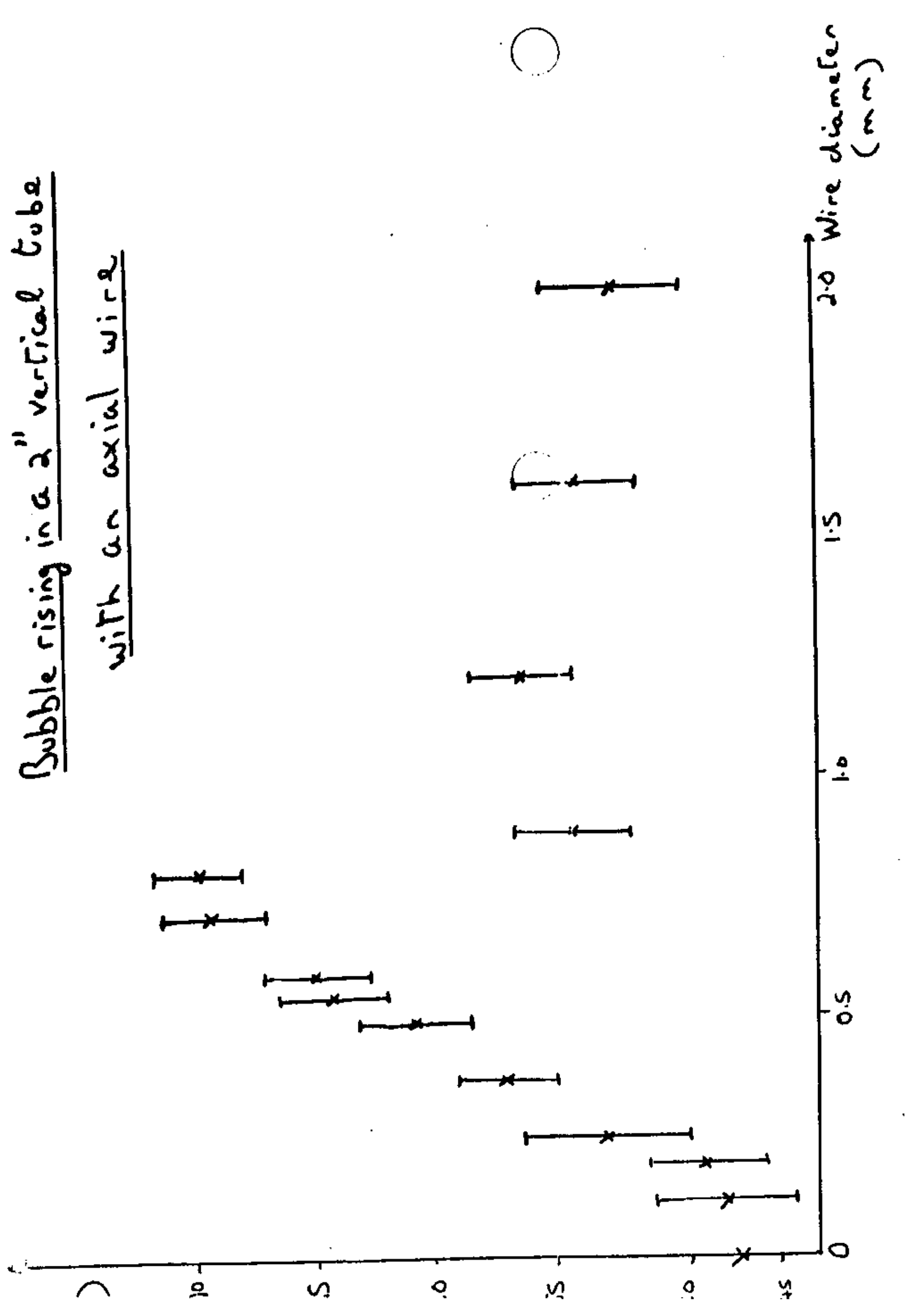
DAVIES + TAYLOR (1950)
THEORY

$$U = 0.464 (g a)^{1/2}$$

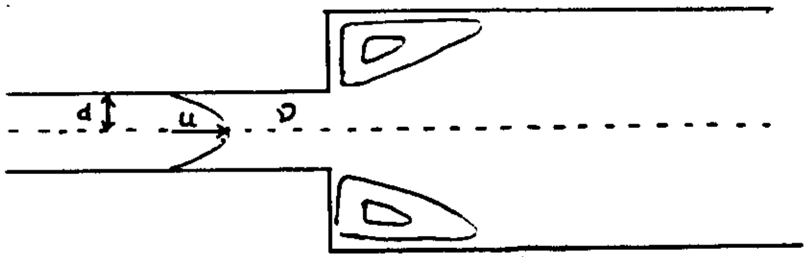
R.M. FEARN (1986)

DUMITRESCU (1943)
EXP.

$$U = 0.488 (g a)^{1/2}$$

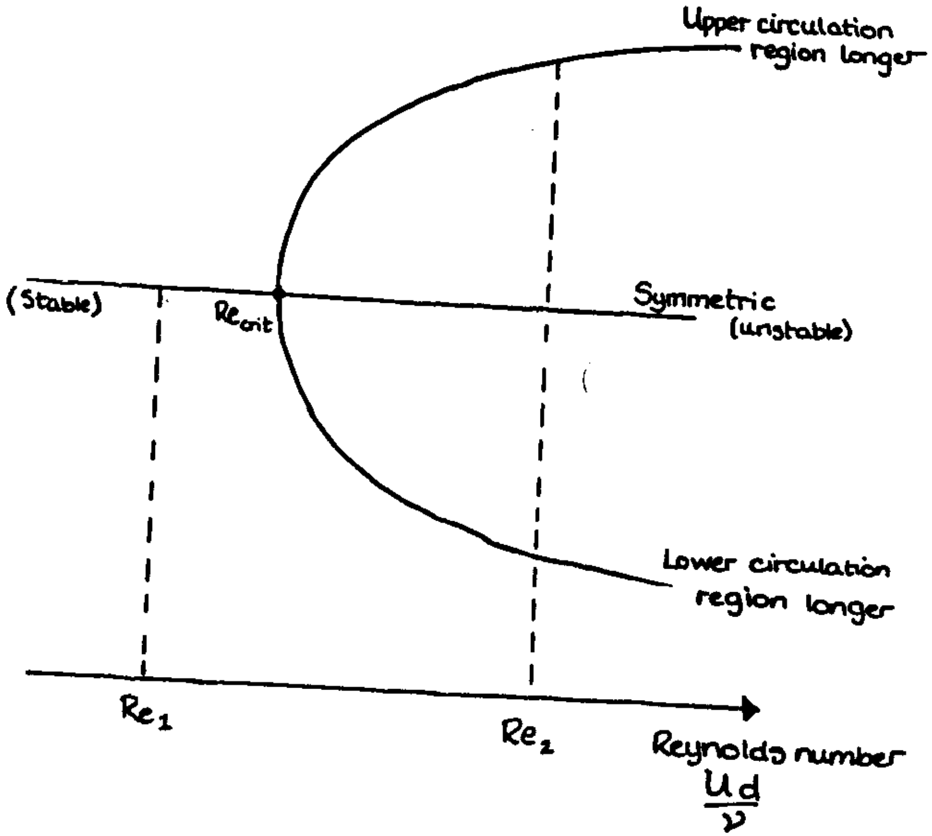


FLOW OVER A PLANE '2-D' SUDDEN EXPANSION



Single controlling parameter

Reynolds number $Re = \dfrac{Ud}{\nu}$



$$y^3 - A\,(Re - Re_{crit})\,y - B = 0$$

where $B = 0$ for unperturbed case.

Then $\quad y^2 = A\,(Re - Re_{crit}) \quad , \; A = \text{const.}$

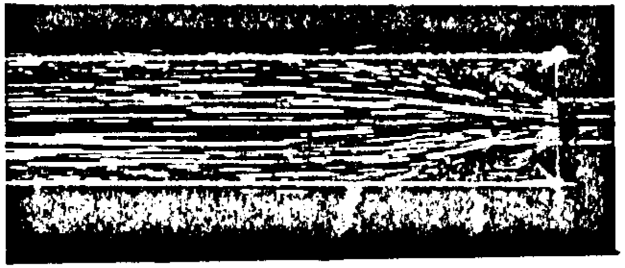


FIGURE 4a. SYMMETRIC FLOW AT Re = 25.

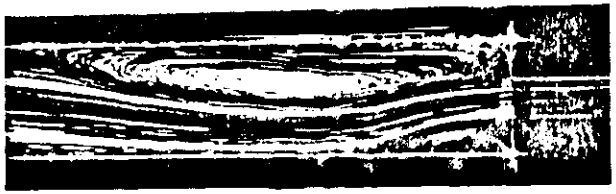
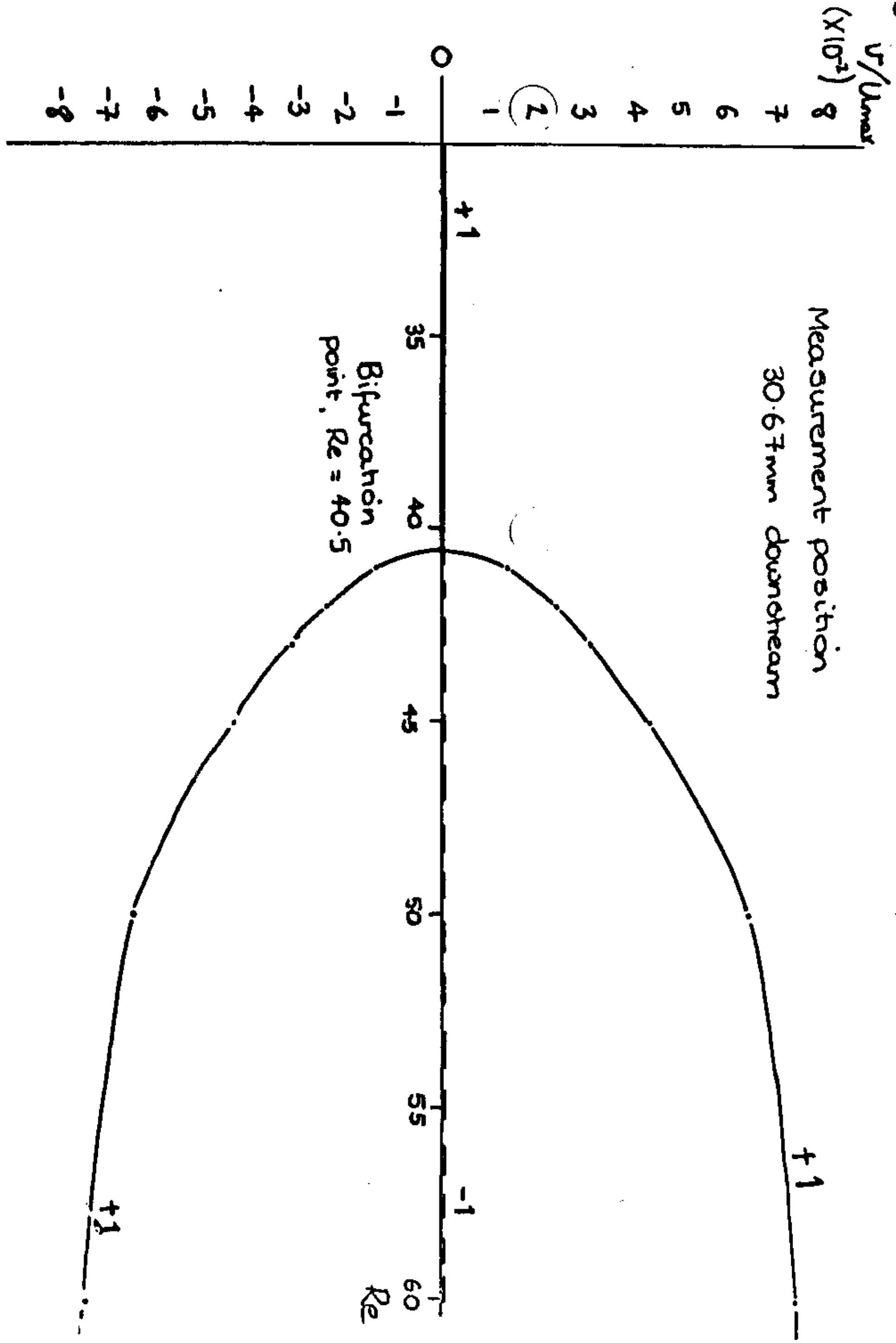
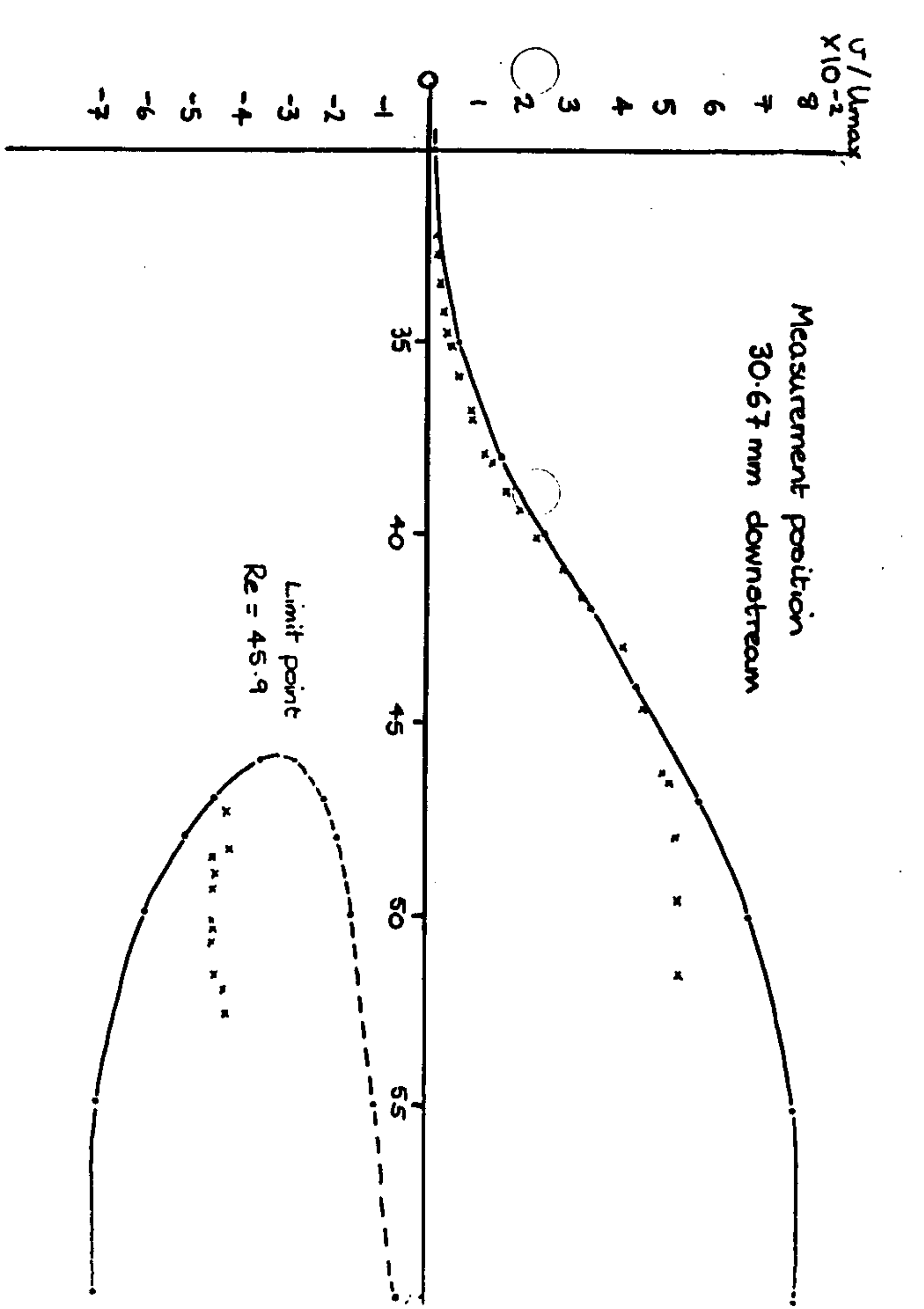
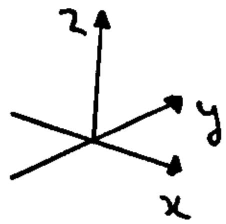


