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34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
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WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

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HYDROMAGNETIC-GRAVITY WAVES

I.A. Eltayeb
Department of Mathematics and Computing
Sultan Qaboos University
Muscat
Sultanate of Oman

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by

Ibrahim A. Eltayeb

Department of Mathematics and Computing, College of Science, Sultan Qaboos University, P.O. Box 32486 Al-Khodh, Muscat, Sultanate of Oman

I. INTRODUCTION

The subject of hydromagnetics deals with the motion of an electrically conducting fluid. The motion of such a fluid is then governed by both the equations of hydrodynamics and those of electromagnetic theory (see, Ferraro and Plumbton: An introduction to magneto-fluid mechanics. Oxford University Press). The applications of hydromagnetics are found in geophysics, astrophysics and engineering. The evolution of the magnetic fields of the Earth, Sun and other planets are governed by the equations of hydromagnetics. Plasma confinement relating to the construction of thermonuclear reactors is another application.

The set of equations of hydromagnetics is

$$\rho\left(\frac{D\underline{u}}{Dt} + 2\,\underline{\Omega}\wedge\underline{u}\right) = -\nabla p + \frac{1}{\mu}\nabla\wedge\underline{B}\wedge\underline{B} + \rho\nu\nabla^2\underline{u} + \rho\underline{F} \tag{1.1}$$

$$\nabla\cdot\underline{u} = 0 \tag{1.2}$$

$$\frac{\partial\underline{B}}{\partial t} = \mathrm{Curl}\,(\underline{u}\wedge\underline{B}) + \eta\nabla^2\underline{B} \tag{1.3}$$

$$\nabla\cdot\underline{B} = 0 \tag{1.4}$$

$$\frac{DT}{Dt} = \kappa\nabla^2 T + \Phi \tag{1.5}$$

$$\rho = \rho(p, T) \tag{1.6}$$

where the mobile operator D/Dt is defined by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$$

and ρ is the density, $\underline{u}$ the velocity in a frame rotating with angular velocity $\underline{\Omega}$, $\underline{B}$ the magnetic field, p the pressure, T the temperature, μ the magnetic permeability, ν the kinetic viscosity and κ, η are the magnetic and thermal diffusivities. $\underline{F}$ is the external force. Φ is the viscous dissipation.

The complexity of these equations both in differential order and in non-linearity makes progress towards a solution very slow. Some insight can be gained by taking some special cases. One such case is the diffusionless fluid, i.e. $\nu = \kappa = \eta = 0$. We shall also set $\underline{\Omega} = 0$ here for simplicity. The ideal fluid case is of some interest in its own right.

We shall then study the propagation of hydromagnetic-gravity waves in an incompressible Boussinesq fluid. The equations are

$$\rho\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \frac{1}{\mu}\nabla\wedge\underline{B}\wedge\underline{B} + \rho\underline{g} \tag{1.7}$$

$$\nabla\cdot\underline{u} = 0 \tag{1.8}$$

$$\frac{\partial\underline{B}}{\partial t} = (\underline{B}\cdot\nabla)\underline{u} - \underline{u}\cdot\nabla\underline{B} \tag{1.9}$$

$$\nabla\cdot\underline{B} = 0 \tag{1.10}$$

$$\frac{D\rho}{Dt} = 0 \tag{1.11}$$

Here g is the constant gravitational acceleration. When we set $\underline{B} = 0$, we recover the equations for gravity waves.

The treatment below is restricted to linear waves. The analysis can be summarized in four main steps:

(i) Consider a simple solution of the governing equations (1.7) - (1.11) and call it the *basic state*.

(ii) Superimpose perturbations of wave-like nature on the basic state. Since the amplitudes of the perturbations are small then all products and cross-products of perturbation variables are neglected in the linear theory.

(iii) Substitute for the variables (in the form basic state + perturbation) into the governing equations and use (i) and (ii) to derive the ***linearized perturbation equations*** which govern the propagation of the waves.

(iv) In the absence of sources these equations are homogeneous and hence they will possess a non-trivial solution only if a consistency condition is satisfied. This condition yields the *dispersion relation* which relates the frequency to the wave-number vector of the waves.

II. THE PERTURBATION EQUATIONS

Consider a coordinate system O(x, z) in which Oz is vertically upwards and Ox horizontal. Take a basic state in which the fluid is stratified vertically upwards and the flow $\underline{u}$ and magnetic field $\underline{B}$ are horizontal and vary along the vertical:

$$\underline{u} = U(z)\,\hat{\underline{x}} \;,\; \underline{B} = B(z)\,\hat{\underline{x}} \;,\; p = p_0(z) \;,\; \rho = \rho_0(z) \tag{2.1}$$

Equations (1.7) - (1.11) then give the magnetostatic balance

$$0 = -\frac{d\Pi_0}{dz} - \rho_0 g \;, \tag{2.2}$$

where

$$\Pi = p + \frac{B^2}{2\mu} \;. \tag{2.3}$$

Π is known as the total pressure (i.e. hydrostatic + magnetic). The density is assumed of the form

$$\rho_0(z) = \rho_\infty e^{-\beta z} \quad ; \beta \equiv \text{constant}. \tag{2.4}$$

We now superimpose the perturbations:

$$\underline{u} = U\hat{\underline{x}} + \epsilon \underline{u}_1 \;,\; \underline{B} = B\hat{\underline{x}} + \epsilon \underline{b}_1 \;,$$
$$\Pi = \rho_0(\Pi_0 + \epsilon\Pi_1) \;,\; \rho = \rho_0(1 + \epsilon\rho_1) \;, \tag{2.5}$$

and use the Boussinesq approximation in which the density variations ρ_1 are neglected *except* when they occur multiplied by g. Thus

