INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR.378/19

WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

CONVECTIVE HEAT TRANSPORT

L.N. Howard
Department of Mathematics
The Florida State University
Tallahassee
Florida 32306-3027
U.S.A.

These are preliminary lecture notes, intended only for distribution to participants

I. CONVECTIVE HEAT TRANSPORT

1. First scientifically recognized by Count Rumford (Benjamin Thompson) in the late 1700's, the fact that moving fluid carries heat was a part of the beginning of the greater understanding of the nature of heat that led to the development of thermodynamics. Rumford and others (notably B. Franklin) made important technical application of this understanding in the development of much more efficient stoves for cooking and heating -- this was particularly important because the increasing population of Europe was rapidly depleting supplies of fuel.

Rumford is even more noted in the scientific sphere for his discovery of the equivalence of mechanical and thermal energy.

2. About 100 years later, with the theory of heat pretty fully developed, a series of striking experiments by Benard and his students began a detailed exploration of the flow structure in one of the simplest forms of convection: a horizontal layer of fluid heated uniformly brom below and cooled above. In this "Benard convection" it is the thermal driving itself which brings about the motion; "Forced convection" is distinguished by the fact that the fluid motion is -- at least in part -- driven by other energy sources, e. g. a water pump in the cooling system of a car. However there are also other forms of "natural convection" driven entirely by heating -- a hot vertical wall produces a current of air up along the wall, which participates strongly in the transport of heat away from the wall. Again, horizontal variations of boundary temperature of any sort lead inevitably to fluid motion and associated convective heat transport -- a good example is the sea breeze.

3. A unique feature of Benard convection is that the boundary conditions are consistent with a state of no motion - and heat flux only by conduction. Yet motion nevertheless (sometimes) occurs. The experiments demonstrated that motion does not occur - and indeed if initially present dies away - provided the temperature difference across the layer is less than a critical value. Above this value, motion always develops, and for sufficiently small supercriticality generally evolves to a steady cellular flow. In many of Benard's experiments this motion took the form of very regular patterns of hexagonal cells; these occur frequently when a layer of oil in a pan is heated from below, with its top surface open to the air. The horizontal size of the cells is found to be about equal to the depth of the layer. In other circumstances, e.g. with the top covered by a conducting plate, long approximately two dimensional cells, commonly called "rolls", are more typical. More complex patterns also can occur, often evolving in time sometimes toward a steady flow, but apparently in some cases never settling down.

4. Benard's experiments led Lord Rayleigh to attempt a theoretical explanation of the onset of convection in a layer. He envisioned the onset as an instability of the state of no motion and pure conduction, and gave a mathematical description based on a simple model of the fluid itself and of the boundary conditions. The

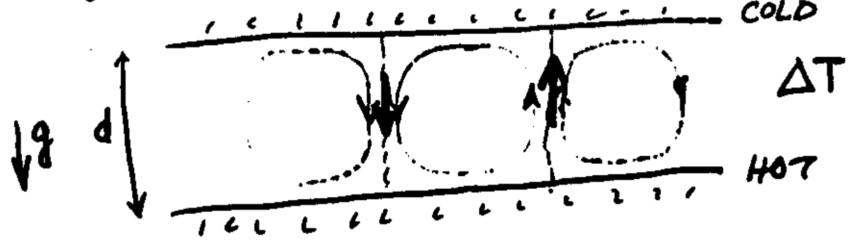
physical mechanisms in Rayleigh's theory are the buoyancy forces produced by thermal expansion of the fluid and gravity, and the dissipative processes of viscosity and heat conduction. The destabilization occurs because horizontal variations in temperature produce buoyancy forces which drive motion, and this motion itself can enhance the temperature differences. Rayleigh found that when the overall temperature difference was small enough then viscosity and heat conduction won out, and any (small) initial motion would decay. But as the overall temperature difference is increased, buoyancy becomes increasingly important and ultimately some motions at least start to grow rather than decay. The important dimensionless parameter in Rayleigh's theory is given on the first transparency; it is now called the Rayleigh Number. Instability occurs when it exceeds a certain numerical value of the order of 1000, depending on the precise boundary conditions.

5. Rayleigh's theory gave a plausible explanation of the threshold of convection, and predicted also a characteristic horizontal scale about equal to the layer thickness. It did not give a prediction of the cell shape, nor did it describe the evolution of the instability to a steady flow of some amplitude dependent on the temperature difference, as generally observed in experiments. It was a linear theory, insensitive to the overall amplitude, and treating cell patterns as eigenfuctions of the horizontal Laplacian -- any two with the same eigenvalue have equal growth rates according to the linear theory.