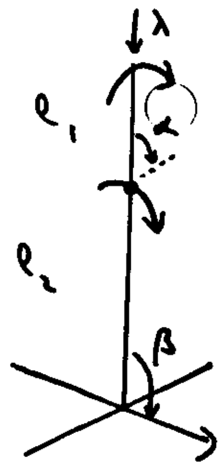
FIGURE 4b. ASYMMETRIC FLOW WITH LONGER UPPER CIRCULATION REGION AT Re = 80.

FIGURE 4c. ASYMMETRIC FLOW WITH LONGER LOWER CIRCULATION REGION AT Re = 80.









TAVENER + MULLIN PHYSICA D (1988)

$$\left.\begin{aligned} &\alpha - \lambda \frac{\mu}{\nu} \sin\alpha \cos\beta = 0 \\ &\beta - \lambda(1 + \nu\cos\alpha)\sin\beta = 0 \end{aligned}\right\} \ldots (1)$$

$\nu = l_1/l_2$ - the length ratio

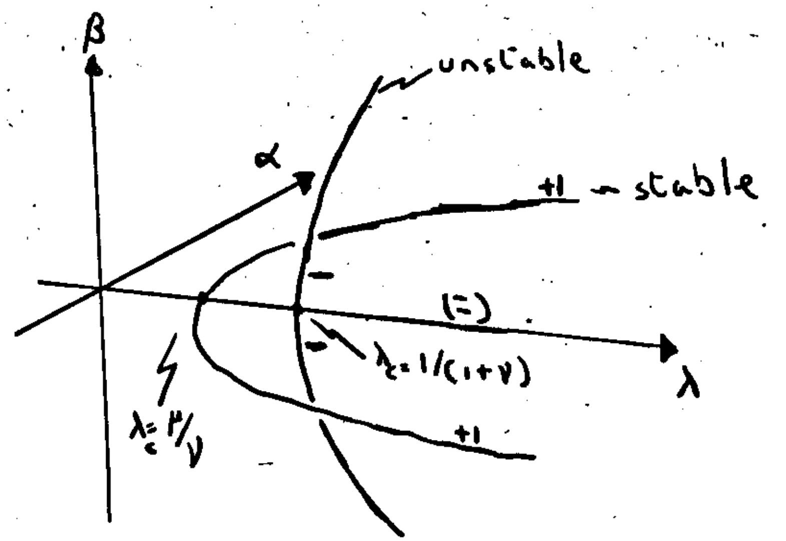
$\mu = k_1/k_2$ - the stiffness ratio

RIGID ROD / SPRING

$\alpha = \beta = 0$ – trivial solution

BUCKLING OF COUPLED EULER STRUTS

(17) | secondary bif^n (Bauer, Keller + Reiss) (18)



How does exchange of mode of buckling occur?

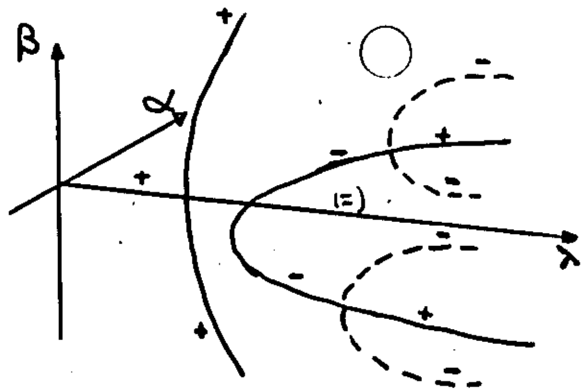
Above picture is incomplete

- it would require simultaneous exchange of stabilities along $(\alpha, 0)$, $(0, \beta)$ branches when

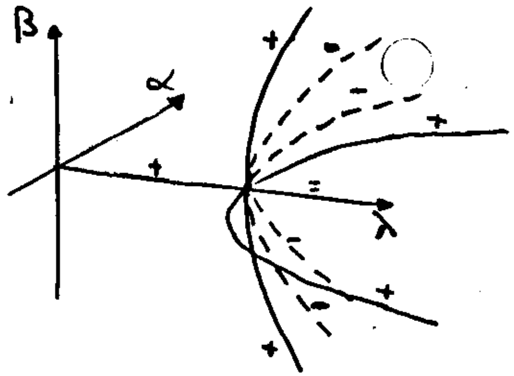
$$\frac{\mu}{\nu} = \frac{1}{1+\nu}$$

Can change either μ or ν. Choose to vary ν as more practical.

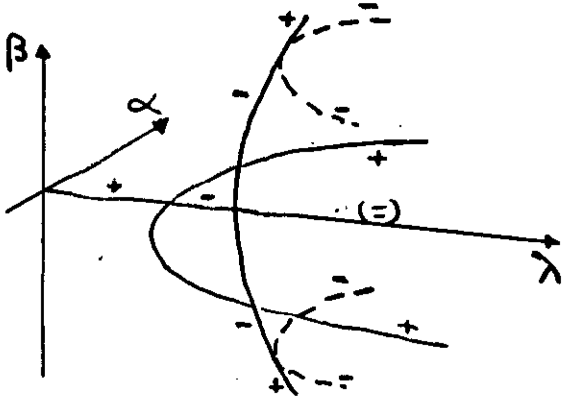
$\nu < \nu_c$



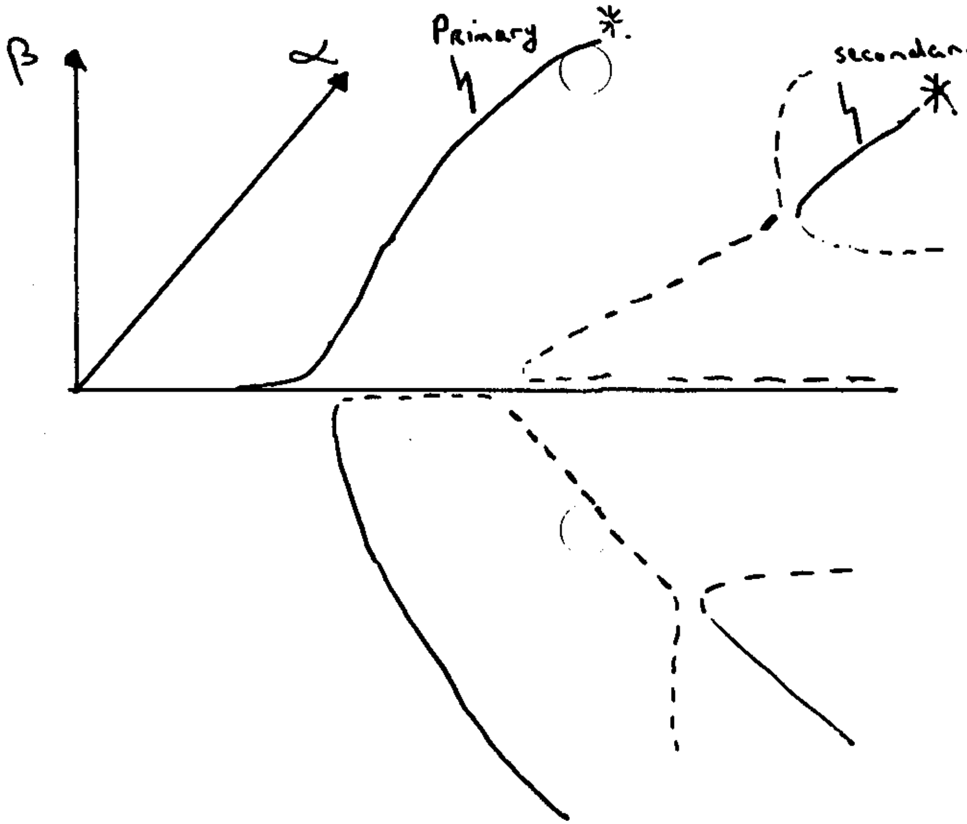
$\nu = \nu_c$



$\nu > \nu_c$



Exchange with imperfections



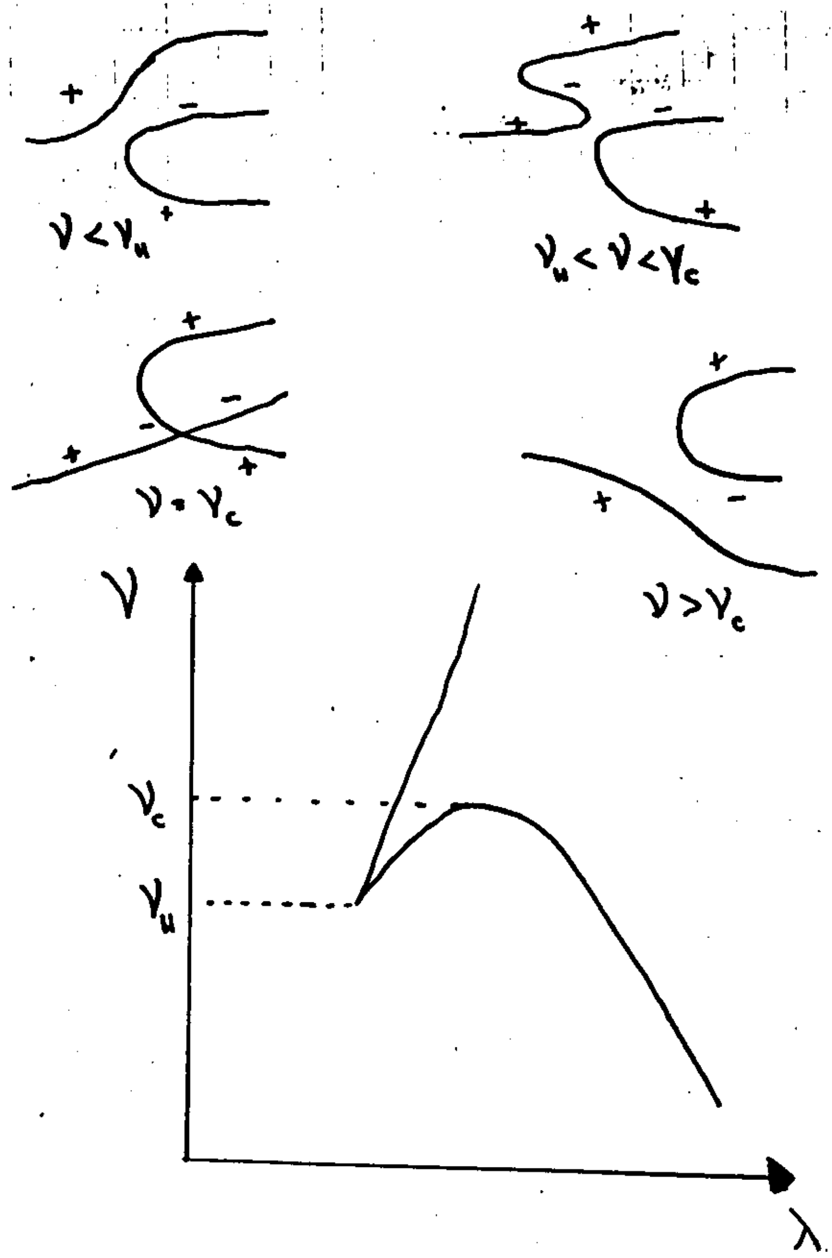
Model equations

$$\left.\begin{aligned}(\alpha - \varepsilon_1) - \lambda \frac{\nu}{\mu} \sin\alpha \cos\beta = 0 \\ (\beta - \varepsilon_2) - \lambda (1 + \nu \cos\alpha) \sin\beta = 0\end{aligned}\right\}$$

ie model imperfections as offsets.

