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WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

BOUNDARY-LAYER FLOWS

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Boundary-Layer Flows

Boundary layers/shear layers/vorticity layers:

Thin regions in high-Reynolds-number flows where viscous forces are important.

Governing Equations

$\hat{\underline{x}}$ position, $\hat{t}$ time

$\hat{\underline{u}}(\hat{\underline{x}},\hat{t})$ velocity vector

$\hat{\rho}(\hat{\underline{x}},\hat{t})$ density

$\hat{p}(\hat{x},\hat{t})$ pressure

Mass Conservation:

$$\frac{\partial\hat{\rho}}{\partial\hat{t}} + \hat{\nabla}.(\hat{\rho}\hat{\underline{u}}) = 0$$

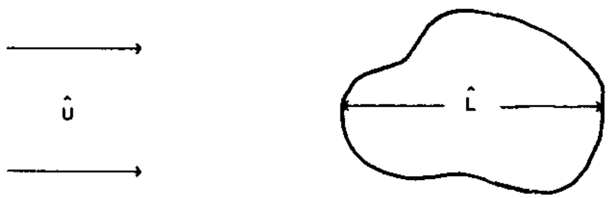
Assume incompressible $\Rightarrow$ $\hat{\nabla}.\hat{\underline{u}}=0$

Assume $\hat{\rho}$ uniform

Navier-Stokes equation:

$$\frac{\partial\hat{\underline{u}}}{\partial\hat{t}} + (\hat{\underline{u}}.\hat{\underline{\nabla}})\hat{\underline{u}} = -\frac{1}{\hat{\rho}}\hat{\underline{\nabla}}\hat{p} + \nu\hat{\nabla}^2\hat{\underline{u}} ,$$

where ν is the kinematic viscosity (assume constant).



Assume: typical velocity $\hat{U}$

typical lengthscale $\hat{L}$

Reynolds number: $R = \frac{\hat{U}\hat{L}}{\nu}$.

Nondimensionalise:

$$\underline{x} = \frac{\hat{\underline{x}}}{\hat{L}} , \qquad t = \frac{\hat{U}\hat{t}}{\hat{L}} , \qquad \underline{u} = \frac{\hat{\underline{u}}}{\hat{L}} , \qquad p = \frac{\hat{p}}{\hat{\rho}\hat{U}^2}$$

then

$$\nabla.\underline{u} = 0 , \qquad \frac{\partial\underline{u}}{\partial t} + (\underline{u}.\underline{\nabla})\underline{u} = -\underline{\nabla}p + \frac{1}{R}\nabla^2\underline{u} ,$$

plus boundary conditions, e.g. no-slip at surface of rigid body.

Vorticity equation:

$$\frac{\partial\underline{\omega}}{\partial t} + (\underline{u}.\underline{\nabla})\underline{\omega} = (\underline{\omega}.\underline{\nabla})\underline{u} + \frac{1}{R}\nabla^2\underline{\omega} , \qquad \underline{\omega} = \text{curl } \underline{u}$$

High Reynolds number $\Rightarrow$ $R \gg 1$.

Euler-limit

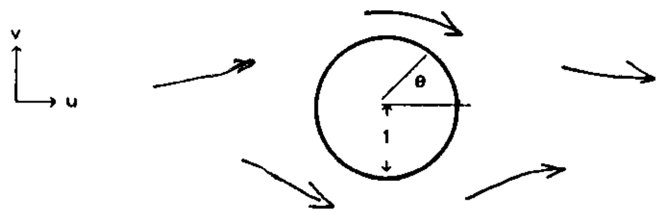
$$\underline{u} = \underline{u}_0 + \frac{1}{R^{1/2}}\underline{u}_{1/2} + \frac{1}{R}\underline{u}_1 + \dots , \qquad p = p_0 + \frac{1}{R^{1/2}}p_{1/2} + \frac{1}{R}p_1 + \dots .$$

$$\frac{\partial\underline{u}_0}{\partial t} + (\underline{u}_0.\underline{\nabla})\underline{u}_0 = -\underline{\nabla}p_0$$

↑ largest derivative term missing

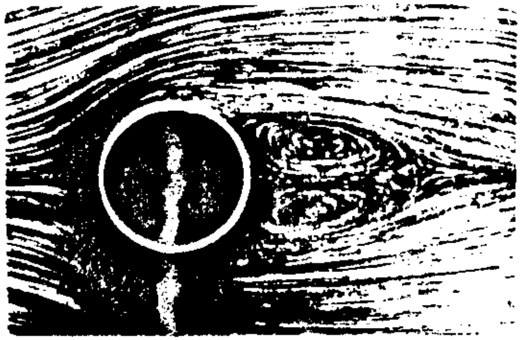
$\Rightarrow$ boundary conditions must simplify