The stability theory itself has been subsequently extended in various ways: first the calculations have been carried out for more realistic boundary conditions than those used originally, by Jeffreys, Pellew and Southwell, and others using various methods. Much of this is described in S. Chandrasekhar's book on Hydrodynamic and Hydromagnetic Stablility, together with other extensions motivated in part by astrophysical applications. Stability problems for other types of convection in which the physical mechanisms are not the same as those utilized in Rayleigh's picture have also been studied. An interesting example of this is the study by J. A. Pearson of convection driven thermally, but by variations of surface tension rather than by buoyancy. This mechanism seems to be sometimes important in forming patterns in drying paint, and was probably in fact important in some of Benard's original experiments.

Numerous (and complex) investigations of finite amplitude convection using bifurcation methods and numerical calculations have been made in attempts to answer some of the questions left open by the linear theories, and to give some theoretical interpretation of experimental studies (also fairly numerous and complex) of flow above the critical Rayleigh number. Very briefly, one may say that as the Rayleigh number is raised, various transitions are encountered which are detectable in the measurements of overall heat flux and are generally associated with changes in the spatial or temporal character of the motion; some of these have been fairly well characterized experimentally, and identified with certain instabilities of evolved flows which have been found theoretically. The situation is made more complex by the dependence of these things on the Prandtl number, and in some cases on relatively subtle boundary effects, and by the rather large degree of non-uniqueness (even of steady flows) which appears to be the rule at elevated Rayleigh number. While a certain overall picture of transitions from steady two-dimensional to three dimensional to time-dependent to "turbulent" flow has emerged, it is almost certaily highly over-simplified. Even the supposedly fully-developed turbulence has been found, in some cases, to be more structured than expected.

THERMAL CONVECTION ("BÉNARD PROBLEM")



$\rho = \rho_0 (1 - \alpha T)$

α: THERMAL EXPANSION COEFF.

RAYLEIGH NUMBER:

$$R = \frac{\alpha g \Delta T d^3}{K_T \nu}$$

K_T: THERMAL DIFFUSIVITY

ν: KINEMATIC VISCOSITY

MOTIONLESS (CONDUCTIVE) STATE IS <u>STABLE</u> IF (AND ONLY IF) $R \leq R_c$

WHERE $R_c \sim 10^3$, (e.g. 1708 OR 657)

(DEPENDS ON BOUNDARY CONDITIONS)

$\sigma = \nu / K$ = PRANDTL NUMBER

BOUSSINESQ MODEL:

(DIMENSIONLESS FORM)

$$\left.\begin{aligned}&\frac{1}{\sigma}\left(\underline{u}_t + \underline{u}\cdot\nabla\underline{u}\right) + \nabla p = \underline{k}RT + \nabla^2\underline{u}\\ &T_t + \underline{u}\cdot\nabla T = \nabla^2 T, \quad \nabla\cdot\underline{u} = 0\end{aligned}\right\} 0<z<1$$

on $z=0$: $\underline{u}=0$, $T=1$
on $z=1$: $\underline{u}=0$, $T=0$

Assuming $\overline{T} = \overline{T}(z)$, $\overline{\underline{u}} = \overline{w}(z)\underline{k}$ etc.

$\overline{\underline{u}\cdot\nabla T} = \overline{\nabla\cdot(\underline{u}T)} = \frac{\partial}{\partial z}\overline{wT}$, so

$(\overline{wT})_z = \overline{T}_{zz}$, $\overline{wT} = \overline{T}_z - \overline{T}_z(0)$

$\therefore$ a) $-\overline{T}_z(0) = -\langle \overline{T}_z\rangle + \langle wT\rangle = 1 + \langle wT\rangle$

b) $-\overline{T}_z = 1 + \langle wT\rangle - \overline{wT}$

Thus NUSSELT NUMBER N (dimensionless heat flux) is given by

$$N = 1 + \langle wT\rangle$$

Now $0 = \langle T\nabla^2 T\rangle = \langle \nabla\cdot(T\nabla T)\rangle - \langle |\nabla T|^2\rangle$
$= -\overline{T}_z(0) - \langle|\nabla T|^2\rangle$, hence

$$1 + \langle wT\rangle = \langle|\nabla T|^2\rangle$$

Similarly, $R\langle wT\rangle = \langle|\nabla\underline{u}|^2\rangle$.

