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WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

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ON THERMAL AND ROTATIONAL STABILITY

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ON THERMAL AND ROTATIONAL STABILITY.

Workshop on Theoretical Fluid Mechanics and Applications

1. Preamble

This discussion will be centred around the onset of instability occasioned by adverse temperature gradient on one hand and adverse gradient of angular momentum on the other hand. Both chemically reacting flow and flow in porous medium will be considered. The methodology follows that of the monumental work of Chandrasekhar [1].

2. Stability of fluid heated from below - an introduction

We consider a horizontal layer of fluid of depth 'd', such that the lower wall is on $z'=0$ while the upper wall is at $z'=d$ in the cartesian (x', y', z') coordinate in which t' is time. Gravitation $\vec{g}$ acts in the reverse direction to the z'-axis. The lower wall is heated to a constant temperature T_0 which is higher than the temperature at the wall $z'=d$. So that we take $d\bar{T}/dz'|_{z'=0} = -\beta$, where β is a constant greater than zero and $\bar{T}$ is the temperature of the fluid. Generally subscript zero will be used to denote conditions at the lower wall.

If we denote the velocity of the fluid by $\vec{q} = (u, v, w)$, the pressure by p, the heat capacity by κ, the viscosity by μ and the density by ρ, then the equations of continuity, momentum and energy under the usual Boussinesq approximation are

$$\nabla\cdot\vec{q} = 0 \tag{1}$$

$$\rho_0\left(\frac{\partial\vec{q}}{\partial t'} + \vec{q}\cdot\nabla\vec{q}\right) = -\nabla p + \mu\nabla^2\vec{q} + \rho\vec{g} \tag{2}$$

$$\frac{\partial\bar{T}}{\partial t} + \vec{q}\cdot\nabla\bar{T} = \kappa\nabla^2\bar{T}, \qquad \nabla = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right) \tag{3}$$

where the equation of state for a Boussinesq fluid is

$$\rho = \rho_0\left[1 - \alpha\left(\bar{T} - T_0\right)\right] \tag{4}$$

in which α is the coefficient of volume expansion. It is pertinent to note that equation (3) may be replaced by concentration gradient may be formulated similarly as discussed in Gershuni and Zhukhovitskii [2]

In the static flow (denoted by subscript s), equations (1) - (4) reduce to

$$\frac{dp_s}{dz'} = -\rho g, \quad \frac{d^2 T_s}{dz'^2} = 0, \quad \rho = \rho_0\left[1-\alpha(T_s - T_0)\right],$$

$$T_s = T_0, \quad \left.\frac{dT_s}{dz'}\right|_{z=0} = -\beta, \tag{5}$$

with solutions

$$T_s = T_0 - \beta z', \quad p_s - p_0 = -\rho_0 g\left(z + \frac{1}{2}\alpha\beta z^2\right). \tag{6}$$

In the perturbed state we set

$$T = T_0 - \beta z' + \theta, \quad p = p_s + p', \quad \vec{q}' = (u', v', w')$$

then substituting in equations (1) - (4) and neglecting the squares and products of perturbed quantities, we have

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0 = \nabla\cdot\vec{q}' \tag{7}$$

$$\frac{\partial \vec{q}'}{\partial t} = -\nabla\left(\frac{p'}{\rho_0}\right) + \nu\nabla^2\vec{q}' + g\alpha\theta\hat{k} \tag{8}$$

$$\frac{\partial\theta}{\partial t} = \beta w' + \quad \nabla^2\theta \tag{9}$$

$(\nu = \mu/\rho_0)$. Taking the curl of equation (8) and writing the vorticity $\vec{\omega} = \nabla_\wedge \vec{q}'$, we get

$$\frac{\partial\vec{\omega}}{\partial t} = \nu\nabla^2\vec{\omega} + g\alpha\left(\hat{i}\frac{\partial\theta}{\partial y'} - \hat{j}\frac{\partial\theta}{\partial x'}\right). \tag{10}$$

Further operation on (10) by curl leads to the result

$$\frac{\partial}{\partial t'}\nabla^2 w' = \nabla^4 w' + g\alpha\left(\frac{\partial^2\theta}{\partial x'^2} + \frac{\partial^2\theta}{\partial y'^2}\right), \tag{11}$$

after use is made of the vector identity

$$\nabla_\wedge\vec{\omega} = \nabla_\wedge\nabla_\wedge\vec{q}' = \nabla(\nabla\cdot\vec{q}') - \nabla^2\vec{q}',$$

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta, \qquad (12)$$

~~in which ζ is the z-component of the vorticity vector $\vec{\omega}$.~~

The problem now reduces to the solution of equations (9), (11) and (12).