Exchange takes place between branches marked *

EXCHANGE PROCESS FOR COUPLED ELASTICAS WITH IMPERFECTIONS



FIVE RULES OF BIFURCATION. BENJAMIN [illegible] (21) NAVIER-STOKES.

1) For every R the solutions of a steady flow problem are isolated i.e. a continuum of solutions does not exist. Solutions are smooth except at critical points. (FOIAS + TEMAM)

2) At sufficiently small values of R the flow has a unique solution which is unconditionally stable. (SERRIN)

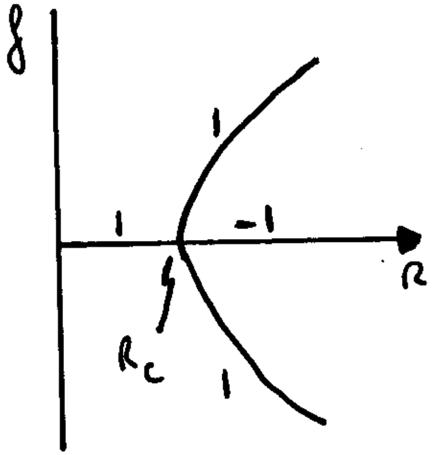
3) Each solution may be assigned a Leray-Schauder index $i = \pm 1$. $\underline{-1}$ necessarily indicates instability. As R changes i remains the same unless a critical point is encountered.

4) The total number of solutions is finite and the sum of the indices $= 1$.

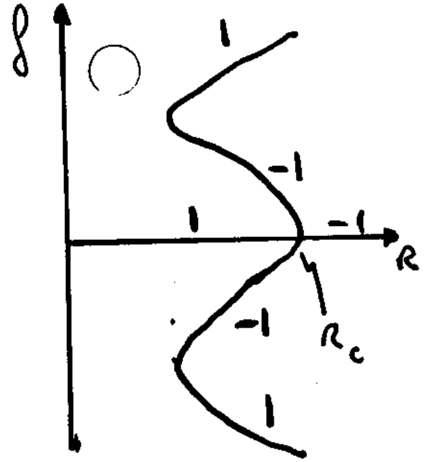
$$\sum_{m=1}^{k} i_m = 1$$

Except at critical values of R where bifurcations occur the number k of possible flows is odd.

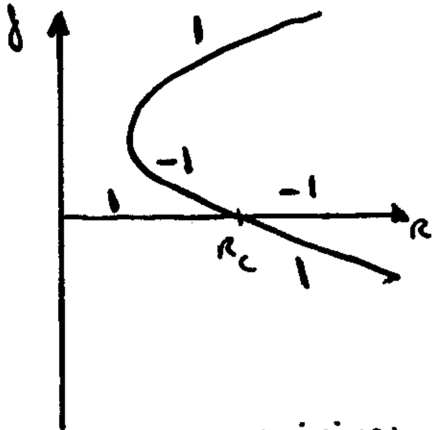
5) At bifurcation points with respect to R, the index of the critical solution is zero unless special conditions of symmetry apply.



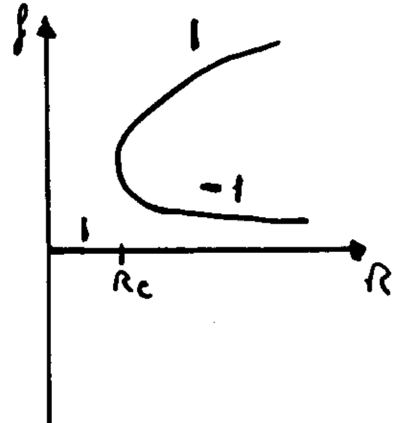
SUPERCRITICAL



SUBCRITICAL

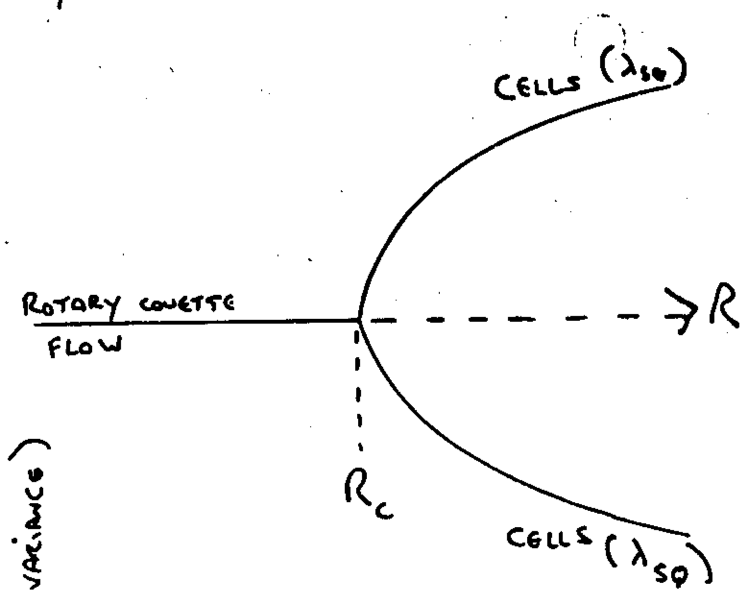


TRANSCRITICAL

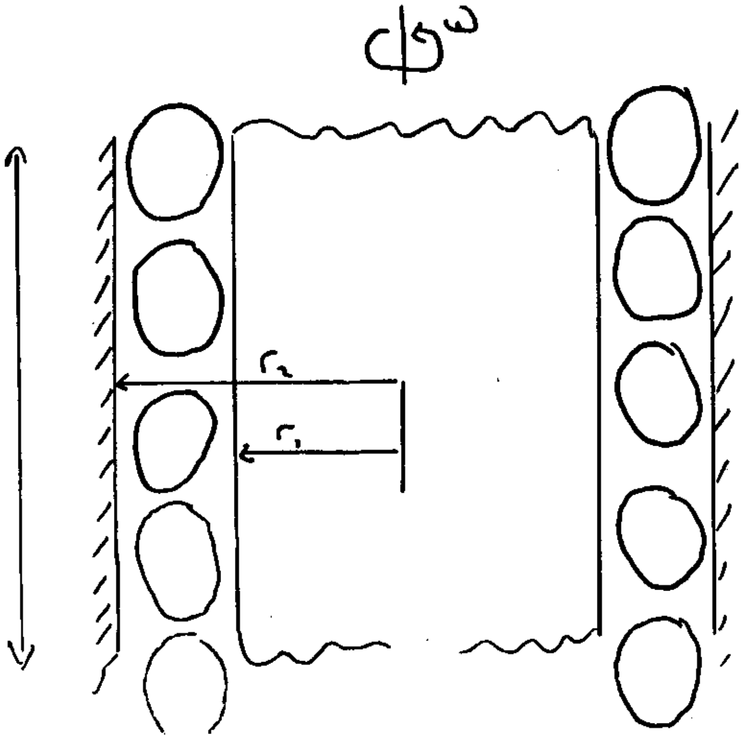


FOLD

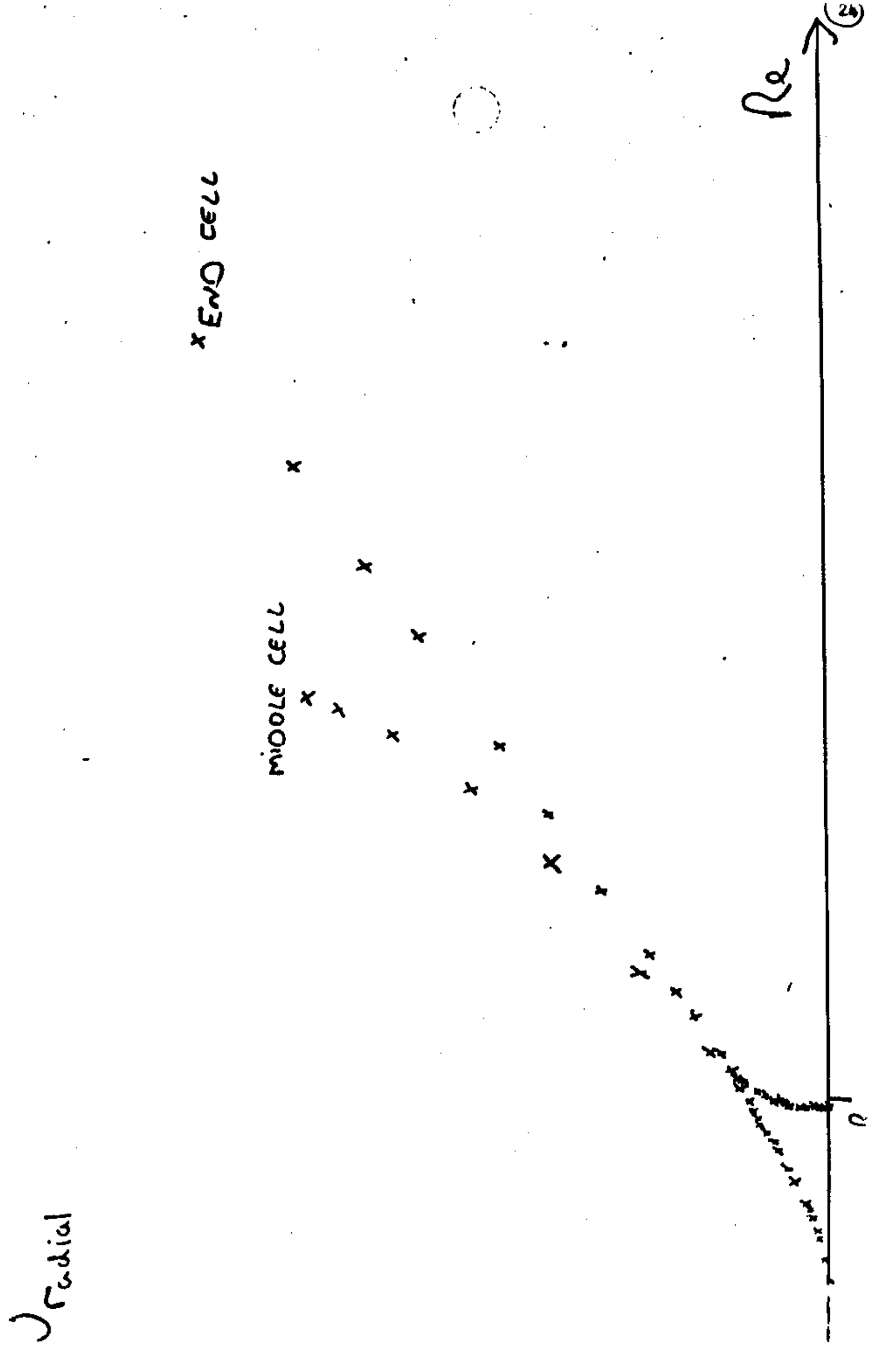
TAYLOR - COUETTE FLOW (TAYLOR 1923)



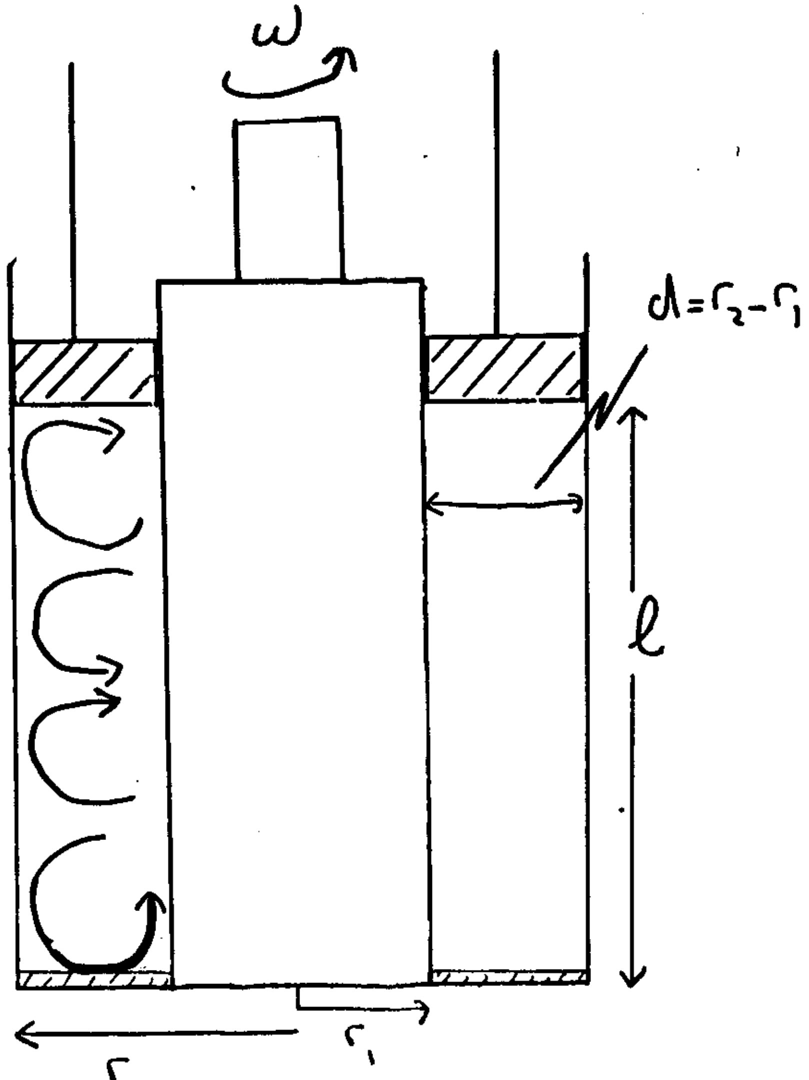
PERIODIC IN L (TRANSLATIONAL INVARIANCE)