Let $T = \overline{T} + \Theta$; Then $\langle|\nabla T|^2\rangle = \langle\overline{T}_z^2\rangle + \langle|\nabla\Theta|^2\rangle$
$= \langle(1 + \langle w\Theta\rangle - \overline{w\Theta})^2\rangle + \langle|\nabla\Theta|^2\rangle$
$= 1 + \langle(\overline{w\Theta} - \langle w\Theta\rangle)^2\rangle + \langle|\nabla\Theta|^2\rangle$

$$\boxed{\begin{aligned}&\langle w\Theta\rangle = \langle(\overline{w\Theta} - \langle w\Theta\rangle)^2\rangle + \langle|\nabla\Theta|^2\rangle\\ &R\langle w\Theta\rangle = \langle|\nabla\underline{u}|^2\rangle\end{aligned}}$$

One consequence of these is that any convective flow with steady averages and $\langle w\Theta\rangle > 0$ must satisfy

$$R \geq \frac{\langle|\nabla\Theta|^2\rangle\langle|\nabla\underline{u}|^2\rangle}{\langle w\Theta\rangle^2}$$

(where $\underline{u}=0$ and $\Theta=0$ on $z=0,1$), so must have

$$R \geq \mathrm{Min}\,\frac{\langle|\nabla\Theta|^2\rangle\langle|\nabla\underline{u}|^2\rangle}{\langle w\Theta\rangle^2} = R_c$$

FROM THE 2 "POWER INTEGRALS" WE GET:

$$N-1 = \langle w\theta\rangle = \langle w\theta\rangle \frac{\langle w\theta\rangle - \langle|\nabla\theta|^2\rangle}{\langle(\langle w\theta\rangle - \overline{w\theta})^2\rangle}$$

$$= \frac{\langle w\theta\rangle^2 - \frac{1}{R}\langle|\nabla \underline{u}|^2\rangle\langle|\nabla\theta|^2\rangle}{\langle(\langle w\theta\rangle - \overline{w\theta})^2\rangle}$$

$$\equiv \mathcal{F}\{\underline{u},\theta;R\}, \text{ say.}$$

$\mathcal{F}$ is a homogeneous functional of $\underline{u}$ and θ, which is equal to $N-1$ for any solution to the Boussinesq equations with steady averages. But, $\mathcal{F}$ can be shown to be bounded above by a certain function $F(R)$, for <u>any</u> $\underline{u}$, θ satisfying the boundary conditions. Thus this F gives an estimate from above of the heat transport in turbulent convection

ESTIMATE OF F(R):

a) Optimal to choose $\underline{u} = w(z)\underline{k}$, $\theta = \theta(z)$

b) Optimal to choose w proportional to θ, i.e. since homogeneous, $w = \theta$

c) <u>Lemma</u>: for $f(z)$ continuous, $f(0)=f(1)=0$, $\langle f^2\rangle = 1$ and f' in $L_2(0,1)$ we always have
$$\langle f'^2\rangle\langle(1-f^2)^2\rangle \geq 16/9$$

d) Thus

$$F(R) \leq \frac{\left(1 - \frac{1}{R}\langle f'^2\rangle\right)^2}{\langle(1-f^2)^2\rangle} \leq \frac{9}{16}\, x\left(1-\frac{x}{R}\right)^2$$

$$\leq \frac{9}{16}\cdot\left(\frac{R}{3}\right)^{1/2}\left(1-\frac{1}{3}\right) = \frac{3}{8}\left(\frac{R}{3}\right)^{1/2} = \left(\frac{3R}{64}\right)^{1/2}$$

$$\therefore\ N \leq 1 + \left(\frac{3R}{64}\right)^{1/2}$$

(Another result, using continuity constraint and only 1 horiz. wavenumber is

$$N-1 \leq \frac{5\sqrt{2}}{22}\left(\frac{3R}{11}\right)^{3/8}$$)

Proof of Lemma

$$0 \le \int_0^t \left(f' - A(1-f^2)\right)^2 dz + \int_t^1 \left(f' + A(1-f^2)\right)^2 dz$$

$$= \langle f'^2\rangle + A^2\langle (1-f^2)^2\rangle - 2A\int_0^t (1-f^2) f'\, dz + 2A \int_t^1 (1-f^2) f'\, dz$$

$$= \langle f'^2\rangle + A^2 \langle(1-f^2)^2\rangle - 4A\left[f(t) - \tfrac13 f^3(t)\right]$$