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\underline{u}_1 + \underline{u}_1\cdot\nabla U\hat{\underline{x}} = -\nabla\Pi_1 + V\frac{\partial \underline{v}_1}{\partial x} + \underline{v}_1\cdot\nabla V\hat{\underline{x}} - \rho_1 g\hat{\underline{z}} \tag{2.6}$$

$$\nabla\cdot\underline{u}_1 = 0 \tag{2.7}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\underline{v}_1 = V\frac{\partial \underline{u}_1}{\partial x} \tag{2.8}$$

$$\nabla\cdot\underline{v}_1 = 0 \tag{2.9}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\rho_1 = \beta\, \underline{u}_1\cdot\hat{\underline{z}} \quad , \tag{2.10}$$

in which the *Alfvén* velocity $\underline{v}$ is defined by

$$\underline{V} = \underline{B}/\sqrt{\mu\rho_0} \quad . \tag{2.11}$$

Take perturbations of wave-like nature

$$\{\underline{u}_1, \underline{v}_1, \pi_1, \rho_1\} = \{\underline{u}, \underline{v}, \pi, \rho\}\, e^{i(\omega t - kx)} \tag{2.12}$$

where $\underline{u}$, $\underline{v}$, p, ρ are functions of z only.

We substitute from (2.12) into (2.6) - (2.10) to get

$$P w'' + P' w' + \left\{\frac{k(PU'' + P'U')}{\hat{\omega}} - k^2\left(P + \frac{N^2}{\hat{\omega}^2}\right)\right\} w = 0 \tag{2.13}$$

where the "dash" represents differentiation with respect to z, and

$$w = \underline{u}\cdot\hat{\underline{z}} \;,\quad P = -1 + k^2V^2/\hat{\omega}^2 \;,\quad \hat{\omega} = \omega - kU$$

$$u = \underline{u}\cdot\hat{\underline{x}} = -iw'/k \;,\quad v_1 = \underline{v}\cdot\hat{\underline{x}} = iVw'/\hat{\omega} \;,$$

$$v_2 = \underline{v}\cdot\hat{\underline{z}} = -kVw/\hat{\omega} \;,\quad \rho = -i\beta w/\hat{\omega} \;,$$

$$ik\pi = i\hat{\omega}u + U'w + ikVv_1 - V'v_2 \;. \tag{2.14}$$

$\hat{\omega}$ is the Doppler-shifted frequency (or intrinsic frequency) and it represents the frequency

of the wave as measured by a stationary observer.

The propagation of the waves is then governed by (2.13). However, (2.13) is not suitable, in its entirity, to study the basic properties of the waves. We shall therefore consider two simple cases first before we return to (2.13).

III. THE DISPERSION RELATION FOR A UNIFORM BASIC STATE

When U and V are both constant, (2.13) reduces to

$$w'' - k\left(1 + N^2/\hat{\omega}^2 P\right) w = 0 \tag{3.1}$$

The solution of this equation has the form

$$e^{imz} \tag{3.2}$$

Then

$$m^2 + k^2 = \frac{k^2N^2}{\hat{\omega}^2 - k^2V^2} \;. \tag{3.3}$$

This is the dispersion relation for hydromagnetic-gravity waves in a uniformly moving medium in the presence of a uniform magnetic field.

When V = 0, it reduces to the dispersion relation for gravity waves

$$m^2 + k^2 = \frac{k^2N^2}{\hat{\omega}^2} \;. \tag{3.4}$$

On the other hand, if $N^2 = 0$, it reduces to the dispersion relation for Alfvén waves

$$\hat{\omega}^2 = k^2 V^2 . \qquad (3.5)$$

The dispersion relation (3.3) can be written as

$$\omega \left(k, m ; U, V, N \right) = \text{constant} \qquad (3.6)$$

which means that for every set of the parameters U, V, N, ω represents a surface in the wave number space. The normal to this surface

$$\underline{v}_g = \nabla \omega = \left(\frac{\partial \omega}{\partial k} , \frac{\partial \omega}{\partial m} \right) , \qquad (3.7)$$

is known as the *group velocity*. This is different from the *phase velocity* $\underline{v}_p$ which represents the velocity of the wave front

$$\omega t - kx - mz = \text{constant}$$

so that

$$\omega - \underline{v}_p \cdot \underline{K} = 0 , \quad \underline{K} = (k, m) . \qquad (3.8)$$

One of the most illuminatiing methods of studying the dispersion realtion is geometrical. We assume ω constant and sketch the curve (3.3) in the (k,m) plane. Where there are curves the waves are propagating and where there are no curves the waves are evanescent. The normal to the curve in the direction of increasing ω represents the direction of the group velocity and the direction of the radius vector gives the direction of phase propagation (see Fig. 1). The figure shows the existence of vertical asymptotes. Near an asymptote the direction of the group velocity is horizontal and the wave propagates with the flow. The vertical asymptote referes to a *critical* level and the wave is said to get absorbed at the critical level. However, the manner in which this takes place cannot be explained when the basic state is uniform. For this reason we shall consider the propagation of the waves in a slowly varying medium.