The boundary conditions basically are

$$u' = 0 = v' = w' = \theta \quad \text{on} \quad z' = 0, d \qquad (13)$$

At a rigid surface, there is no slip and the condition $u' = 0 = v'$ correspond to

$$\frac{\partial w'}{\partial z'} = 0 \quad \text{on} \quad z' = 0, d. \qquad (14)$$

For a free surface, the shear stresses

$$\tau_{xz} = \mu\left(\frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'}\right), \quad \tau_{yz} = \mu\left(\frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'}\right)$$

demand that $\partial u'/\partial z'$ and $\partial v'/\partial z'$ be zero on $z' = 0, d$. Hence

$$\frac{\partial^2 w'}{\partial z'^2} = 0 \quad \text{on} \quad z' = 0, d \qquad (15).$$

For the vorticity, we have that

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Therefore

$$\zeta = 0 \quad \text{on} \quad z' = 0, d \quad \text{for rigid surfaces} \qquad (16a)$$

$$\frac{\partial \zeta}{\partial z'} = 0 \quad \text{on} \quad z' = 0, d \quad \text{for free surfaces}. \qquad (16b)$$

We now analyse disturbance into normal mode. Neglecting an uncoupled vorticity term, we write

$$w = W(z') \exp\left[i(k_x x' + k_y y') + pt'\right]$$

$$\theta = \Theta(z') \exp\left[i(k_x x' + k_y y') + pt'\right], \qquad (17)$$

then putting

$$k^2 = k_x^2 + k_y^2, \quad z = z'/d, \quad a = kd, \quad \sigma = pd^2/\nu, \quad D = d/dz \tag{18}$$

the governing equations reduce to

$$(D^2 - a^2)(D^2 - a^2 - \sigma)W = \left(\frac{g\alpha}{\nu} d^2\right) a^2 \Theta \tag{19}$$

$$(D^2 - a^2 - P_r\sigma)\Theta = -\left(\frac{\beta}{K} d^2\right) W \tag{20}$$

$$\Theta = 0 = W \quad \text{on} \quad z = 0, 1 \tag{21}$$

$$\left.\begin{aligned} \frac{dW}{dz} &= 0 \quad \text{on a rigid surface} \\ \frac{d^2W}{dz^2} &= 0 \quad \text{on a free surface} \end{aligned}\right\} \tag{22}$$

P_r is the Prandtl number,
a is the wave number vector and σ is the time constant.

Eliminating Θ between equations (19) and (20), we obtain

$$(D^2 - a^2)(D^2 - a^2 - \sigma)(D^2 - a^2 - P_r\sigma)W = -R a^2 W \tag{23}$$

$$R = \frac{g\alpha\beta}{K\nu} d^4 \quad \text{the Rayleigh number.}$$

For the problem at hand the principle of exchange of stabilities are valid, that is σ is real, and the marginal states are characterized by $\sigma = 0$. Hence the solution of

$$(D^2 - a^2)^3 W = -R a^2 W \tag{24}$$

is sought subject to conditions (21) and (22), and $(D^2 - a^2)^2 W = 0$ on $z = 0, 1$.