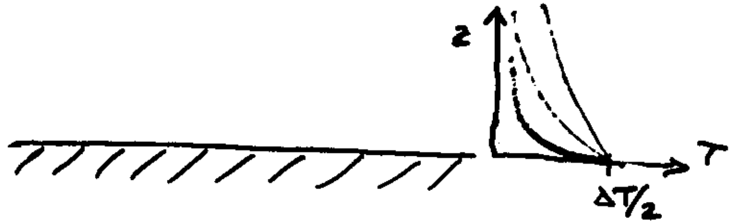
Let $A = \dfrac{2 f(t)\left(1-\frac13 f^2(t)\right)}{\langle(1-f^2)^2\rangle}$ (minimize)

to get: $0 \le \langle f'^2\rangle - \dfrac{4 f^2(t)\left(1-\frac13 f^2(t)\right)^2}{\langle(1-f^2)^2\rangle}$

Since $\langle f^2\rangle = 1$ and f is continuous there is a t such that $f^2(t)=1$. Choose this t to get

$$0\le \langle f'^2\rangle - \frac{4\cdot\left(\frac23\right)^2}{\langle(1-f^2)^2\rangle} \quad \text{or}$$

$$\langle f'^2\rangle\langle(1-f^2)^2\rangle \ge \frac{16}{9}$$



$$T = \tfrac12 \Delta T \operatorname{erfc}\left(\frac{z}{2\sqrt{\kappa t}}\right) \qquad 0<t<t_*$$

$$\bar T = \tfrac12 \Delta T\left[(1+2\xi^2)\operatorname{erfc}\xi - \tfrac{2}{\pi}\xi e^{-\xi^2}\right]$$

$$N = d/\sqrt{\pi\kappa t_*} = \frac{d}{\delta} = \left(\frac{R}{R_\delta}\right)^{1/3}$$

$$\xi = \frac{z}{2\sqrt{\kappa t_*}} = \frac{\sqrt{\pi}}{2}\,\frac{z}{\delta}, \quad \delta = \sqrt{\pi\kappa t_*}$$

Take $R_\delta \cong 10^3$.

A.M. TURING "THE CHEMICAL BASIS OF MORPHOGENESIS" PHIL. TRANS. ROY. SOC. LOND (B) 237, 37-72 (1952)

A ALL EIGENVALUES HAVE NEGATIVE REAL PARTS

$A - pI$ p A POSITIVE NUMBER

(EIGENVALUES EVEN MORE TO THE LEFT)

BUT:

P POSITIVE DEFINITE SYMMETRIC

$A - P$ MAY HAVE SOME EIGENVALS. IN THE RIGHT HALF PLANE

Ex: $A = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$ (eigenvals $-1, -2$)

$P = \begin{bmatrix} 1/2 & 0 \\ 0 & 10 \end{bmatrix}$

$A - P = \begin{bmatrix} 1/2 & -3 \\ 2 & -14 \end{bmatrix}$; $\det(A-P) = -1$

ANOTHER EXAMPLE: $A_1 = \begin{bmatrix} 0 & -1 & 0.8 \\ 1 & 0 & 0 \\ 0.8 & 0 & 0 \end{bmatrix}$

(CHAR. POLY.: $\lambda^3 + (1 - .8^2)\lambda = 0$ OR $\lambda^3 + .6^2\lambda = 0$)

$\therefore A = \begin{bmatrix} -0.01 & -1 & 0.8 \\ 1 & -0.01 & 0 \\ 0.8 & 0 & -0.01 \end{bmatrix}$

HAS EIGENVALUES -0.01, $-0.01 \pm 0.6i$

BUT $A - \begin{bmatrix} .01 & 0 & 0 \\ 0 & .01 & 0 \\ 0 & 0 & .69 \end{bmatrix} = \begin{bmatrix} -0.02 & -1 & 0.8 \\ 1 & -0.02 & 0 \\ 0.8 & 0 & -0.7 \end{bmatrix}$

HAS EIGENVALUES -1.02 and $+0.14 \pm \sqrt{0.6544}\,i \approx 0.14 \pm 0.81i$

A−P MAY SOMETIMES HAVE A CONJUGATE PAIR OF EIGENVALUES IN THE RIGHT HALF PLANE.