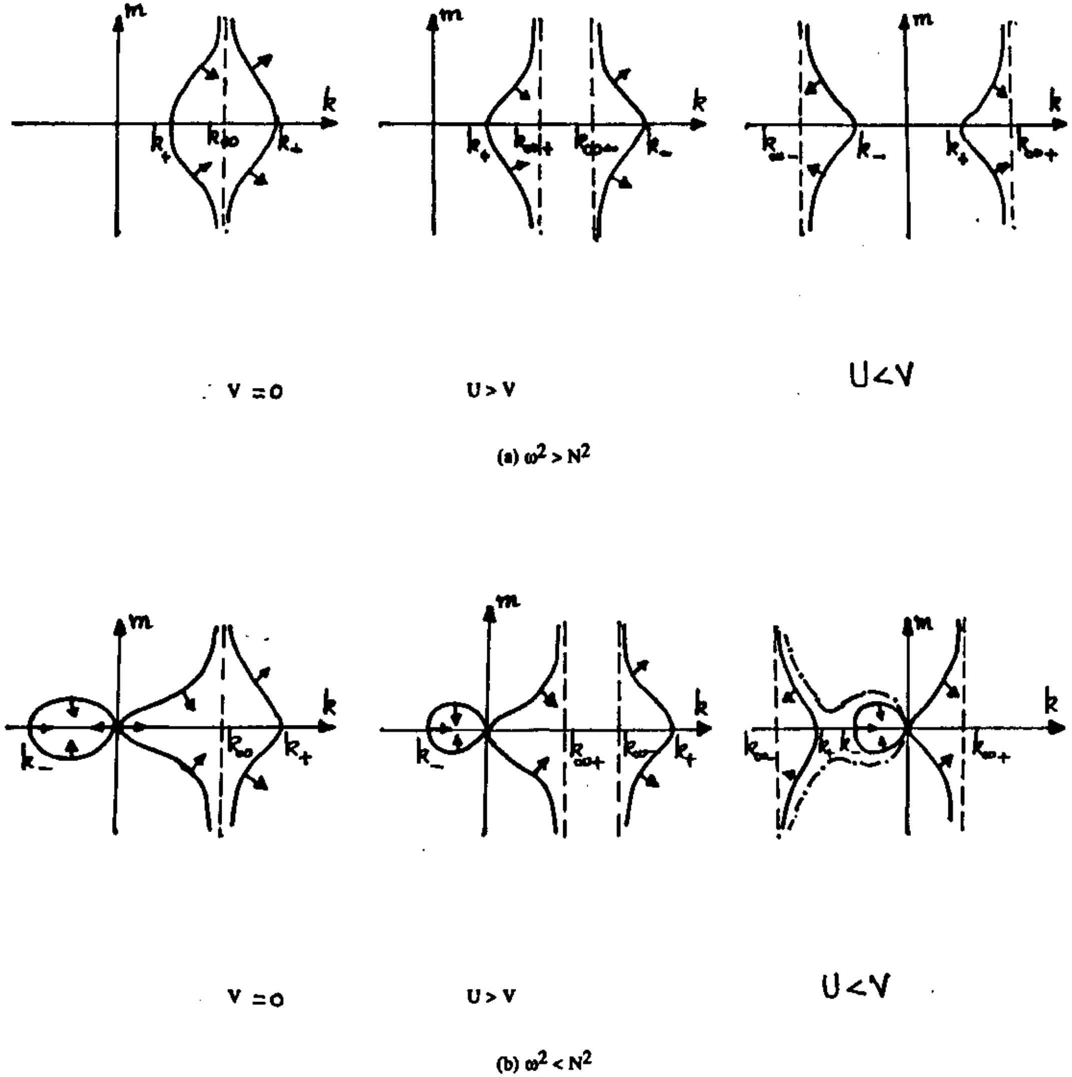


Figure 1.

The wave normal curves for a uniform basic state

$$k_{\infty\pm} = \omega/(U\pm V) \quad , \; k_\infty = \omega/U \, ,$$

$$k_\pm = \{\omega U \pm [\omega^2 V^2 + N^2(U^2 - V^2)]^{\frac{1}{2}}\}/(U^2 - V^2)$$

IV. PROPAGATION IN A SLOWLY VARYING MEDIUM

The generalization of the dispersion relation obtained for a uniform background to more realistic states can be made when the length scale L of variations of U and V is much larger than the horizontal wavelength λ (= $2\pi/k$) of the waves. This can be achieved by using a *multiple scale method* (or WKBJ method) which makes use of the small parameter

$$\epsilon = \lambda / L \, . \tag{4.1}$$

In this case the treatment can be made for general basic states. Define the position vectors and time by

$$\underline{x} = (x, z) \; , \; \underline{X} = \epsilon \underline{x} \; , \; T = \epsilon t \tag{4.2}$$

Consider a basic state

$$\rho = \rho_0(\underline{X},T) \, , \; \Pi = \epsilon^{-1}\Pi_0(\underline{X},T) \, , \; \underline{u} = \underline{U}(\underline{X},T), \; \underline{v} = \underline{V}(\underline{X},T), \tag{4.3}$$

and let

$$\underline{u} = \underline{U} + \epsilon \underline{u}_1 , \; \underline{v} = \underline{V} + \epsilon \underline{v}_1 \, , \; \Pi = \epsilon^{-1}\Pi_0 + \epsilon \Pi_1 \, , \; \rho = \rho_0(1+\epsilon\rho_1) \tag{4.4}$$