The simplest solution to this characteristic value problem corresponds to the free boundary — the Bénard problem. In this case the problem has the simple solution

$$W = A \sin n\pi z \quad (n = 1, 2, 3, \ldots)$$

such that

$$R = (n^2\pi^2 + a^2)^3 / a^2.$$

For given a^2 the minimum value of R occurs when $n=1$ and the critical Rayleigh number is determined from the condition

$$\frac{\partial R}{\partial a^2} = 0 \quad \text{or} \quad a^2 = \pi^2/2$$

and

$$R_c = \frac{27}{4}\pi^4 = 657.5 .$$

For the more realistic two rigid boundaries, equation (24) could be solved exactly for this simplest of stability problems. However we shall illustrate its solution by Chandrasekhar's variational method by expressing (24) in the form

$$\begin{aligned}(D^2-a^2)F &= -Ra^2W \\ F &= (D^2-a^2)^2W\end{aligned} \qquad (25)$$

and moving the origin so that the lower and upper walls are now confined between $z = \mp 1/2$, the boundary conditions are

$$F = 0 = W = DW \quad \text{for } z = \pm 1/2 . \qquad (26)$$

Now since the differential operator in (25) is even and the boundary conditions (26) are identical at the two boundaries, the problem is split into non-combining groups of even and odd parts. Thus for the even solution we set

$$\begin{aligned}F &= \sum_m A_m \cos[(2m+1)\pi z] \\ W &= \sum_m A_m W_m\end{aligned} \qquad (27)$$

therefore the boundary condition on F is identically satisfied and from the second equation in (26) we have

$$(D^2-a^2)^2 W_m = \cos[(2m+1)\pi z] \quad , \; W_m = 0 = DW_m \quad \text{on } z = \pm 1/2$$

with solution

$$\gamma_{2m+1} = \frac{1}{(2m+1)^2\pi^2 + a^2}$$

$$P_m = (-1)^{m+1} \frac{(2m+1)\pi\gamma^2_{2m+1}}{a + \sinh a} \sinh \frac{1}{2}a$$

$$Q_m = (-1)^m \frac{2(2m+1)\pi\gamma^2_{2m+1}}{a + \sinh a} \cosh \frac{1}{2}a .$$

Next substituting (27) and (28) into the first equation in (25) and using the orthogonality property of the cosine function, it is possible to obtain the infinite circular determinant

$$\Bigg\| \left(\frac{1}{\pi^2 R \gamma^2_{2m+1}} - \gamma^3_{2m+1} \right) \delta_{mn} +$$

$$+ (-1)^{m+n} 16a\pi^2 \frac{\cosh^2 \frac{1}{2}a}{\sinh a + a} (2n+1)(2m+1)\gamma^2_{2n+1}\gamma^2_{2m+1} \Bigg\| = 0 , \qquad (29)$$

The first term usually gives a good enough accuracy owing to the variational nature of the problem.

For more general flows involving rotation and magnetic effects, the principle of exchange of stabilities is not valid and the case $\sigma = 0$ corresponds to instability via stationary convection. It is also possible to have instability when σ is purely imaginary in which case instability is via overstability i.e. oscillations of increasing amplitude.

3. Stability of chemically reacting gas in a porous medium

As observed earlier instabilities may also arise as a result of adverse concentration gradients. We consider this effect in a porous sphere of radius r_o. For flows in porous media, the velocities are usually very small so that the nonlinear convection terms in (2) are negligible. The momentum equation could then be written in the form

$$0 = -\nabla(p - p_\infty) + \mu \nabla^2 \vec{q} - \frac{\mu}{K}\vec{q} - \rho\alpha\vec{g}(c - c_\infty) \qquad (30)$$

while for the mass transfer we write

$$\frac{\partial c}{\partial t} + \vec{q}\cdot\nabla c = D_m \nabla^2 c - k_p'^2 c - \epsilon' \qquad (31)$$

which is appropriate for a dilute mixture of binary fluid such that D_m is the mass diffusivity, $k_r'^2$ is the chemical rate constant, K is the permeability of the porous medium and ϵ' is the chemical source strength. In the spherical polar coordinate system (r', θ, φ) we assume that the gravitational field is radially symmetric ie $\vec{g} = g(r)\,\hat{r}$.