REACTION–DIFFUSION EQUATIONS

$$c_t = F(c) + K\nabla^2 c$$

c: VECTOR OF CONCENTRATIONS

$F(c)$: CHEMICAL KINETICS

K: POS. DEF. MATRIX OF DIFFUSIVITIES

HOMOGENEOUS EQUILIBRIUM: $c = c_0$ (A CONSTANT VECTOR) WITH $F(c_0) = 0$.

LINEARIZATION:

$$c'_t = Ac' + K\nabla^2 c' \qquad \left(A = \frac{\partial F}{\partial c} \text{ at } c_0\right)$$

$$c' = \gamma(t)e^{ikx}, \quad \text{or } \gamma(t)\phi(x,y),\ \nabla^2\phi = -k^2\phi$$

OR SUPERPOSITION OF SUCH.

$$\gamma_t = (A - k^2K)\gamma$$

(MAY BE UNSTABLE, FOR A LIMITED RANGE OF k^2: $0 < k_1^2 < k^2 < k_2^2 < \infty$)

EQUATIONS DESCRIBING DOUBLE DIFFUSIVE CONVECTION:

$$\left[\rho = \rho_0(1 - \alpha T + \beta S)\right]$$

$$\underline{u}_t + \underline{u}\cdot\nabla\underline{u} + \nabla p = g\underline{k}(\alpha T - \beta S) + \nu\nabla^2\underline{u}$$

$$\nabla\cdot\underline{u} = 0$$

$$T_t + \underline{u}\cdot\nabla T = \kappa_T\nabla^2 T$$

$$S_t + \underline{u}\cdot\nabla S = \kappa_S\nabla^2 S$$

$0 \le z \le d$, $T(0) = \Delta T$ (OR $-\Delta T$), $T(1) = 0$

$S(0) = \Delta S$ (OR $-\Delta S$), $S(1) = 0$

DIMENSIONLESS FORM:

$$\frac{1}{\sigma}(\underline{u}_t + \underline{u}\cdot\nabla\underline{u}) + \nabla p = \underline{k}(R_T T - R_S S) + \nabla^2\underline{u}$$

$$\nabla\cdot\underline{u} = 0$$

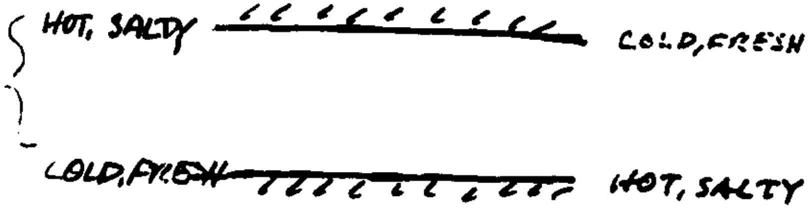
$$T_t + \underline{u}\cdot\nabla T = \nabla^2 T - w$$

$$S_t + \underline{u}\cdot\nabla S = \tau\nabla^2 S - w$$

$\sigma = \nu/\kappa_T \qquad \tau = \kappa_S/\kappa_T$

$$R_T = \frac{\alpha g \Delta T d^3}{\kappa_T \nu}, \qquad R_S = \frac{\beta g \Delta S d^3}{\kappa_T \nu}$$

$S = T = 0$ on $z = 0, 1$



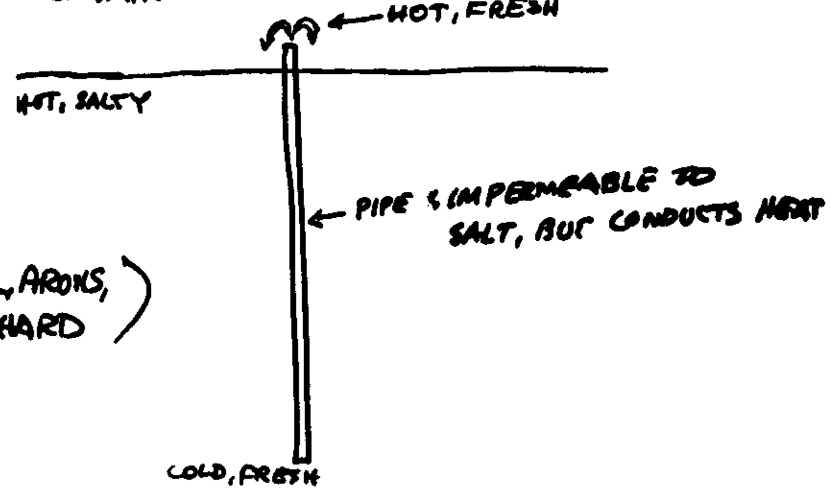
→ MOTIONLESS STATE BECOMES UNSTABLE WHEN

$$\frac{g\beta \Delta S d^3}{K_S \nu} - \frac{g\alpha \Delta T d^3}{K_T \nu} > R_c$$

i.e. $$\left(\frac{\beta \Delta S}{K_S} - \frac{\alpha \Delta T}{K_T}\right)\frac{g d^3}{\nu} > R_c$$