Assume

$$\{\underline{u}_1, \underline{v}_1, \Pi_1, \rho_1\} = \{\underline{u}(\underline{X},T), \underline{v}(\underline{X},T), \Pi(\underline{X},T), \rho(\underline{X},T)\} \, e^{i\epsilon^{-1}\theta(\underline{X},T)} \, . \tag{4.5}$$

The phase function $\epsilon^{-1}\Theta$ defines the *local* wave-number and frequency by

$$\underline{K} = (k, m) = \left(-\frac{\partial\theta}{\partial k} , -\frac{\partial\theta}{\partial m}\right) , \; \omega = \frac{\partial\theta}{\partial T} \, . \tag{4.6}$$

Substitution of (4.4) - (4.6) in the equations (1.1) - (1.6) yields a basic state governed by

$$\left(\frac{\partial}{\partial T} + \underline{U}\cdot\nabla\right)\underline{U} = -\nabla\Pi_0 + \underline{V}\cdot\nabla\underline{V} - g\,\hat{\underline{z}} \tag{4.7}$$

$$\left(\frac{\partial}{\partial T} + \underline{U}\cdot\nabla\right)V = \underline{V}\cdot\nabla\underline{U} \tag{4.8}$$

$$\nabla\cdot\underline{U} = \nabla\cdot\underline{V} = 0 \tag{4.9}$$

$$\left(\frac{\partial}{\partial T} + U\cdot\nabla\right)\rho_0 = 0 \tag{4.10}$$

and perturbation equations

$$i\hat{\omega}\,\underline{u} - i\underline{K}\,\Pi - i(\underline{K}\cdot\underline{V})\,\underline{v} + \rho g\,\hat{\underline{z}} = -\epsilon\underline{Q} \tag{4.11}$$

$$i\hat{\omega}\,v + i(\underline{K}\cdot\underline{V})\,\underline{u} = -\epsilon\underline{R} \tag{4.12}$$

$$i\hat{\omega}\,\rho - \beta\,\underline{u}\cdot\hat{\underline{z}} = -\epsilon I \tag{4.13}$$

$$i\underline{K}\cdot\underline{u} = \epsilon\,\nabla\cdot\underline{u} \tag{4.14}$$

$$i\underline{K}\cdot\underline{v} = \epsilon\,\nabla\cdot\underline{v} \tag{4.15}$$

where

$$\underline{Q} = \partial\underline{u}/\partial T + \underline{U}.\nabla\underline{u} + \underline{u}.\nabla\underline{U} + \nabla\pi - \underline{V}.\nabla\underline{v} - \underline{v}.\nabla\underline{V},$$

$$\underline{R} = \partial\underline{v}/\partial T - \underline{V}.\nabla\underline{u} + \underline{U}.\nabla\underline{v} + i(\underline{K}.\underline{v})\underline{u} - i(\underline{K}.\underline{u})\underline{v},$$

$$I = \partial\rho/\partial T + \underline{U}.\nabla\rho \quad . \tag{4.16}$$

Further, we let

$$\{\underline{u}, \underline{v}, \pi, \rho\} = \sum_{n=0}^{\infty} \epsilon^n \{\underline{u}^{(n)}, \underline{v}^{(n)}, \pi^{(n)}, \rho^{(n)}\} \tag{4.17}$$

and substitute in (4.11) - (4.16) and equate the coefficients of ϵ^n ($n = 0, 1, 2, \ldots$) to zero. For every value of n we obtain a system of equations which can be solved before the next value of n is considered.

Problem ϕ

Terms of order ϵ^0 give

$$i\hat{\omega}\,\underline{u}^{(0)} - i\underline{K}\pi^{(0)} + i(\underline{K}.\underline{V})\,\underline{v}^{(0)} + \rho^{(0)} g\,\hat{\underline{z}} = 0 \tag{4.18}$$

$$i\hat{\omega}\,\underline{v}^{(0)} + i(\underline{K}.\underline{V})\,\underline{u}^{(0)} = 0 \tag{4.19}$$

$$i\hat{\omega}\,\rho^{(0)} - \beta\,\underline{u}^{(0)}.\hat{\underline{z}} = 0 \tag{4.20}$$

$$i\underline{K}.\underline{u}^{(0)} = i\underline{K}.\underline{v}^{(0)} = 0 \tag{4.21}$$

This is the same set of equations obtainable for a uniform basic state. Thus

$$\underline{v}^{(0)} = -(\underline{K}.\underline{V})\underline{u}^{(0)}/\hat{\omega}\ , \quad \pi^{(0)} = -N^2(\underline{K}.\hat{\underline{z}})(\underline{u}^{(0)}.\hat{\underline{z}})/\hat{\omega}\,\underline{K}^2,$$

$$\rho^{(0)} g = N^2(\underline{u}^{(0)}.\hat{\underline{z}})/i\hat{\omega}\ ,$$

$$\underline{u}^{(0)}.\hat{\underline{x}} = (\underline{K}.\hat{\underline{x}})\,\pi^{(0)}/[\hat{\omega} - (\underline{K}.\underline{V})^2/\hat{\omega}]\ , \tag{4.22}$$

provided that the dispersion relation

$$\hat{\omega}^2 - (\underline{K}.\underline{V})^2 = (\underline{K}.\hat{\underline{x}})^2 N^2/\underline{K}^2 . \tag{4.23}$$

is satisfied.