Now under equilibrium we assume that ϵ' maintains a radial concentration gradient and for the homogeneous sphere, g is constant and the equilibrium concentration satisfies the equation

$$\frac{d^2c}{dr'^2} + \frac{2}{r'}\frac{dc}{dr'} - k_r'^2 c = -\epsilon'/D_m \qquad (32)$$

If for simplicity we take $c = c_0$ on $r' = r_0$, then the solution for (32) becomes

$$c = \frac{\epsilon'}{D_m k_r'^2} + \left(c_0 - \frac{\epsilon'}{D_m k_r'^2}\right)\frac{i_0(k_r' r')}{i_0(k_r' r_0)} = \frac{\epsilon'}{D_m k_r'^2} + \left(c_0 - \frac{\epsilon'}{D_m k_r'^2}\right)\frac{r_0}{r'}\frac{\sinh(k_r' r')}{\sinh(k_r' r_0)}$$

Thus

$$\frac{dc}{dr'} = \left(c_0 - \frac{\epsilon'}{D_m k_r'}\right)\frac{k_r' i_{-1}(k_r' r')}{i_0(k_r' r_0)} = -c_0 r_0(1-\epsilon)\frac{k_r^2}{\sinh k_r}\left(\frac{\sinh k_r r}{k_r^2 r^2} - \frac{\cosh k_r r}{k_r r}\right)$$
$$= \frac{c_0}{r_0}\beta(r) \qquad (33)$$

where the unprimed terms in $\beta(r)$ are suitably non-dimensionalized.

$$i_n(z) = (\pi/2z)^{1/2}\, I_{n+\frac{1}{2}}(z)$$

is the modified spherical Bessel function of the first kind, in which I_n is the modified Bessel function of the first kind while k_n denotes the second kind. In the subsequent analysis that follows, $j_n(z)$ will denote the spherical Bessel function of the first kind.

On superimposing small disturbance on the basic or equilibrium flow, and following the usual stability analysis we can obtain the eigenvalue equations

$$\left(D_c - k_r^2 - S_c\sigma\right)C = -\left(\frac{c_0 r_0}{D_m}\right)\beta(r)\,W$$

$$D_l(D_l-\chi^2-\sigma)W = \frac{\alpha g}{\gamma} r_0^4\, l(l+1)C$$

$$r=1:\ W = 0 = \frac{dW}{dr} = C \qquad (34)$$

where

$$D_l = \frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2},\quad S_c = \nu/D_m,\quad \chi = r_0/K^{1/2}.$$

In equation (34), l is an integer, S_c and χ are the Schmidt number and the porosity parameter. W corresponds to the radial component of $r'\vec{q}$, C is concentration and we have discarded an uncoupled vorticity term. The parameter 'a' is replaced by l since in this problem we have expanded the perturbed quantities in the form

$$c' = C(r)\, Y_l^m(\theta,\varphi)\, e^{pt}$$

and so on. Y_l^m are the spherical harmonics.

We now consider (34) for the marginal state $\sigma=0$, which corresponds to instability via stationary convection. We set

$$C = \sum_j{}' A_j\, j_l(\alpha_{l,j} r),\quad W = \sum_j{}' A_j W_j \qquad (35)$$

such that $j_l(\alpha_{l,j}) = 0$. The $\alpha_{l,j}$ are also the zeros of the more familiar Bessel function $J_{l+1/2}(z)$. The boundary condition on C are automatically satisfied. We obtain the solution for W from the equation

$$D_l(D_l-\chi^2)W_j = \frac{\alpha g}{\gamma} r_0^4\, l(l+1)\, j_l(\alpha_{l,j} r),\quad W_j = 0 = \frac{dW_j}{dr}.$$

Hence

$$\frac{\quad}{\alpha_{l,j}^{3}(\alpha_{l,j}^{2}+\chi^{2})}\left[\;\cdots\;\right]$$

$$A_{l,j}=\frac{(l+1)\left(\frac{1}{\alpha_{l,j}}-1\right)i_l(\chi)j_l(\alpha_{l,j})-i_l(\chi)j_{l+1}(\alpha_{l,j})-\chi\, i_{l+1}(\chi)j_l(\alpha_{l,j})}{i_l(\chi)+\chi\, i_{l+1}(\chi)}$$

$$B_{l,j}=\frac{\left(l-\frac{l+1}{\alpha_{l,j}}\right)j_l(\alpha_{l,j})-j_{l+1}(\alpha_{l,j})}{i(\chi)+\chi\, i_{l+1}(\chi)} \qquad (36)$$