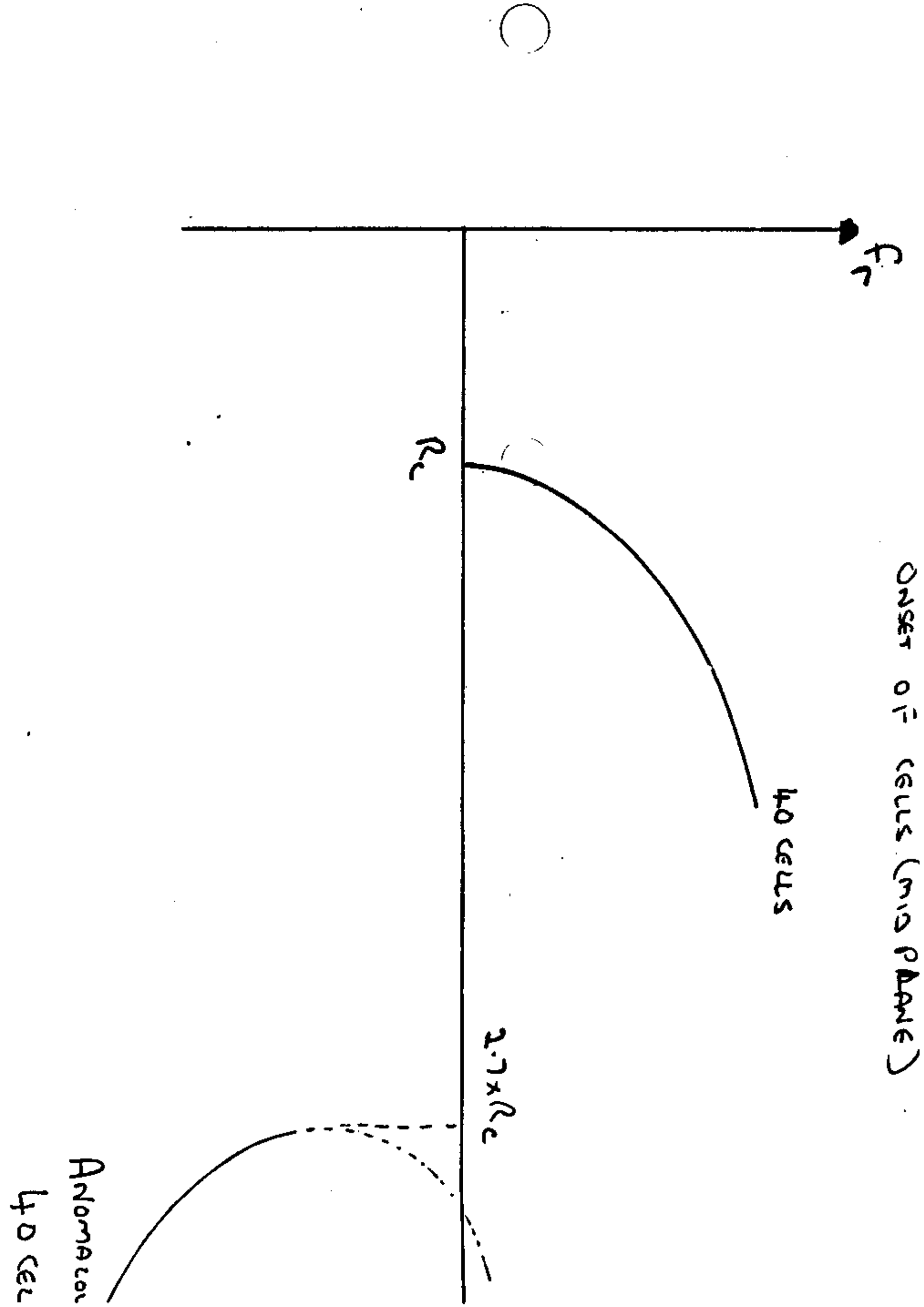
ONSET OF CELLS IN EXPERIMENT



TAYLOR FLOW APPARATUS
(SCHEMATIC)



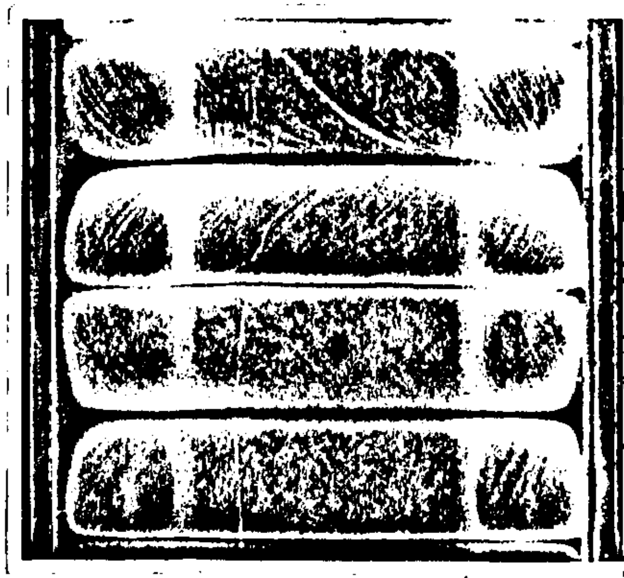
$$\eta = \frac{r_1}{r_2} \qquad R = \frac{\omega r_1 d}{\nu} \qquad \Gamma = l/d$$



← SECTIONAL VIEW

TAYLOR CELLS

FRONT VIEW



STEADY FLOW TAYLOR CELLS

NORMAL	ANOMALOUS.
INWARD FLOW AT END WALLS	OUTWARD FLOW AT END WALL.
EVEN Nº OF CELLS (2,4,6 ....)	EVEN/ODD Nº OF CELLS (2,3 ... 40 ..) ↖ WORLD RECORD
FORMED IN A CTS. WAY BY TS. INCREASE IN Re. Nº. PRIMARY	FORMED BY DISCTS. INCREASE IN RE. Nº. ONLY SECONDARY
DISCTS. SECONDARY	1-CELL EXCEPTIONAL

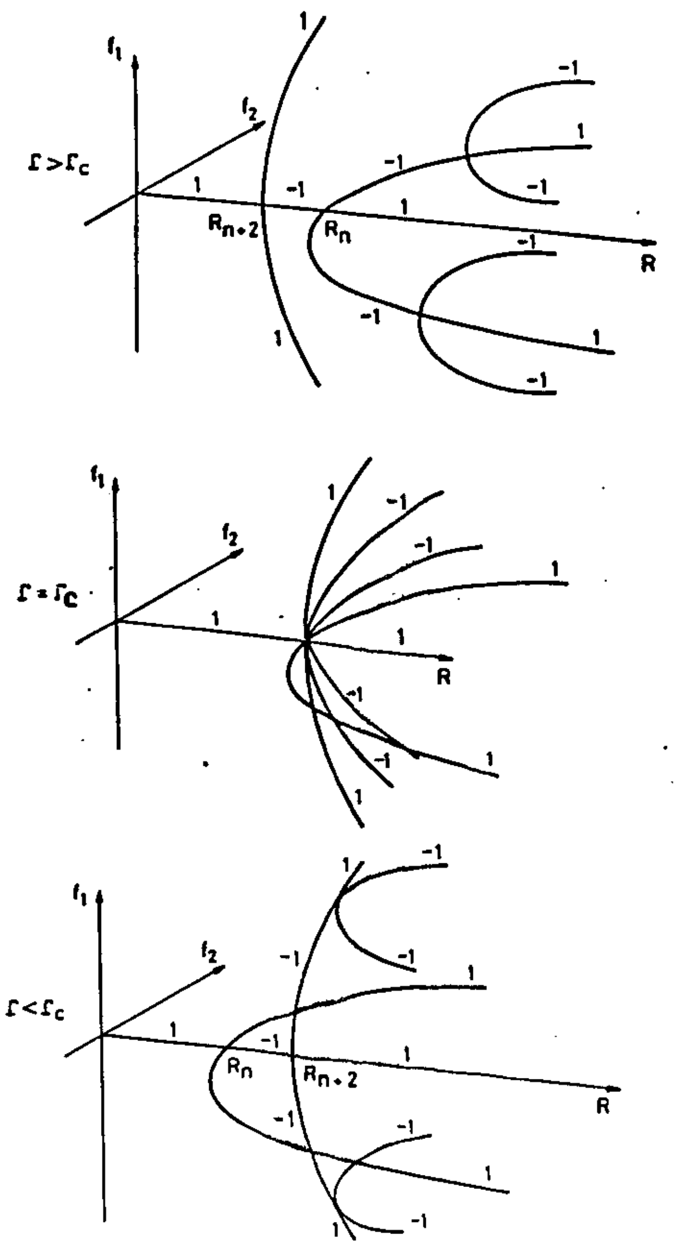


Fig. 3

IDEALIZED "SCHAEFFER MODEL"

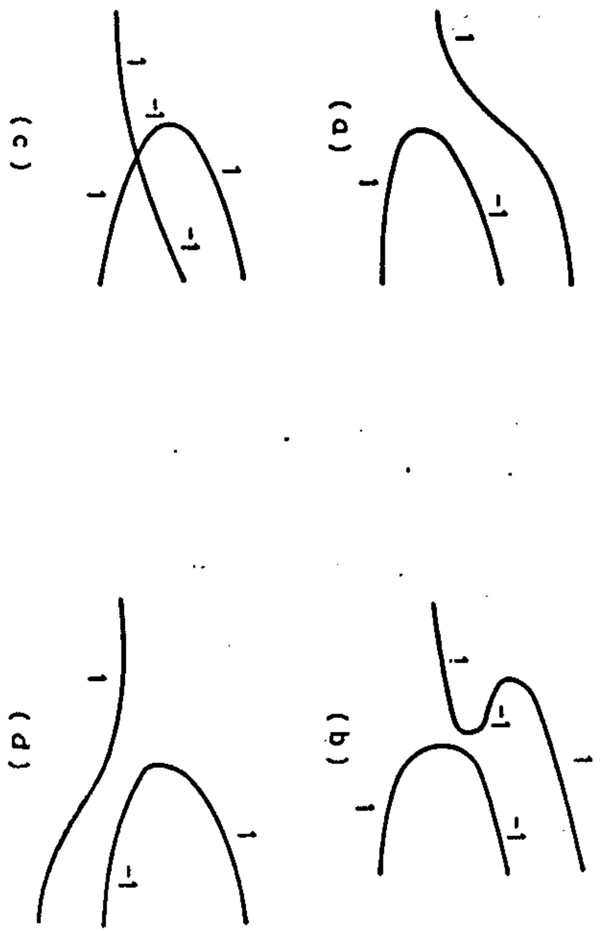


Fig. 4

BENJAMIN MODEL

SCHAEFFER HOMOTOPY τ

Periodic B.C.

$$U_z = \frac{\partial U_r}{\partial z} = \frac{\partial U_\phi}{\partial z} = 0$$

at $z = \pm l \quad 0 \geq r \geq 1$

Modified B.C.

$$U_z = 0$$

$$(1-\tau)\frac{\partial U_r}{\partial z} \pm \tau U_r = 0$$

$$(1-\tau)\frac{\partial U_\phi}{\partial z} \pm \tau \left(U_\phi - F(r)\right) = 0$$

$\tau = 0 \Rightarrow$ Periodic

$\tau = 1 \Rightarrow$ realistic

$F(r)$ introduced by B+M to cope with corner singularity.