Problem I

If we consider the terms of order ϵ^1 we obtain a set of non-homogeneous equations with a homogeneous part identical to problem ϕ. The solvability condition yields the conservation law

$$\partial A/\partial T + \nabla\cdot\underline{E} = 0\ , \tag{4.24}$$

in which

$$\mathcal{A} = E/\hat{\omega} \quad , \quad \mathcal{F} = \mathcal{A}\left(\underline{v}_g + \underline{U}\right) . \tag{4.25}$$

Here the energy density E is given by

$$E = \frac{1}{2}\left\{ |\underline{u}^{(0)}|^2 + |\underline{v}^{(0)}|^2 + N^{-2}|\rho^{(0)}g|^2 \right\} , \tag{4.26}$$

and the group velocity $\underline{v}_g$ is obtained from the dispersion relation

$$\underline{v}_g = \frac{(\underline{k}\cdot\underline{V})}{\hat{\omega}}\,\underline{V} - \frac{N^2(\underline{k}\cdot\hat{\underline{x}})^2}{\hat{\omega}\,\underline{k}^4}\,\underline{k} - \frac{(\underline{k}\cdot\hat{\underline{x}})N^2}{\underline{k}^2}\,\hat{\underline{x}} . \tag{4.27}$$

$\mathcal{A}$ is called wave-action density and $\mathcal{F}$ is called wave-action flux. The equation (4.25) in the absence of the magnetic field was derived by Bretherton and Garrett (1968).

Let us examine the dispersion relation (4.23) as the profiles of $\underline{V}$ and $\underline{U}$ vary. For simplicity let

$$\underline{U} = U\hat{\underline{x}} \quad , \quad \underline{V} = V\hat{\underline{x}} . \tag{4.28}$$

Then (4.23) take the form

$$m^2 + k^2 = k^2N^2/(\hat{\omega}^2 - k^2V^2) , \quad \hat{\omega} = \omega - kU . \tag{4.29}$$

The asymptotes occur at $k_{\infty\pm}$ where

$$k_{\infty\pm} = \omega/(U \pm V)$$

As z varies the positions of $k_{\infty\pm}$ vary while the values $k_\pm$ (corresponding to m = 0) also vary. Noting that the direction of a ray is given by the direction of the group velocity which is normal to the wave normal curve we can trace the ray trajectories by constructing the wave normal curves at different heights. Some of the ray trajectories are shown in Fig. 2. Here we assumed that V is constant and non-zero and U increases from small positive values for z = 0 to large values at $z \to \infty$. The types of ray trajectories fall into two main categories:

(i) rays which propagate away from a critical level but get reflected back towards the critical level where they are absorbed. These rays propagate on one side of the critical level with the region between the critical leves being a region of evanescence.

(ii) rays which propagate upwards against the flow, get bent and finally get reflected by

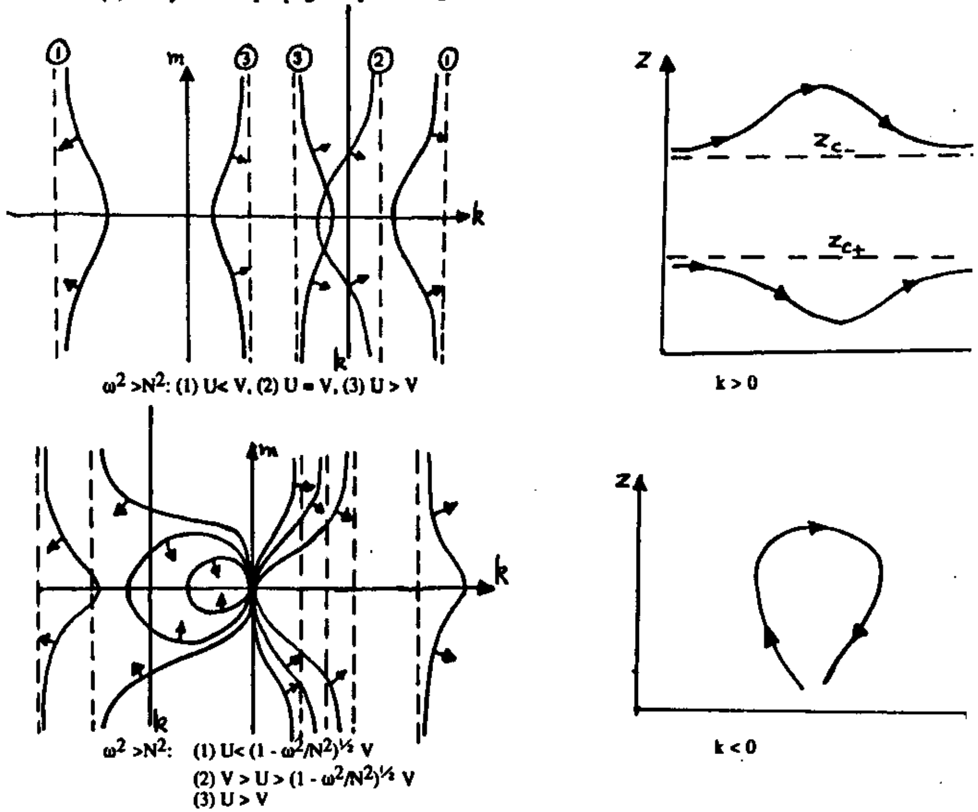


Figure 2

The increasing numbers indicate the increasing height the strong flow to propagate downwards. Other types of rays arise for different velocity profiles. Let us now justify absorption at a critical level. The vertical component of group velocity w_g is

$$w_g = -\frac{N^2 k^2 m}{\hat{\omega}(k^2+m^2)^2} = -m(\hat{\omega}^2 - k^2 V^2)^2 / k^2 N^2$$

using the dispersion relation.