Next we substitute (35) and (36) in the first equation in (34), then employing the orthogonality relationship

$$\int_0^1 t^2 j_l(\alpha_m t)\, j_l(\alpha_n t)\, dt=\begin{cases}0\\ \frac{1}{2}\left\{\frac{[\alpha_n^{1/2} j_l(\alpha_n)]'}{\alpha_n^{1/2}}\right\}^2\end{cases}$$

such that a dash now denotes differentiation with respect to the argument of j_l we can deduce that

$$\frac{(\alpha_{l,k}+k_r^2)\left([\alpha_{l,k}^{1/2} j_l(\alpha_{l,k})]'\right)^2}{\alpha_{l,k}}A_k=R\,\chi\, l(l+1)\sum_j A_j\frac{(k|j)}{\alpha_{l,j}^2(\alpha_{l,j}^2+\chi^2)}$$

$$R=\frac{2c_0 r_0^5\alpha g}{D_m\gamma},\quad (k|j)=\int_0^1 r^2\beta(r)\, W_j(r)\, j_l(\alpha_{l,j}\, r)\, dr\,.$$

We can therefore obtain the infinite circular determinant

$$\left\|R\chi\, l(l+1)\frac{(k|j)}{\alpha_{l,j}^2(\alpha_{l,j}^2+\chi^2)}-\frac{(\alpha_{l,j}+k_r^2)\left([\alpha_{l,j}^{1/2} j_l(\alpha_{l,j})]'\right)^2}{\alpha_{l,j}}\delta_{jk}\right\|=0 \qquad (37)$$

for the Rayleigh-type stability parameter R. Details of computation by Bestman [3] shows that porosity enhances stability while chemical reaction provokes instability.

4. Couette flow with heating

The problem of instability provoked by an adverse angular momentum is amply illustrated in [1] for flow between two rotating cylinders. However in this section we shall briefly and temperature on the stability of flow in a porous medium between two cylinders in a radially symmetrical gravitational field. Let the cylinder radii be R_1 and $R_2 (>R_1)$ with temperatures T_1 and T_2 and angular velocities Ω_1 and Ω_2. In the cylindrical coordinate system (r', φ, z'), the gravitational field is given by

$$|\underset{\sim}{\vec{g}}(r')| = G\left(\frac{M'}{r'^2} + N'\right)$$

where G is the constant of gravitation and M' and N' are material constants. The equations for a highly permeable medium may then be taken as (1) – (3) with the permeability term incorporated as in (3a).

Stationary solutions to these equations exist in the form

$$u' = 0, \; v' = V'(r'), \; w' = 0, \; T = T^{(0)}(r'), \; p = P$$

where

$$\frac{d}{dr'}\left(\frac{P}{\rho}\right) = \frac{V'^2}{r'^2} - G\left(\frac{M'}{r'^2} + N'\right)$$

$$\left(\frac{d^2}{dr'^2} + \frac{1}{r'}\frac{d}{dr'} - \frac{1}{K}\right)V' = 0, \quad \frac{d^2 T^{(0)}}{dr'} + \frac{1}{r'}\frac{dT^{(0)}}{dr'} = 0$$

$$r' = R_1, R_2 : V' = \Omega_1 R_1, \Omega_2 R_2, \; T^{(0)} = T_1, T_2. \qquad (38)$$

Thus defining the non-dimensional parameters

$$r = r'/R_2, \; V = V'/\Omega_1 R_1, \; \theta^{(0)} = T^{(0)}/T_1$$

$$\eta = R_2/R_1, \; \mu = \Omega_2/\Omega_1, \; \theta_w = T_2/T_1, \; \chi = R_2/K^{1/2}$$

the solutions for the velocity and temperature in (3) could be put in the form

$$V = \frac{K_0(\chi) - (\mu/\eta)K_0(\eta\chi)}{I_0(\eta\chi)K_0(\chi) - I_0(\chi)K_0(\eta\chi)} I_0(\chi r) + \frac{(\mu/\eta)I_0(\eta\chi) - I_0(\chi)}{I_0(\eta\chi)K_0(\chi) - I_0(\chi)K_0(\eta\chi)} K_0(\chi r)$$