EXPERIMENTAL RESULTS FOR FOUR-SIX EXCHANGE

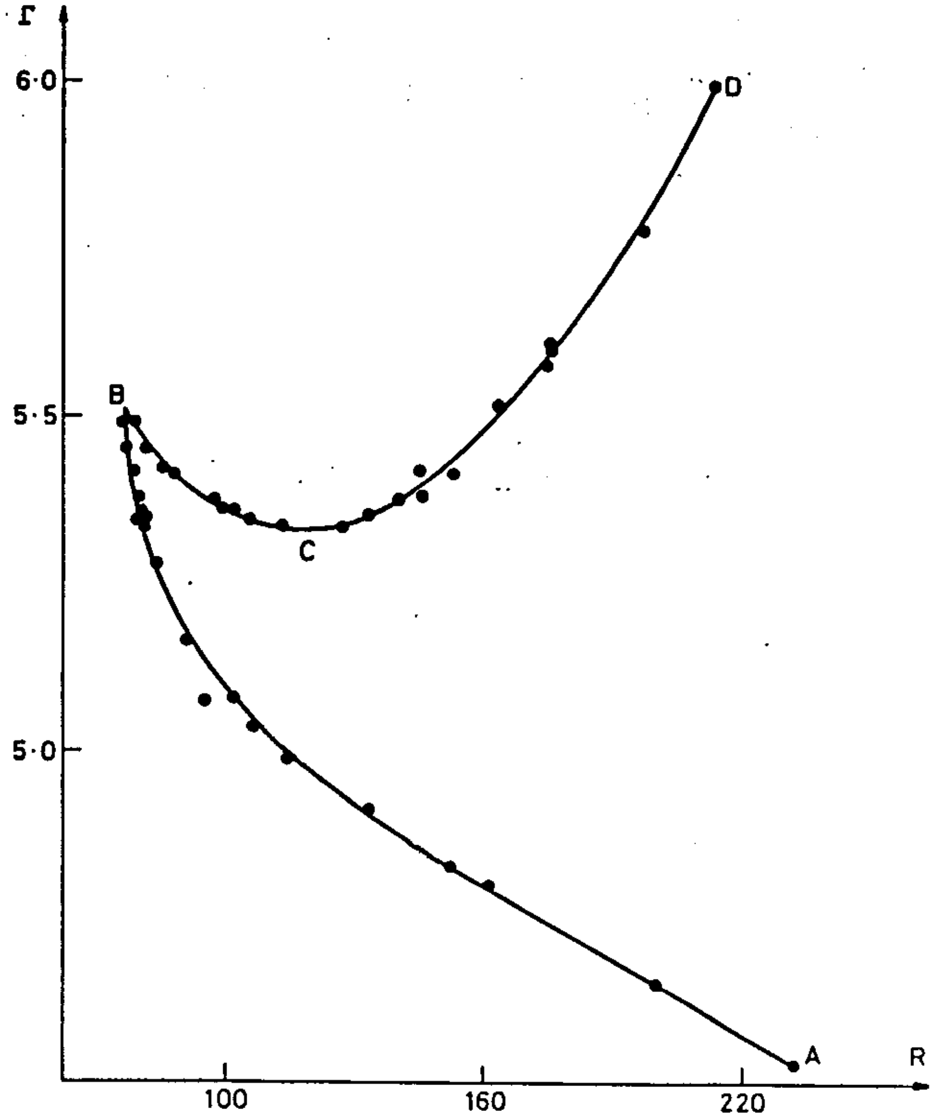
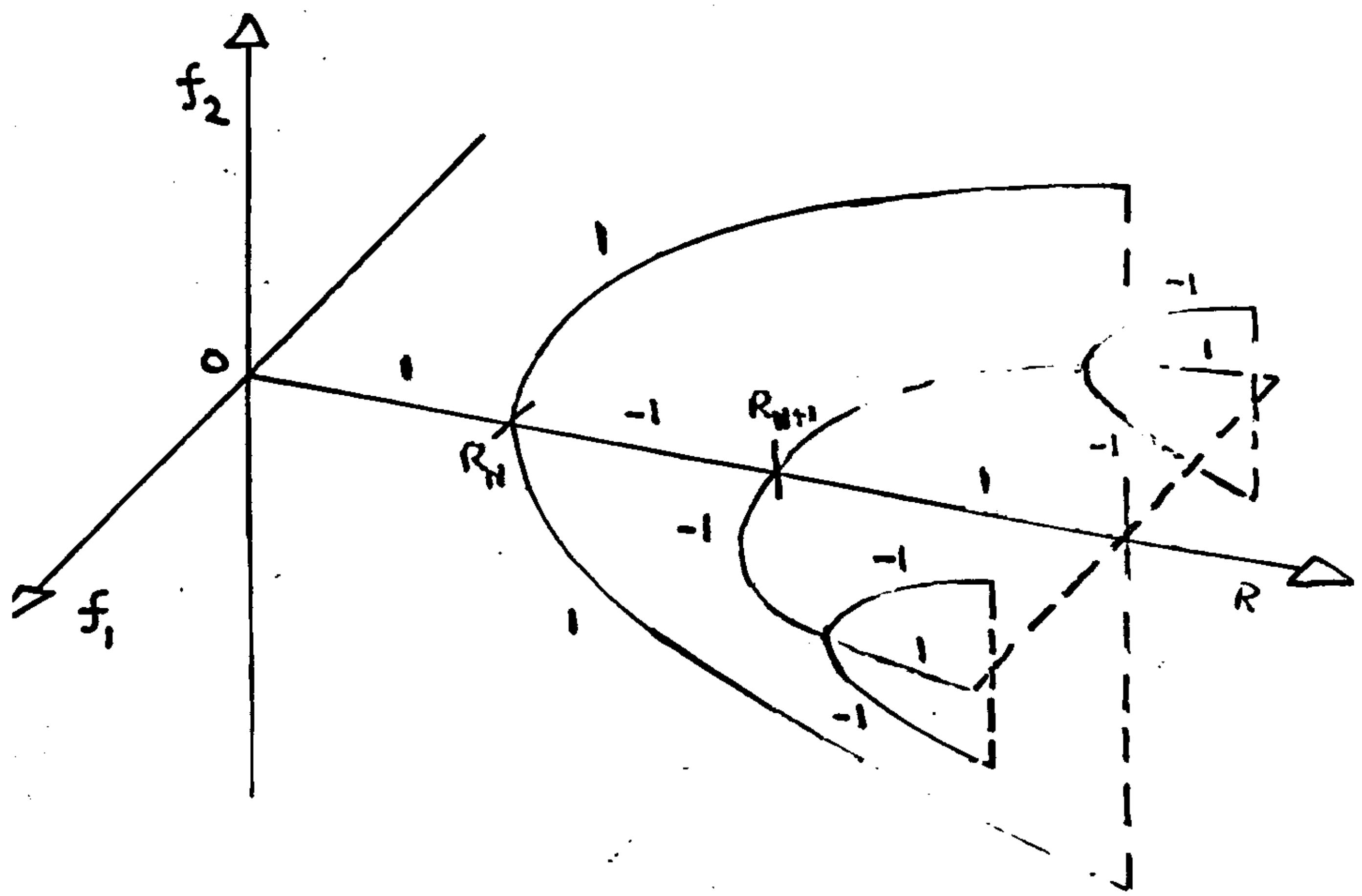
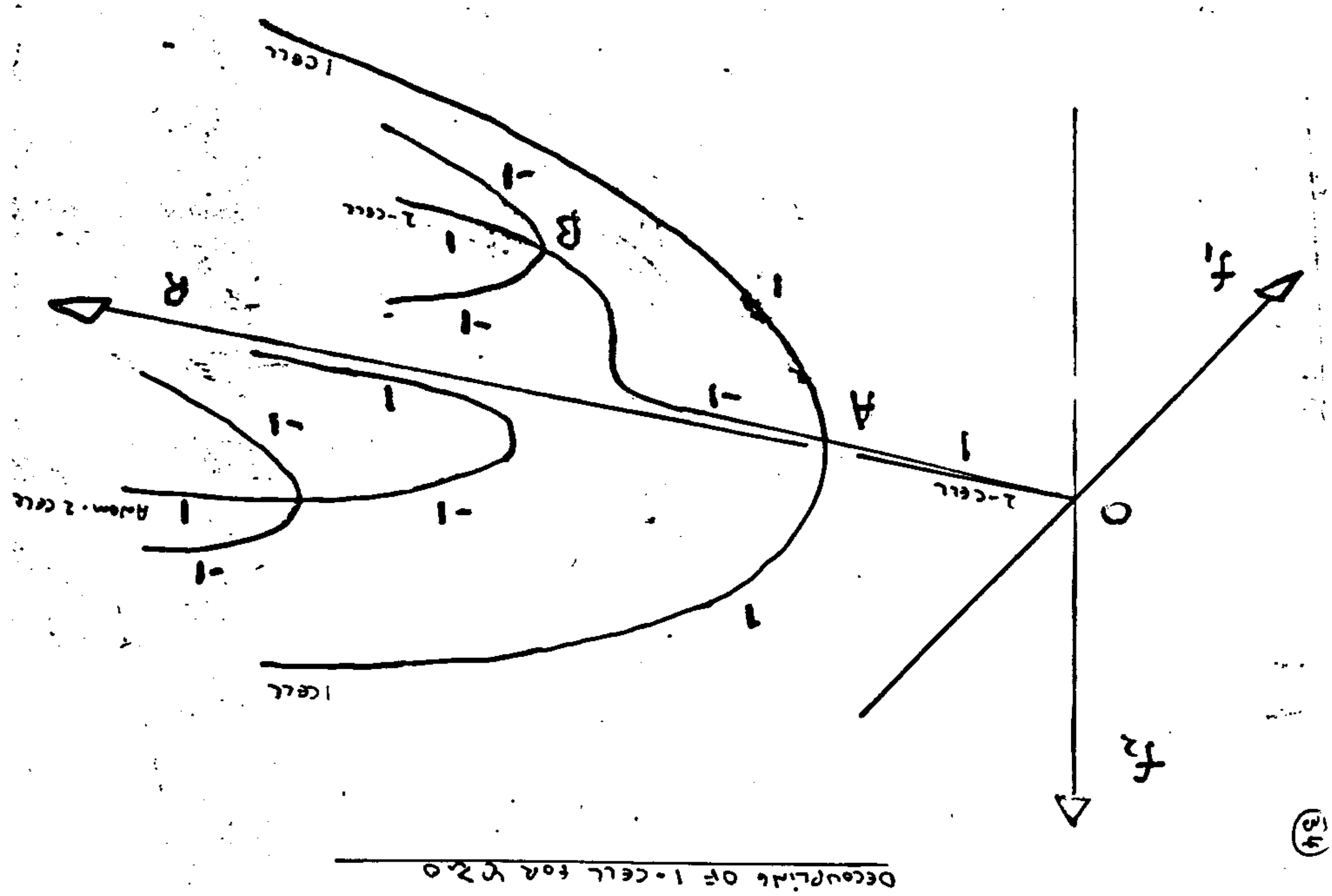


Fig. 5

IDEALISED EVEN/ODD EXCHANGE

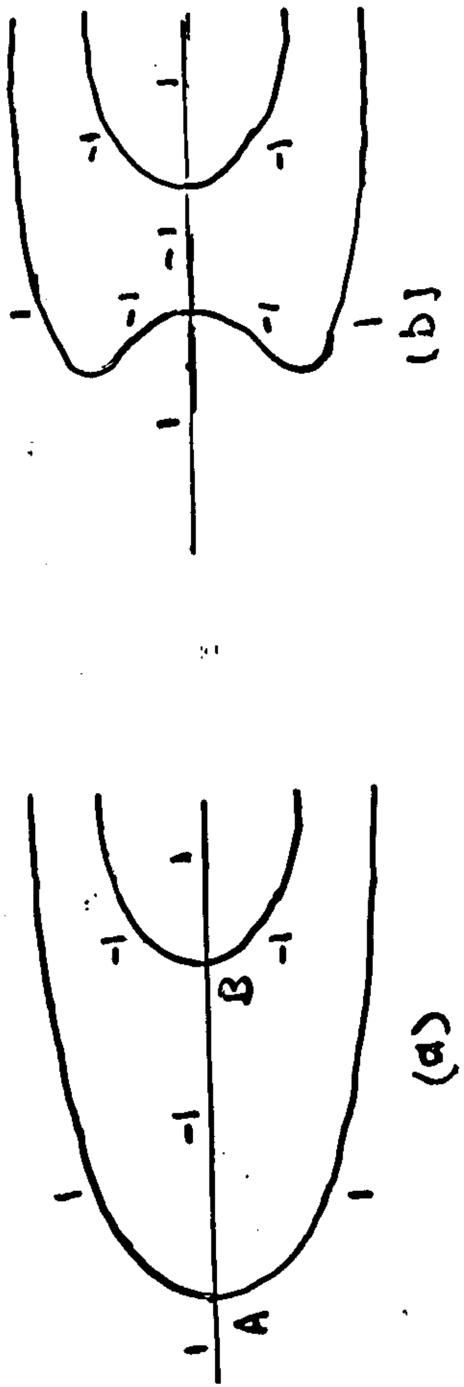


(33)