Near a critical level $\omega = k(U_c \pm V_c)$, we have

$$U = U_c + U_c'(z - z_c) + \cdots$$

$$V = V_c + V_c'(z - z_c) + \cdots$$

Then

$$\hat{\omega} = \hat{\omega}_c - k U_c'(z - z_c) + \cdots$$

Thus

$$w_g = \frac{dz}{dt} = A(z - z_c)^2, \quad A \equiv \text{constant}$$

Integrating we get

$$At + B = \frac{1}{z_c - z}, \quad B \equiv \text{constant}$$

The wave takes an infinite time to reach the critical level.

V. FULL WAVE TREATMENT

We now return to equation (2.13):

$$P w'' + P' w' + \left\{ \frac{k(PU'' + P'U')}{\hat{\omega}} - k^2 (P + N^2/\hat{\omega}^2) \right\} w = 0 \tag{5.1}$$

This equation can be transformed into normalized form by

$$w = P^{-\frac{1}{2}} \phi \tag{5.2}$$

so that

$$\phi'' + g(z)\phi = 0 \tag{5.3}$$

where

$$g(z) = -\frac{\{k\omega U'' + k^2(V'^2 + VV'')\}}{\hat{\omega}^2 P} - k^2\left(1 + \frac{N^2}{\hat{\omega}^2 P}\right)$$
$$+ k^3 V(kVV'^2 + 2\hat{\omega} V'U' + kVU'^2)/\hat{\omega}^4 P^2. \tag{5.4}$$

If ω is real then $g(z)$ is real and the result of multiplying (5.3) with the complex conjugate ϕ^* of ϕ and subtracting from the equation obtained by multiplying the complex conjugate of (5.3) by ϕ is (see Eltayeb 1977, §2)

$$\phi^* \phi'' - \phi \phi^{*''} = 0$$

Hence the quantity

$$\mathcal{A} = \mathrm{Re}\left(-i\phi^*\phi'\right) \tag{5.5}$$

is a constant and is closely related to the wave action density.

For any legitimate solution of (5.3) $\mathcal{A}$ is conserved everywhere except possibly at the singular points of (5.3). We note here that $\hat{\Omega} = 0$ is a singular point of (5.1) but not a singular point of (5.3). Accordingly $\mathcal{A}$ is continuous across $\hat{\Omega} = 0$.

To relate $\mathcal{A}$ to the physics of the problem, we consider the energy flux in the vertical direction. Now the energy flux is composed of three parts: (i) the rate of working of the pressure forces $p\underline{u}$; the flux of kinetic energy $\frac{1}{2}\rho\underline{u}^2\underline{u}$ and (iii) the Ponting vector flux $\frac{1}{\mu}\underline{E}\wedge\underline{B}$ where $\underline{E}$ is the electric field ($\underline{E} = -\underline{u}\wedge\underline{B}$). Hence the *mean* energy flux, F, in the vertical direction is given by

$$F = \overline{pw} + \tfrac{1}{2}\rho_0\left(2U\overline{uw} + U^2\overline{\rho w}\right) + 2V\overline{v_1 w} - V\overline{u v_2} - U\overline{v_1 v_2} \tag{5.6}$$

where the overbar denotes the average over a period. Thus

$$F = -\left(\frac{\omega}{k}\right)\mathcal{A} \tag{5.7}$$

For a given wave (i.e. fixed horizontal phase speed $c = \omega/k$) $\mathcal{A}$ represents the downward flux of energy.

What happens to $\mathcal{A}$ at critical levels? We must look at the solution of (5.3) in the vicinity of critical levels z_c occurring where $\hat{\Omega} = \pm kV$. Expand V and U about z_c to reduce (5.3) to

$$\phi'' + \frac{\frac{1}{4}}{(z-z_c)^2}\phi = 0 \tag{5.8}$$

with a general solution

$$\phi = (z-z_c)^{\frac{1}{2}}\left[A + B\ln(z-z_c)\right]. \tag{5.9}$$

The solution has a logarithmic singularity which means that the solution experiences a phase junp across $z = z_c$. In order to ascertain the value of this jump, we can appeal to a number of methods. The easiest and less demanding method invokes causality (Miles 1961). It simulates an initial value problem by assuming that at the level z which the wave is approaching, the amplitude is increasing so that ω has a small negative imaginary part.

$$\omega = \omega_r - i\omega_i \quad , \; |\omega_i| << |\omega_r| \tag{5.10}$$

Then (5.8) becomes

$$\phi'' + \frac{\frac{1}{4}}{\left[z - z_c + \frac{i\omega_i}{k(U'_c \pm V'_c)}\right]^2}\phi = 0 \tag{5.11}$$

The argument of $z - z_c + i\omega_i/k(U'_c \pm V'_c)$ is θ

$$\tan\theta = \frac{\omega_i}{k(U'_c \pm V'_c)(z-z_c)}. \tag{5.12}$$

As z decreases from large positive values to large negative values, θ varies continuously from 0 to $\pm\pi$ depending on whether $k(U'_c \pm V'_c) >< 0$. The correct solution on either side then is

$$\phi = \begin{cases} \mp i(z-z_c)^{\frac{1}{2}}\left[A + B\ln(z-z_c) \mp i\pi B\right] & ; \; z > z_c \\ (z_c-z)^{\frac{1}{2}}\left[A + B\ln(z_c-z)\right] & ; \; z < z_c \end{cases} \tag{5.13}$$