$$= A_0 I_0(\chi r) + B_0 K_0(\chi r)$$

$$\theta^{(0)} = \frac{\theta_w - 1}{\ln R_2 - \ln R_1}\ln r + \theta_w \equiv C_0 \ln r + \theta_w. \qquad (39)$$

we can deduce the equations for the axisymmetric disturbances as

$$\frac{\partial u'}{\partial r'} + \frac{1}{r'}u' + \frac{\partial w'}{\partial z'} = 0$$

$$\frac{\partial u'}{\partial t'} - 2\frac{V'}{r'}v' = -\frac{\partial p'}{\partial r'} + \nu\left(\nabla'^2 - \frac{1}{r'^2} - \frac{1}{K}\right)u' + \alpha g(r)T'$$

$$\frac{\partial v'}{\partial t'} + \left(\frac{dV'}{dr'} + \frac{V'}{r'}\right)u' = \nu\left(\nabla'^2 - \frac{1}{r'^2} - \frac{1}{K^2}\right)v'$$

$$\frac{\partial w'}{\partial t'} = -\frac{\partial p'}{\partial z'} + \nu\left(\nabla'^2 - \frac{1}{K}\right)w'$$

$$\frac{\partial \theta}{\partial t'} + \frac{dT^{(0)}}{dr'}u' = k\nabla'^2 T' \qquad (40)$$

where $\nabla'^2 = \partial^2/\partial r'^2 + \partial/r'\partial r' + \partial^2/\partial z'^2$. Analysing disturbances into normal modes by setting

$$u' = e^{pt'}u'(r')\cos kz', \quad v' = e^{pt'}v'(r')\cos kz'$$
$$w' = e^{pt'}w'(r')\sin kz', \quad T' = e^{pt'}T'(r')\cos kz', \quad p' = e^{pt'}p'(r')\cos kz'$$

and introducing the additional non-dimensional quantity

$$\hat{\imath} = \frac{\Omega_1 \Omega_2 R_0}{\nu^2}$$

(the rotation parameter), equations (40) may be reduced to

$$(DD_* - a^2 - \chi^2 - \sigma)(DD_* - a^2)u = 2\hat{\imath}a^2\frac{V}{r}\cdot v + \frac{GR_2^2 T_1\beta}{\nu}\cdot a^2\left(\frac{M}{r^2} + N\right)\theta$$

$$(DD_* - a^2 - \chi^2 - \sigma)v = \hat{\imath}(D_*V)u$$

$$(DD_* - a^2 - P_r\sigma)\theta = P_r\hat{\imath}\frac{d\theta^{(0)}}{dr}u$$

$$u = 0 = v = \theta = Du \quad \text{on} \quad r = \eta, 1 \qquad (41)$$

where M and N are non-dimensional material constants and

$$D = \frac{d}{dr}, \quad D_* = \frac{d}{dr} + \frac{1}{r}.$$

for the situation when $R_2 - R_1 << (R_2 + R_1)/2$ (the narrow gap problem). With

$$\zeta = \frac{r - \eta}{1 - \eta}$$

the approximate form of (41) is

$$\left(\frac{d^2}{d\zeta^2} - q^{*2} - \chi^2\right)\left(\frac{d^2}{d\zeta^2} - q^{*2}\right)u = \frac{2\tau^* q^{*2}}{\eta} V v + \frac{G\beta R_2^2 \bar{T}_1 q^{*2}}{\nu}\left(\frac{M}{\eta^2} + N\right)\theta$$

$$\left(\frac{d^2}{d\zeta^2} - q^{*2} - \chi^2\right)v = \tau^*\left(\frac{dV}{d\zeta}\right)u$$

$$\left(\frac{d^2}{d\zeta^2} - q^{*2}\right)\theta = \frac{P_r \tau^* C_o}{\eta}u$$

$$u = 0 = v = \theta = du/d\zeta \quad \text{on} \quad \zeta = 0, 1 \qquad (42)$$

and in this case

$$V = \cosh\chi^*\zeta + \left[(\mu/\eta) - \cosh\chi^*\right]\frac{\sinh\chi^*\zeta}{\sinh\chi^*} \qquad (43)$$

and the star is used to designate the corresponding unstarred quantity multiplied by $1-\eta$.