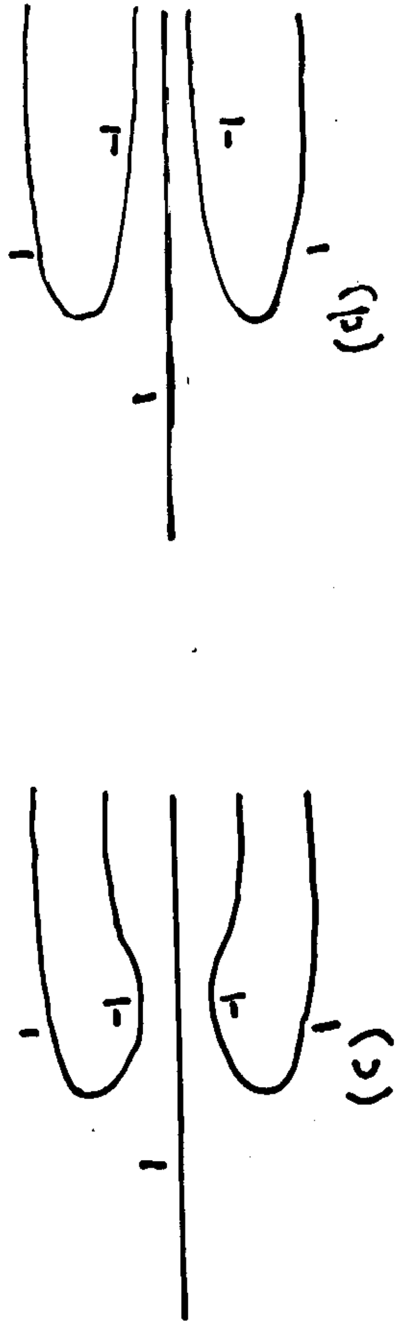


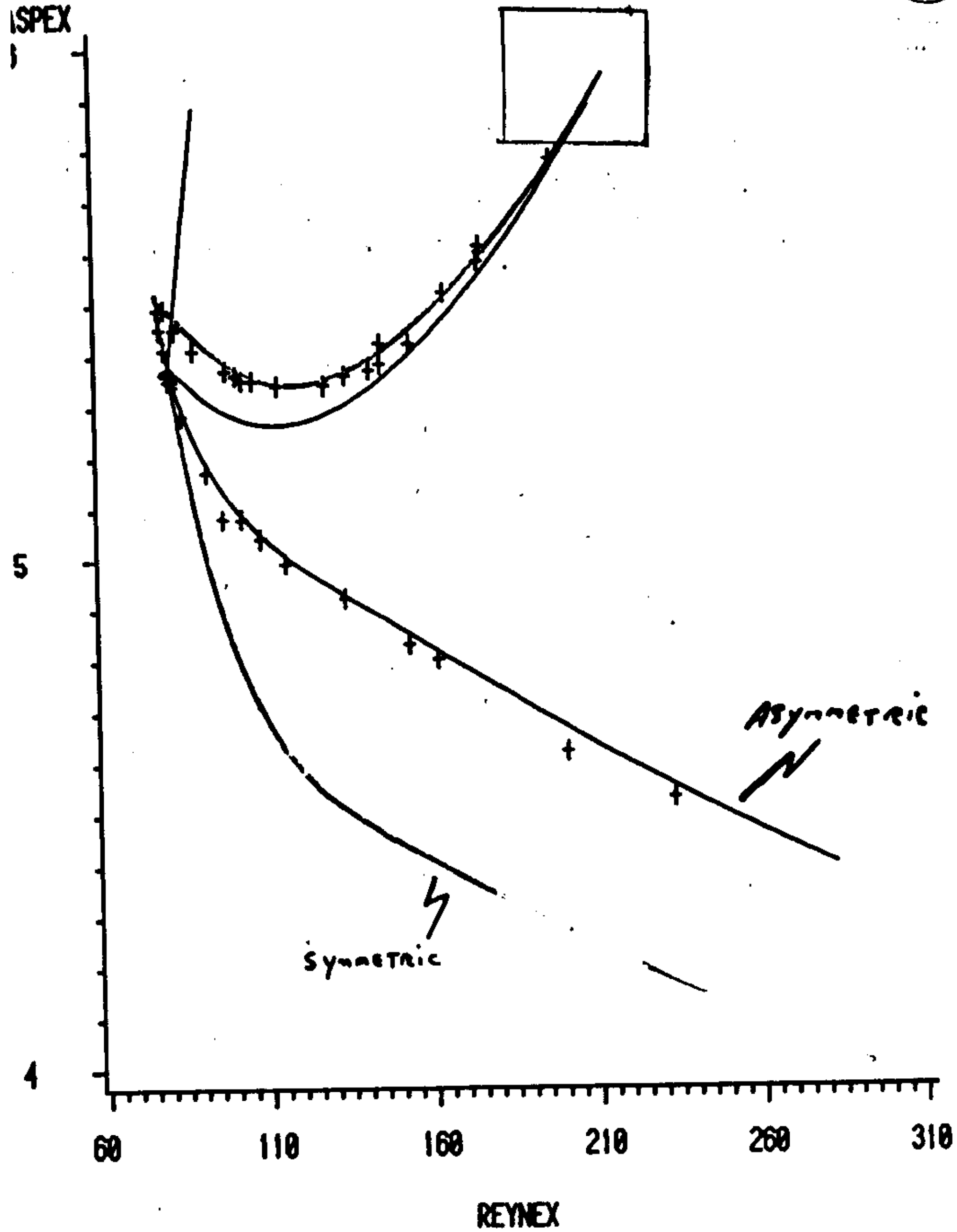
(34)

DECOUPLING OF 1-CELL FOR $\gamma \geq 0$



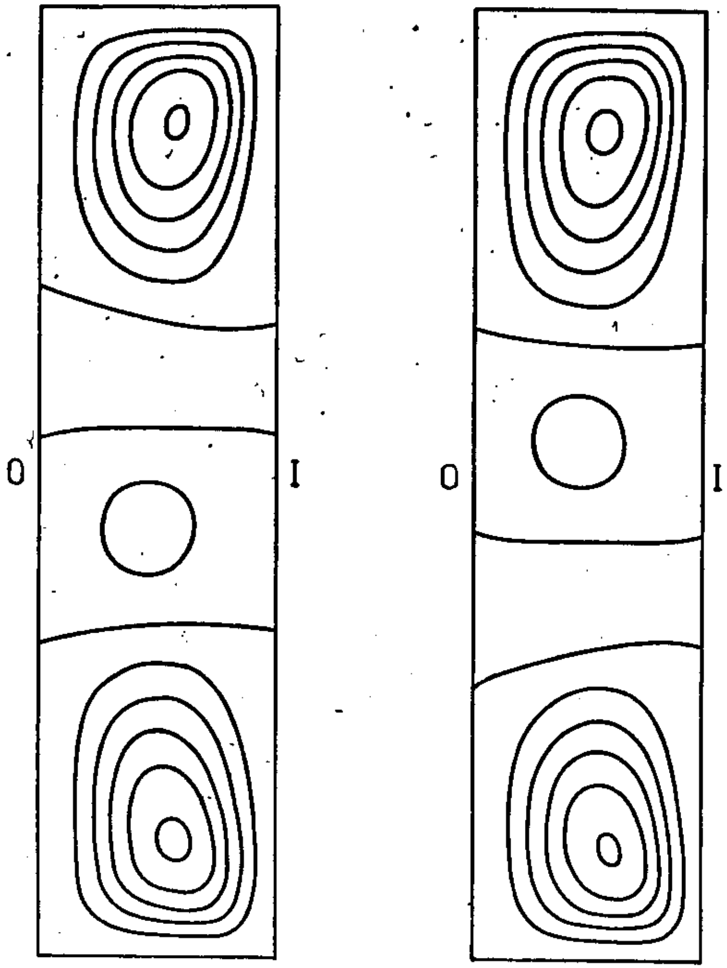
DECOUPLING OF 1-CELL STATE AS Γ INCREASED

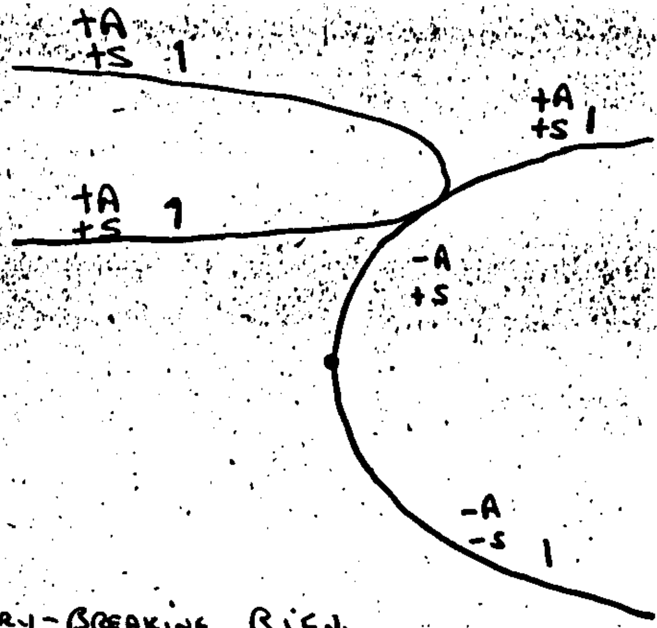




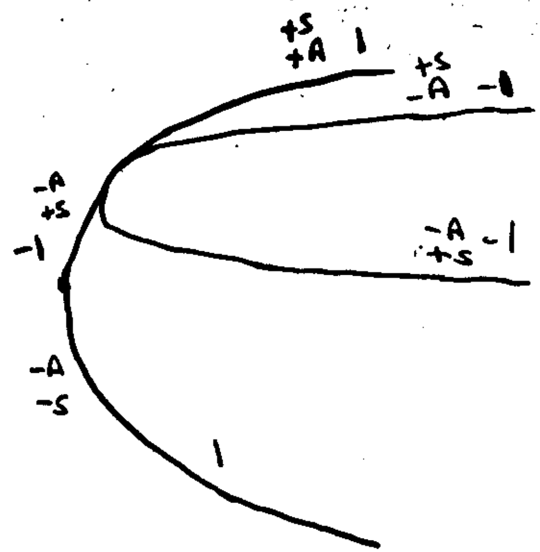
4/6 EXCHANGE : NUMERICS CF EXPERIMENT

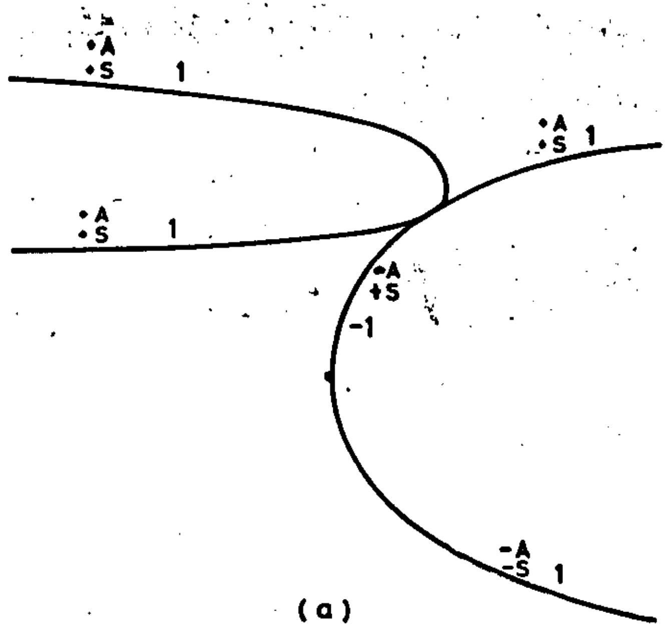
Computed streamlines for a pair of asymmetric flows at equal aspect ratio and Reynold's number





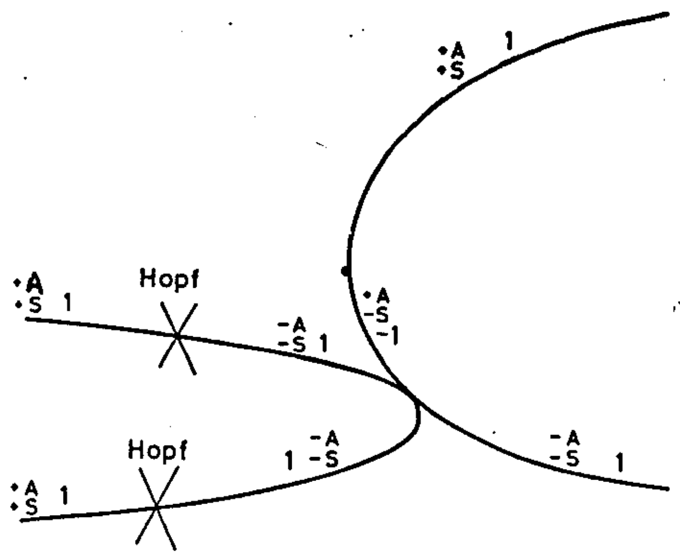
SYMMETRY-BREAKING BIFN.





(a)

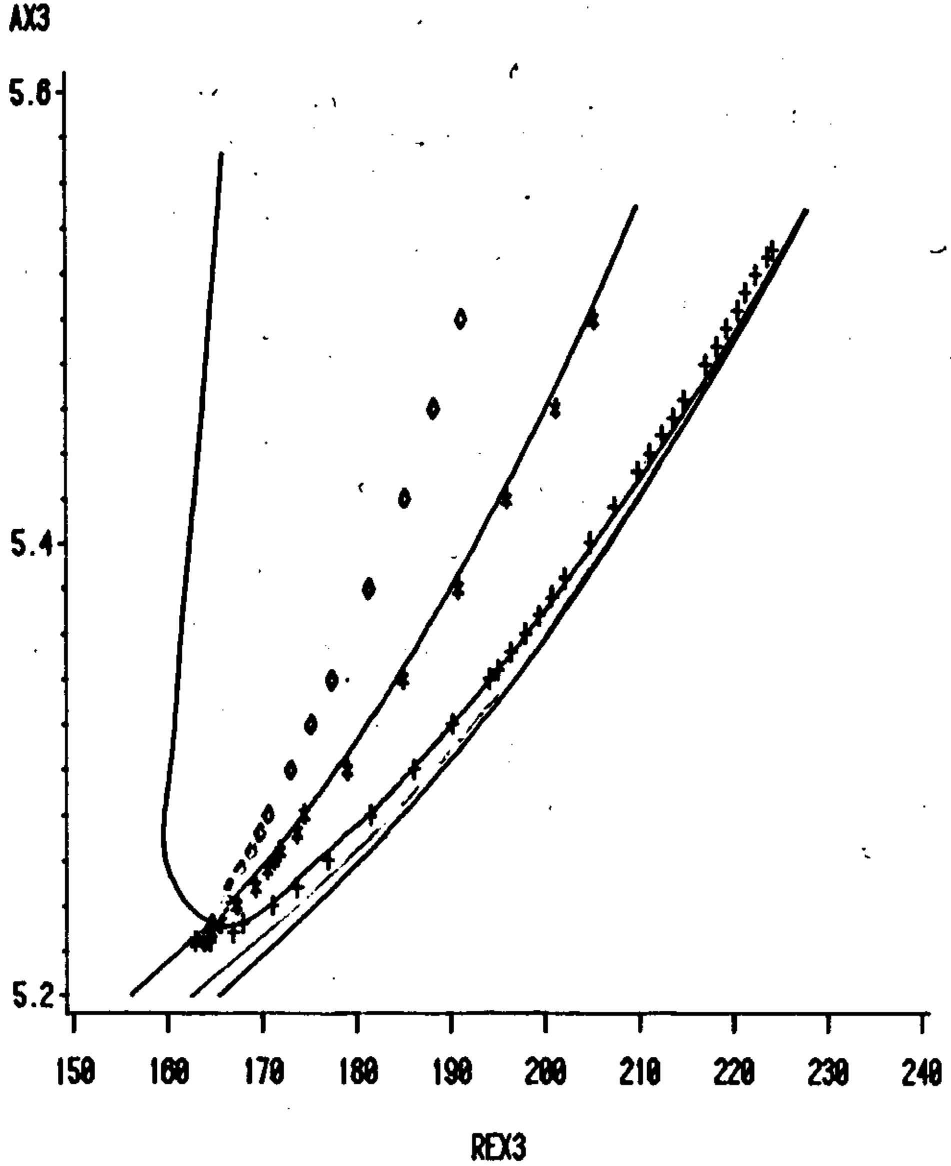
PRODUCTION OF t-dep. BY INTERACTION OF STEADY BIF.