Evaluation of $\mathcal{A}$ on either side of the critical level gives

$$\mathcal{A} = \begin{cases} Re\,(i A^* B) \mp \pi |B|^2 & ; \; z > z_c \\ Re\,(i A^* B) & ; \; z_c \geq z \end{cases} \tag{5.14}$$

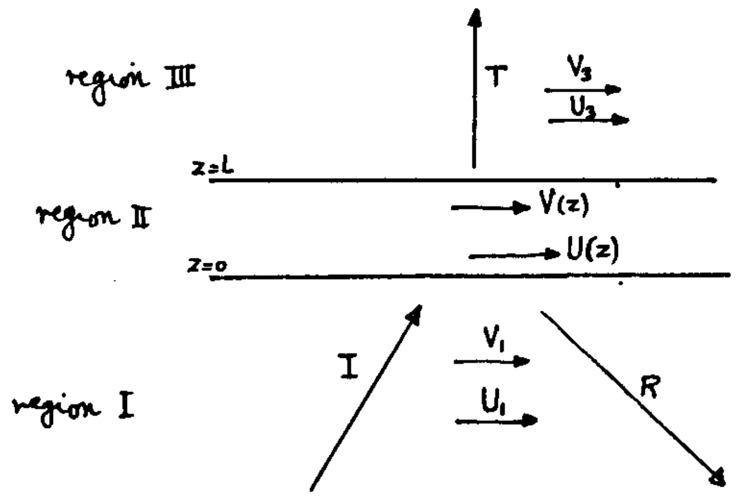
Then

$$\mathcal{A}_{above} = \mathcal{A}_{below} \mp \pi |B|^2 . \tag{5.15}$$

This result, which is general and applies to any flow V and Alfvén speed V, can be used to study the reflexion of hydromagnetic-gravity waves by shear layers.

For example, consider a layer of fluid of thickness L bounded above and below by uniform states i.e.

$$U, V = \begin{cases} U_1, V_1 & ; \; z \leq 0 \quad (\text{region I}) \\ U(z), V(z) & ; \; 0 \leq z \leq L \quad (\text{region II}) \\ U_3, V_3 & ; \; z \geq L \quad (\text{region III}) \end{cases} \tag{5.16}$$



A wave of amplitude I is incident on the layer. It gives rise to a reflected wave, amplitude R, and a transmitted wave, amplitude T. Then

$$\begin{aligned} w_1 &= I\, e^{i m_1 z} + R\, e^{-i m_1 z} , \\ w_3 &= T\, e^{i m_3 z} , \end{aligned} \tag{5.17}$$

where

$$\hat{\omega}_j^2 - k^2 V_j^2 = \frac{k^2 N^2}{k^2 + m_j^2} \quad ; \; j = 1, 3. \tag{5.18}$$

and m_1, m_3 are chosen to satisfy the radiation condition.

Now evaluation of $\mathcal{A}$ below and above the layer gives

$$\mathcal{A}_{below} = m_1 (I^2 - |R|^2) .$$

$$\mathcal{A}_{above} = Re(m_3)\, |T|^2 . \tag{5.19}$$

If there are no critical levels within the layer, then $\mathcal{A}$ is conserved and hence

$$m_1 (I^2 - |R|^2) = Re(m_3)\, |T|^2,$$

$$|R|^2 = I^2 - \frac{Re(m_3)}{m_1} |T|^2. \tag{5.20}$$

If m_3 is imaginary then $|R|^2 = I^2$ and total reflexion takes place. However, if m_3 is real then the wave is partially reflected and partially transmitted across the layer.

If there is one critical level within the layer then we use (5.5) and (5.19) to get

$$Re(m_3)\, |T|^2 = m_1 (I^2 - |R|^2) \mp \pi |B|^2 \; ; \; k(U'_c \pm V'_c) \gtrless 0 \tag{5.21}$$

Thus

$$|R|^2 = I^2 - \frac{Re(m_3)}{m_1} |T|^2 \mp \frac{\pi}{m_1} |B|^2 \; ; \; k(U'_c \pm V'_c) \gtrless 0. \tag{5.22}$$

It is possible to isolate situations in which the presence of the critical level enhances the reflexion coefficient and others in which it depletes it.

Suppose that

(i) $U_c + V_c > 0$

(ii) $U'_c + V'_c < 0$

(iii) $k_{\infty -} < k < k_-$ in region I

(iv) $k_{\infty +} < k < k_{\infty -}$ in region III.

Then $m_1 > 0$ and the transmitted wave is evanescent so that m_3 is purely imaginary. Also $k > 0$ so that $k(U'_c + V'_c) < 0$ and a critical level occurring where $\hat{\omega} = kV_c$ is present in the layer. The lower sign in (5.22) is relevant (see Fig.3). Thus

$$|R|^2 = I^2 + \frac{\pi}{m_1} |B|^2 > I^2$$

Thus the amplitude of the reflected wave is *greater* than that of the incident wave. The wave is said to be *over-reflected*. Over-reflexion must be due to absorption of energy by the wave at the critical level. This cannot be ascertained without non-linear study.