IF $K_S < K_T$, THIS CAN HAPPEN EVEN WHEN $\beta\Delta S - \alpha\Delta T < 0$

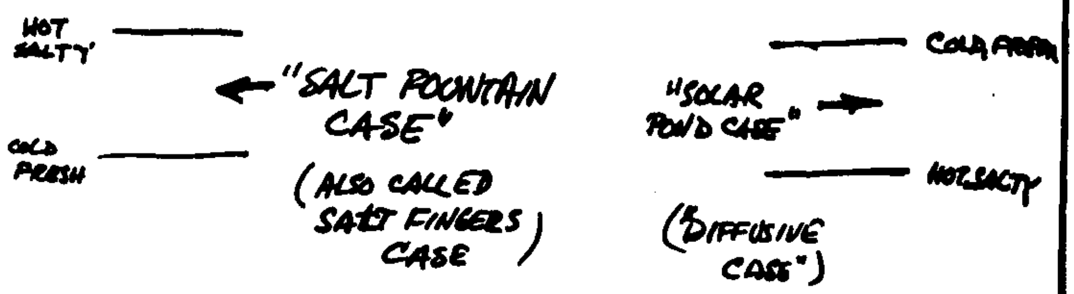
"SALT FOUNTAIN"



(STOMMEL, ARONS, & BLANCHARD)

APPROACHES TO UNDERSTANDING THE NONLINEAR FLOWS RESULTING FROM THESE INSTABILITIES

1. AMPLITUDE EXPANSIONS (BIFURCATION METHODS)
2. 'ENERGY METHOD' CONDITIONS FOR GLOBAL STABILITY OF MOTIONLESS STATE
3. NUMERICAL CALCULATIONS
4. PHYSICAL EXPERIMENTS ALSO OBSERVATIONS, E.G. LAKE VANDA
5. IDEALIZED STRONGLY NONLINEAR SOLUTIONS EXACT, ASYMPTOTIC, SEMINUMERICAL



BIFURCATION STUDIES INDICATE:

SALT FOUNTAIN CASE: BIFURCATION WITH CROSSING OF REAL EIGENVAL.

GENERALLY SUPERCRITICAL

STEADY CONVECTING SOLUTIONS PROBABLY EXIST AND ARE STABLE TO HIGH AMPLITUDE, BEYOND THE USEFUL RANGE OF AMPLITUDE EXPANSIONS

SOLAR POND CASE: HOPF BIFURCATION

OFTEN SUBCRITICAL, ESPECIALLY AT HIGH OR MODERATE R_S AND SMALL τ.

UNSTABLE SUBCRITICAL BRANCH OF PERIODIC SOLUTIONS MAY END BY MERGING WITH A BRANCH OF STEADY SOLUTIONS

APPARENTLY SOME ATTRACTOR(S) OTHER THAN MOTIONLESS STATE EXIST FOR SOME SUBCRITICAL RANGE

?

ENERGY METHOD (SOLAR POND CASE)

MOTIONLESS STATE GLOBALLY STABLE IF

$$\sqrt{R_T} < \sqrt{\tau R_S} + \sqrt{R_B(1-\tau^2)}$$

(D.D. JOSEPH, 'STABILITY OF FLUID MOTIONS' VOL II)

LINEAR OSCILLATORY INSTABILITY POINT (SLIPPERY BOUNDARIES)

$$R_{T\,crit} = R_B(1+\tau)\left(1+\frac{\tau}{\sigma}\right) + \frac{\sigma+\tau}{1+\sigma} R_S$$

(For water $\sigma \cong 7$, SALT IN WATER, $\tau \sim 10^{-2}$,
so $R_{T\,crit} \cong R_B + \frac{7}{8} R_S$)

[FOR LARGE R_S THIS IS FAR ABOVE THE 'ENERGY' BOUND

EXPERIMENTS ARE IN BETWEEN, NOT REALLY CLOSE TO EITHER