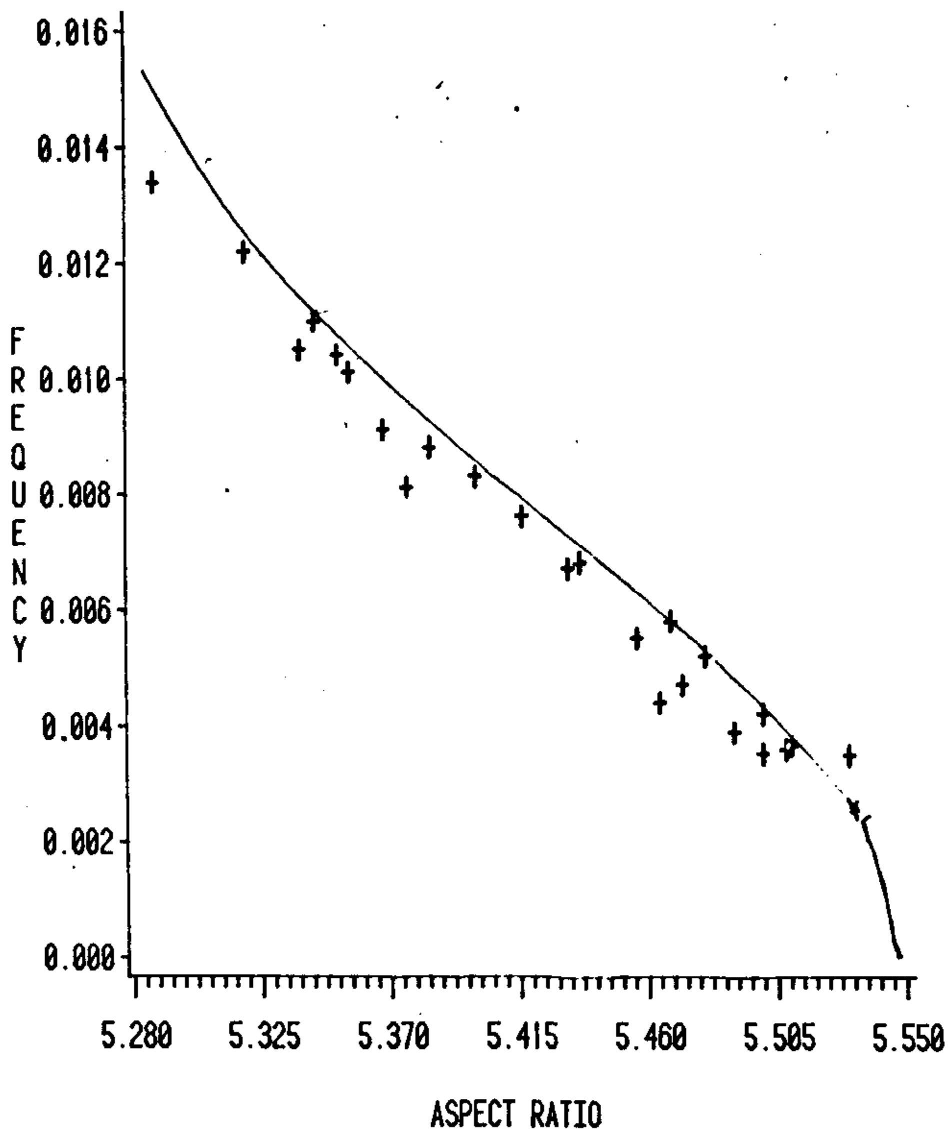
(b)

Fig. 7

ONSET OF t-dep. NEAR CD-1 point: numerics cf exp.

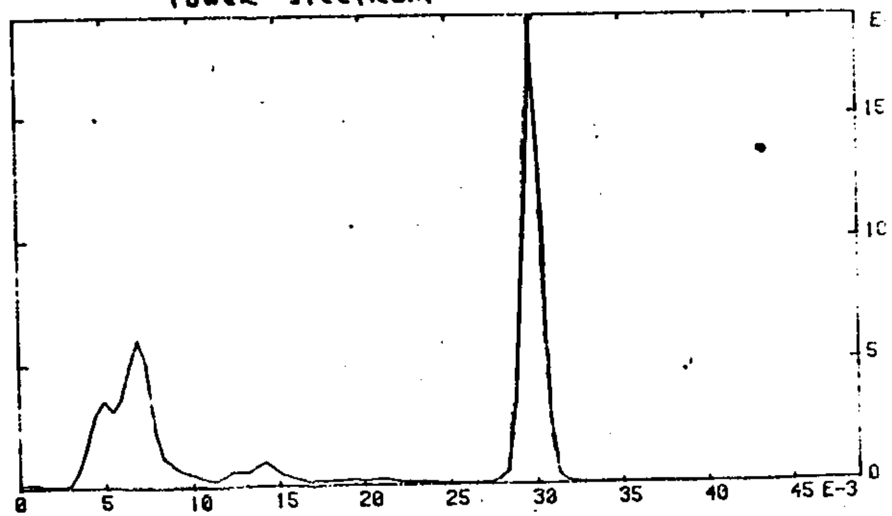


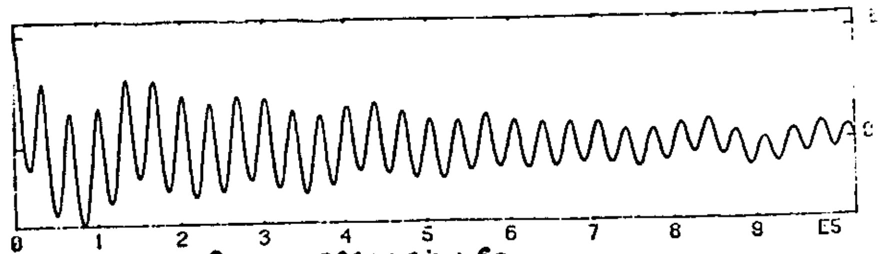
FREQUENCY VARIATION NEAR CD-1 pt: Numerics cf expt.



"CHAOS" NEAR CD-1 Point

Power spectrum



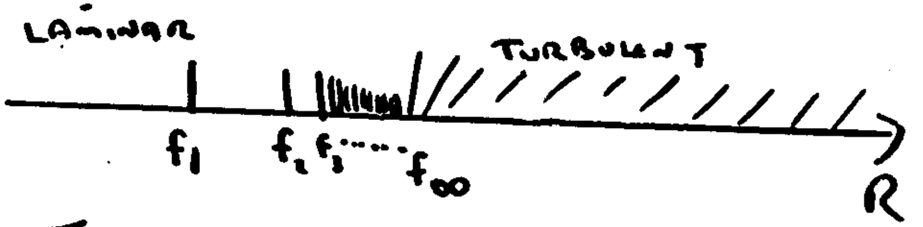


Autocorrelation fn.

TRANSITION TO TEMPORAL CHAOS

SPATIAL CORRELATION MAINTAINED TO HIGHEST R.

LANDAU

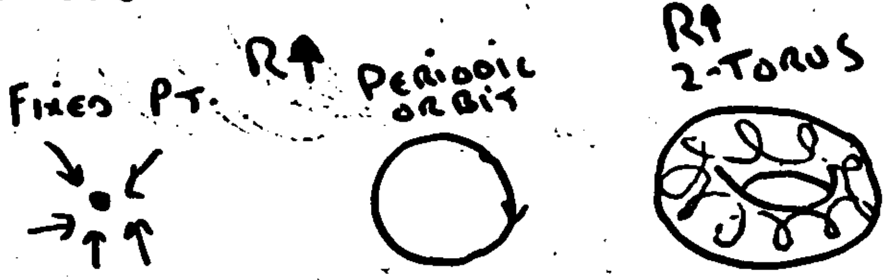


TURBULENCE – SUM OF infinite number OF DISCRETE modes with random amplitude and phase

LANDAU

CONVERGING SEQUENCE OF BIFNS AS $R\uparrow$. EACH INTRODUCES INCOMMEN. FREQ.

RUELLE + TAKENS



STRANGE ATTRACTOR

SENSITIVITY TO INITIAL CONDITIONS

Which (if either) applies to TAYLOR Couette flow?

PERIOD DOUBLING (FEIGENBAUM) 45

$$y = 4\lambda x(1-x) = f(x)$$

ITERATIVE

$$x_{n+1} = 4\lambda x_n(1-x_n)$$

$$0 < x < 1 \qquad 0.7 < \lambda < 1.0$$

$$\delta_n = \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}}$$

$$= 4.6692016 \ldots$$

UNIVERSAL CONST. FOR SINGLE HUMPED MAPS

$$\alpha = 2.502907875$$

SINGLE CELL T-C FLOW

(G. PFISTER)