To solve (42), it is convenient to expand u in the form

$$u = \sum_{m=1}^{\infty} A_m C_m(\xi) + B_m S_m(\xi) \qquad (44)$$

where C_m and S_m are functions defined and tabulated in Harris and Reid [4] and Reid and Harris [5] and $\xi = \zeta - 1/2$. Further writing

$$v = \chi^*\tau^* \sum_{m=1}^{\infty} A_m f_m(\xi) + B_m g_m(\xi), \quad \theta = \frac{P_r \tau^* C_o}{\eta}\sum_{m=1}^{\infty} A_m h_m(\xi) + B_m l_m(\xi)$$

then

$$\left(\frac{d^2}{d\xi^2} - q^{*2} - \chi^{*2}\right)(f_m, g_m) = \left\{\sinh\chi^*(\xi - \tfrac{1}{2}) + \left[(\mu/\eta) - \cosh\chi^*\right]\frac{\cosh\chi^*(\xi - \frac{1}{2})}{\sinh\chi^*}\right\}(C_m, S_m)$$

$$\left(\frac{d^2}{d\xi^2} - q^{*2}\right)(h_m, l_m) = (C_m, S_m)$$

$$f_m(\pm\tfrac{1}{2}) = 0 = g_m(\pm\tfrac{1}{2}) = h_m(\pm\tfrac{1}{2}) = l_m(\pm\tfrac{1}{2}). \qquad (45)$$

be solved in closed form. Hence substituting (44) and (45) in the first equation in (42) leads to the results

$$\sum_{m=1}^{\infty} A_m\left[(\lambda_m^2-a^{*2}-\chi^{*2})(\lambda_m^2-a^{*2})C_m-R_r^*\chi^* a^{*2} V f_m - R_t^* a^{*2}\left(\frac{M}{\eta^2}+N\right)h_m\right] +$$

$$+\sum_{m=1}^{\infty} B_m\left[(\mu_m^2-a^{*2}-\chi^{*2})(\mu_m^2-a^{*2})S_m-R_r^*\chi^* a^{*2} V g_m - R_t a^{*2}\left(\frac{M}{\eta^2}+N\right)l_m\right] = 0 \quad (46)$$

in which

$$R_r^* = \frac{2\zeta^{*2}}{\eta}, \quad R_t = G P_r R_e^2 C_o T_1 \beta \zeta^*$$

are the rotation and free convection parameters respectively. Since A_m and B_m are independent each summation in (46) could be set equal to zero and on employing the orthogonality property of C_m and S_m we finally reduce the problem to the infinite circular determinants

$$\left\|(\lambda_m^2-a^{*2}-\chi^{*2})(\lambda_m^2-a^{*2})\delta_{mn}-R_r^*\chi^* a^{*2}(m|n)_f-R_t^* a^{*2}\left(\frac{M}{\eta^2}+N\right)(m|n)_h\right\| = 0$$

$$\left\|(\mu_m^2-a^{*2}-\chi^{*2})(\mu_m^2-a^{*2})\delta_{mn}-R_r^*\chi^* a^{*2}(m|n)_g-R_t^* a^{*2}\left(\frac{M}{\eta^2}+N\right)(m|n)_l\right\| = 0 \quad (47)$$

where the matrices

$$(m|n)_f = \int_{-1/2}^{1/2} f_m(\xi) V(\xi) C_n(\xi)\, d\xi, \quad (m|n)_g = \int_{-1/2}^{1/2} g_m(\xi) V(\xi) S_n(\xi)\, d\xi,$$

$$(m|n)_h = \int_{-1/2}^{1/2} h_m(\xi) C_n(\xi)\, d\xi, \quad (m|n)_l = \int_{-1/2}^{1/2} l_m(\xi) S_n(\xi)\, d\xi$$

could be evaluated in a close form.

The problem now reduces to the determination of R_r^* and R_t^* from (47). Details of the computation and extension to rotation for arbitrary gap are given in Bestman [6]. In that paper it is shown that, apart from the delay of instability by porosity, rotation and heating. And thermal instability is more fatal than rotational instability.

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