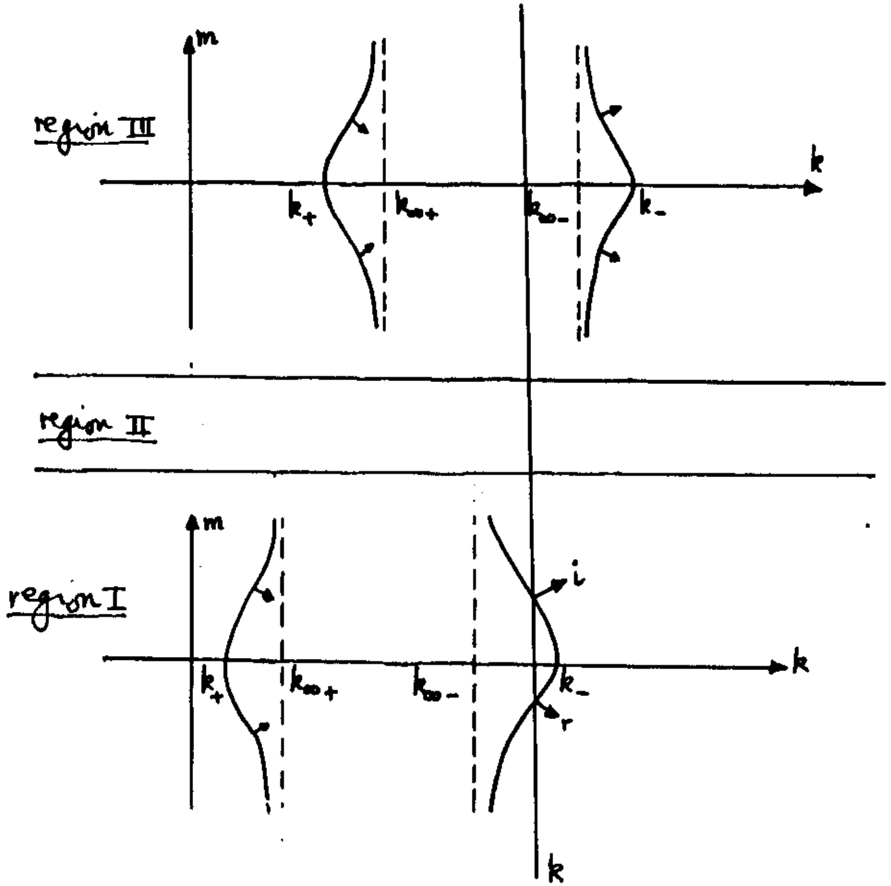


Figure 3

Another case arises when

(i) $k_- < k < 0,\ U_c < V_c$

(ii) $U'_c - V'_c > 0$

(iii) $U_3 - V_3 > \frac{\omega}{k}$

Then a critical level with $\hat{\omega} = -kV$ occurs in region II and $k(U'_c - V'_c) < 0$ and $m_I < 0$ while the wave in region III is evanescent. Then the lower sign in (5.22) is relevant and

$$|R|^2 = I^2 + \frac{\pi}{m_1}|B|^2 < I^2.$$

The critical level depletes R because of energy absorption (see Fig. 4).

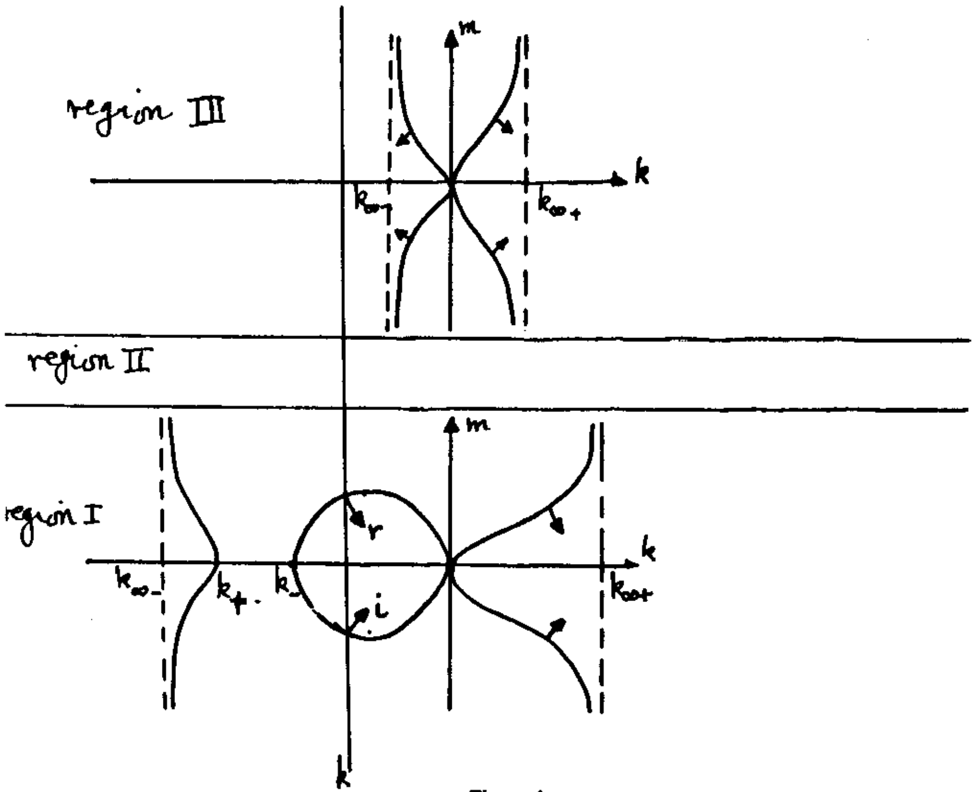


Figure 4

In order to determine R completely we must find B. This can only be done by solving equation (5.1) in region II (see Eltayeb 1981, J.F.M. 105, pp. 1 - 18).