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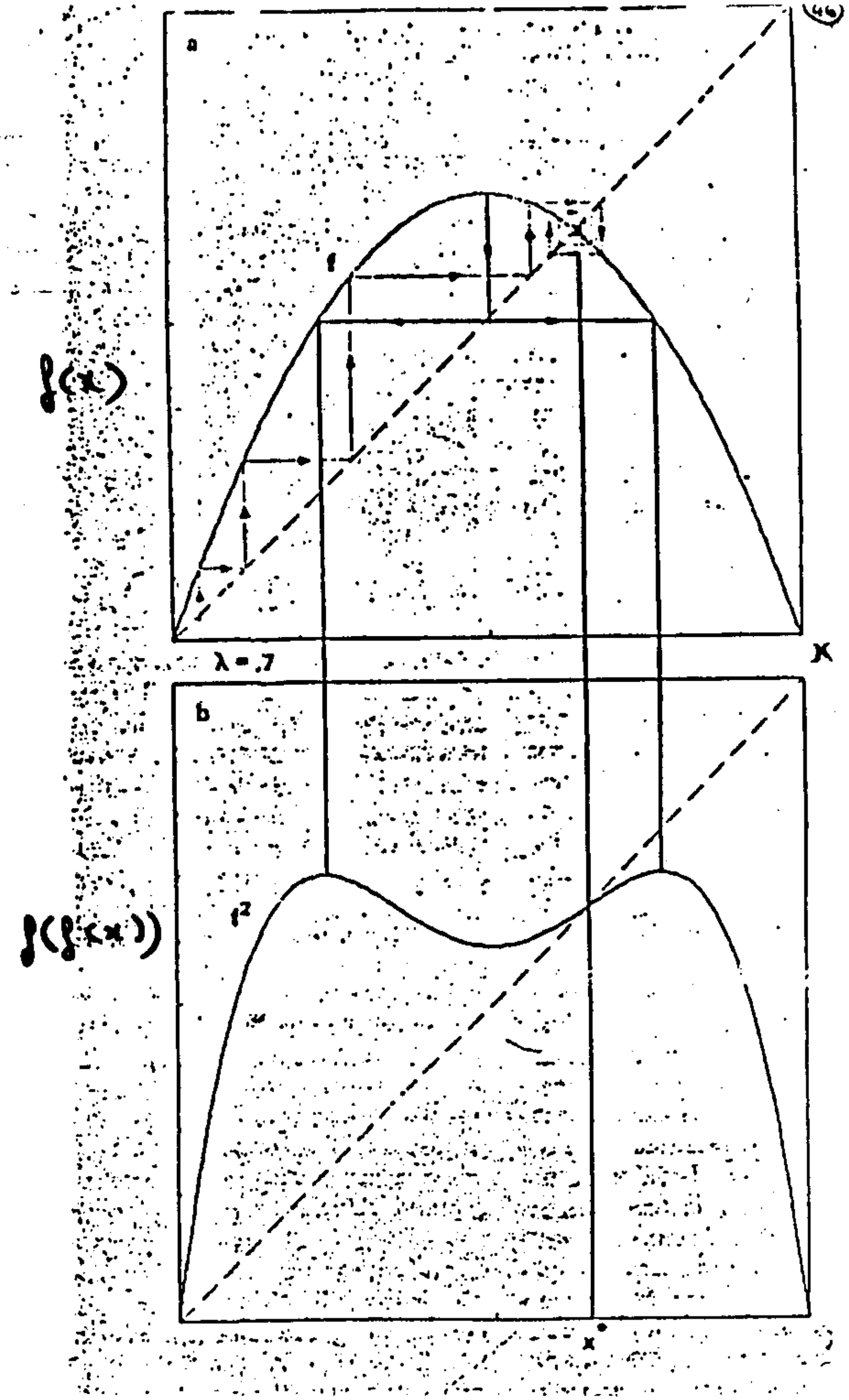
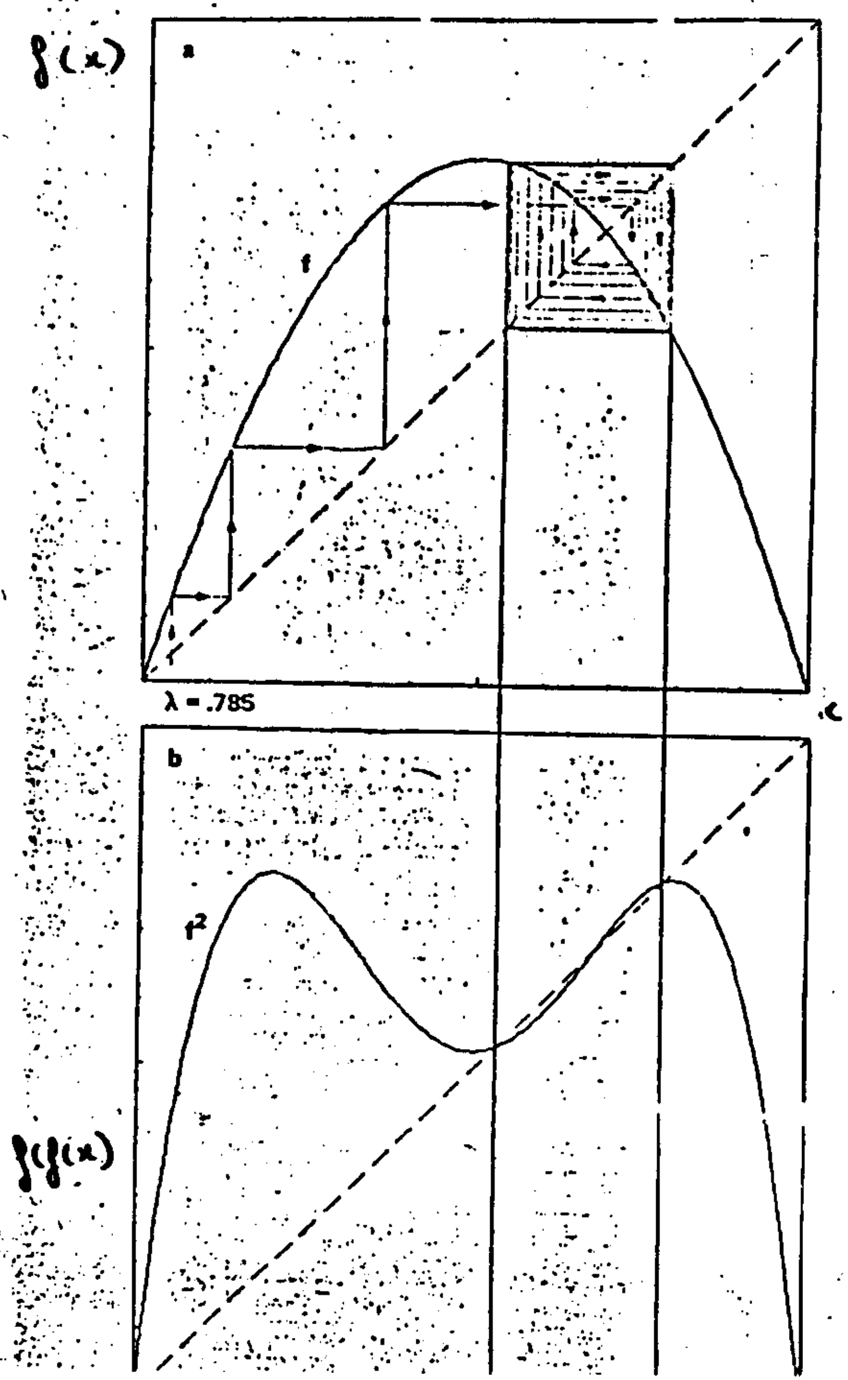
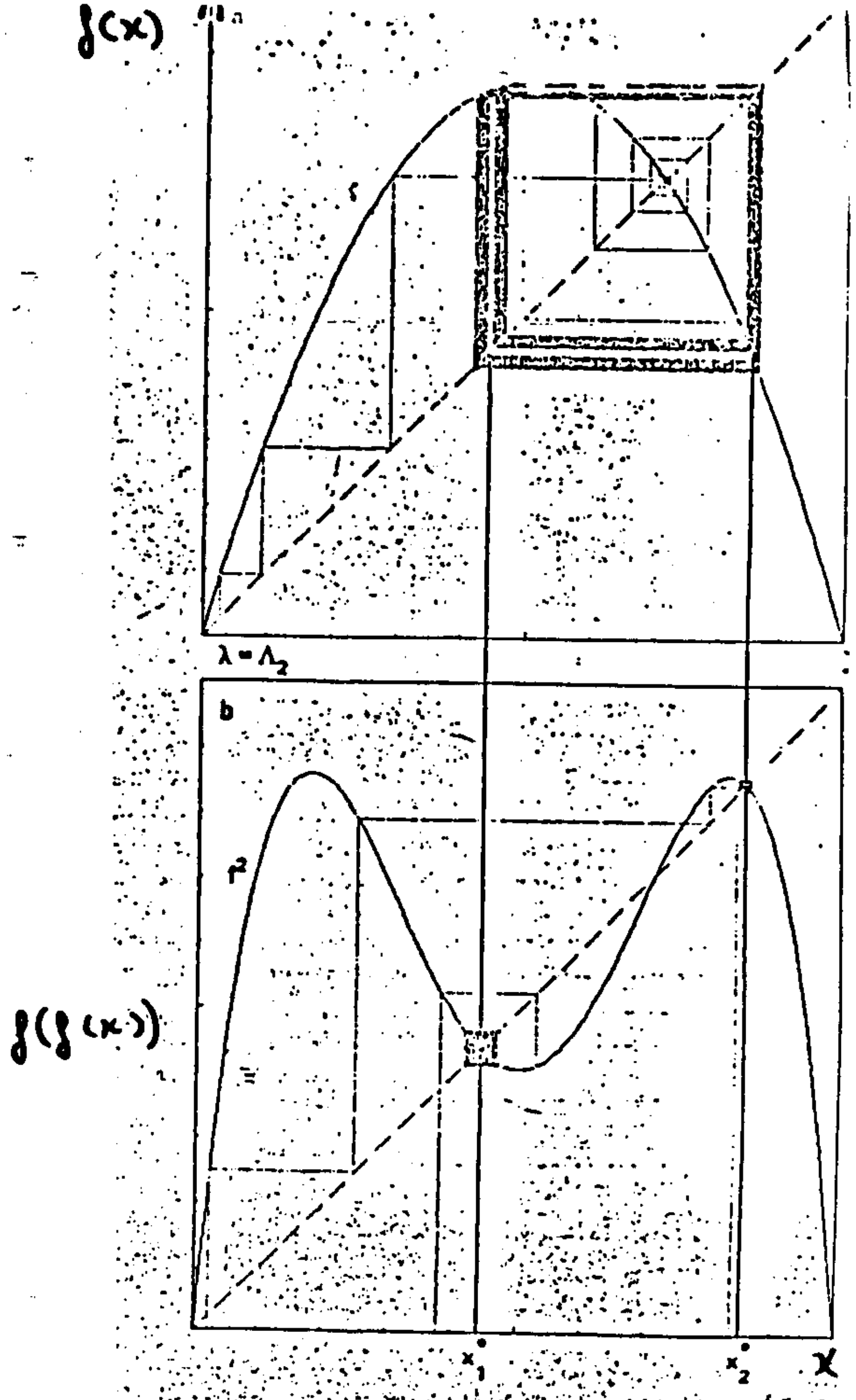
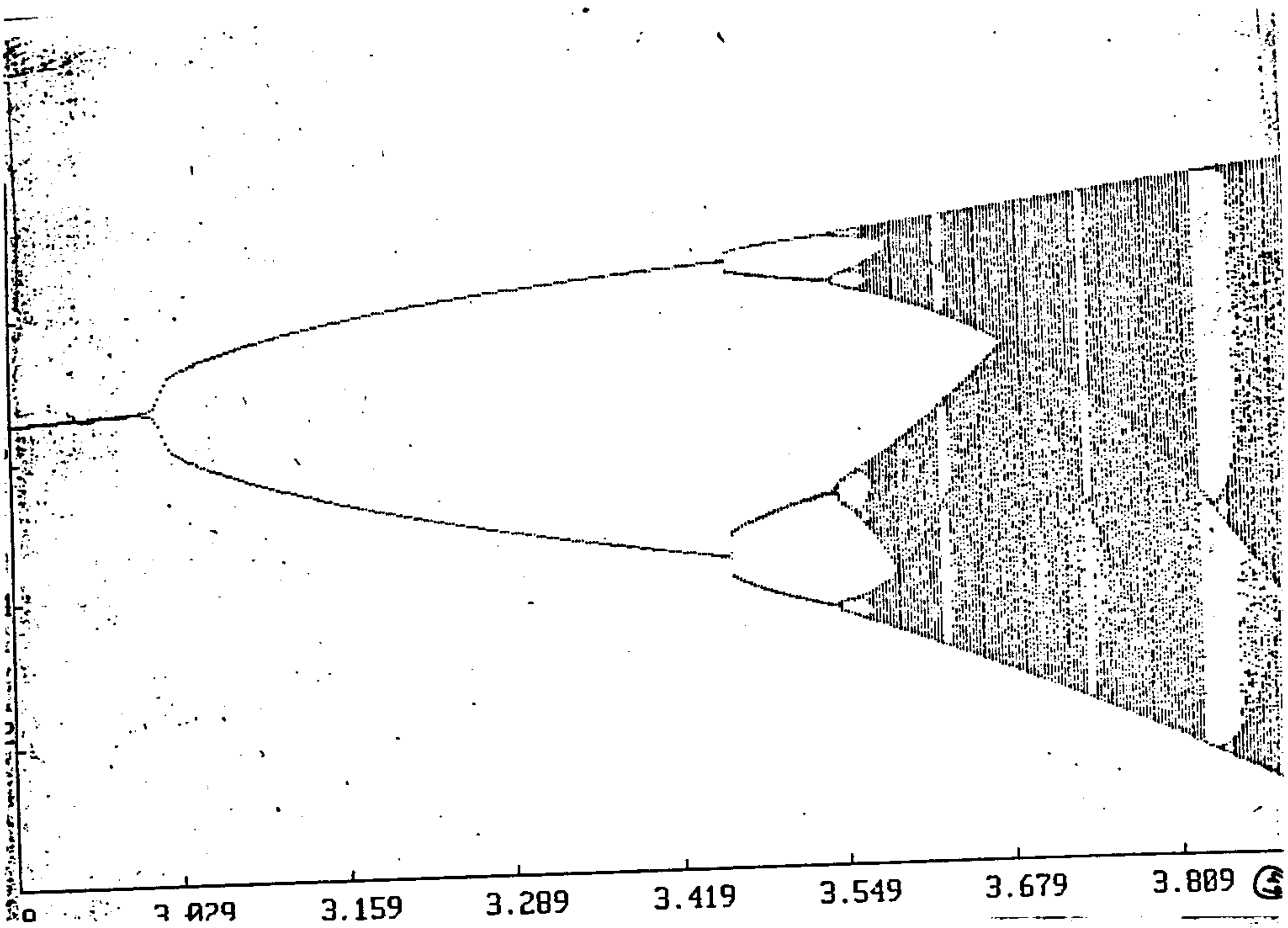
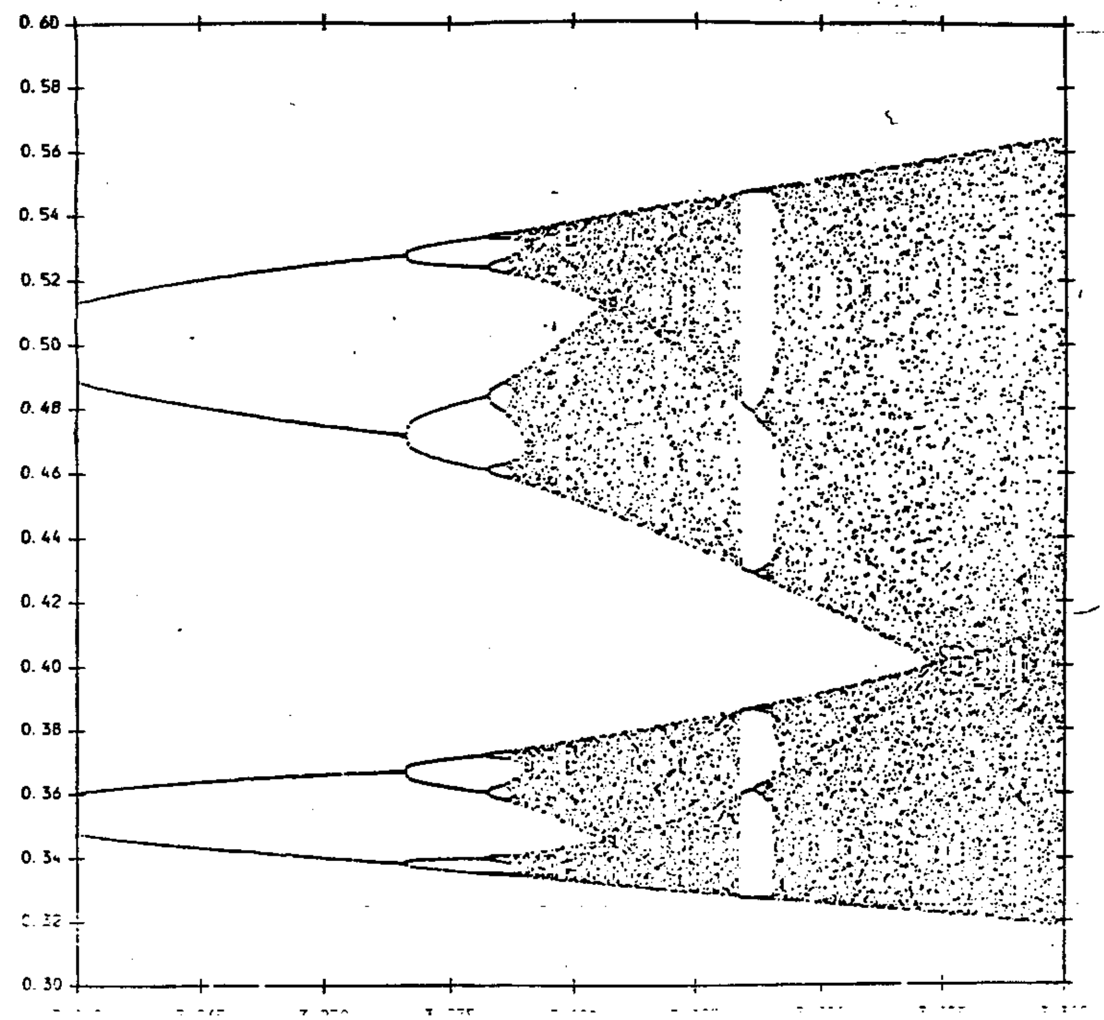



f(x)
a
f
λ = .785
b
f^2
f(f(x))


f(x)
f
λ = Λ_2
b
f^2
f(f(x))
x_1^*
x_2^*
x


0
3.029
3.159
3.289
3.419
3.549
3.679
3.809


0.60
0.58
0.56
0.54
0.52
0.50
0.48
0.46
0.44
0.42
0.40
0.38
0.36
0.34
0.30

CHAOS IN TAYLOR-COUETTE FLOW

1/ PERIOD DOUBLING ON A TORUS
— "TORUS DOUBLING" - TYPE I

2/ TRAJECTORY DOUBLING ON A TORUS
— "TORUS DOUBLING" - TYPE II

3/ HOMOCLINIC ORBITS ON A TORUS
- THE SILNIKOV MECHANISM

4/ INTERMITTENCY - TYPES I AND II.