Example: Steady irrotational flow past cylinder

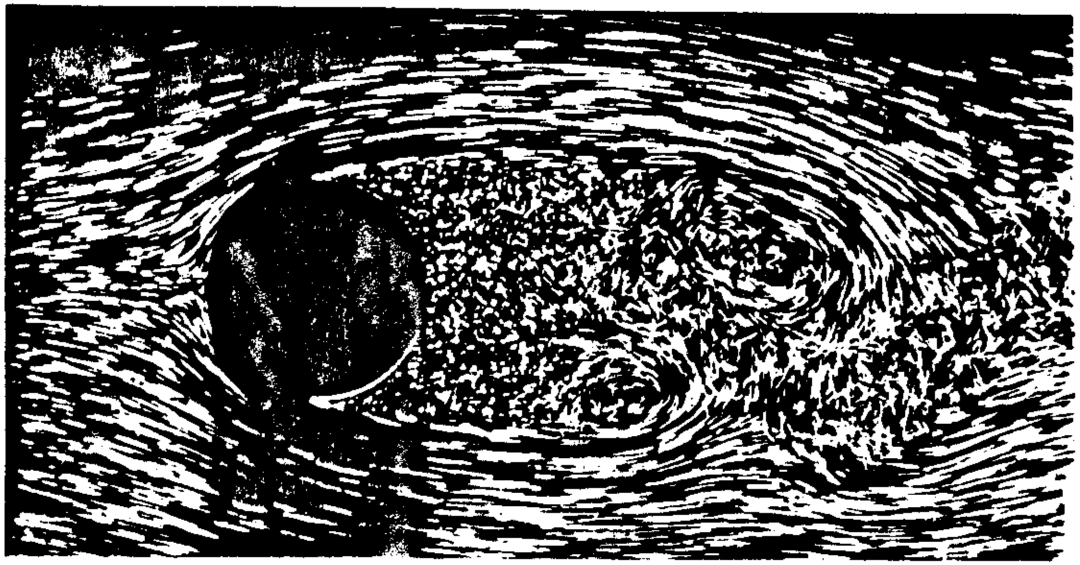


$$\underline{u}_0 = \underline{\nabla}\phi_0 , \qquad \nabla^2\phi_0 = 0$$

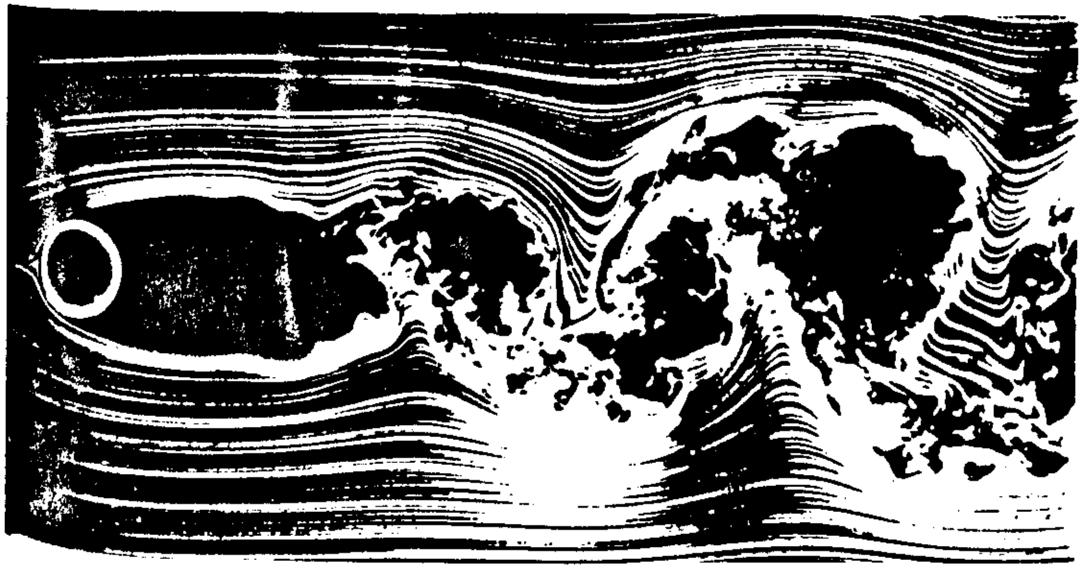
$$\underline{u}_0.\underline{n} = \frac{\partial\phi_0}{\partial n} = 0 \qquad \text{on } r = 1 :$$ zero normal velocity into cylinder



42. Circular cylinder at $R = 26$. The downstream distance to the cores of the eddies also increases linearly with Reynolds number. However, the lateral distance between the cores appears to grow more nearly as the square root. *Photograph by Sadatoshi Taneda*



47. Circular cylinder at $R = 2000$. At this Reynolds number one may properly speak of a boundary layer. It is laminar over the front, separates, and breaks up into a turbulent wake. The separation points, moving forward as the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. *ONERA photograph, Werlé & Gallon 1972*



48. Circular cylinder at $R = 10,000$. At five times the speed of the photograph at the top of the page, the flow pattern is scarcely changed. The drag coefficient consequently remains almost constant in the range of Reynolds number spanned by these two photographs. It drops later when, as in figure 57, the boundary layer becomes turbulent at separation. *Photograph by Thomas Corke and Hassan Nagib*

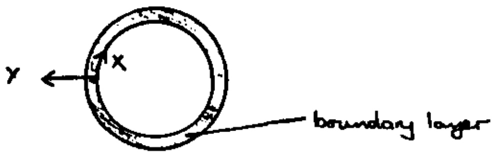
$$u_0 = \frac{\partial \phi_0}{\partial x} \to 1 \qquad \text{as } r \to 1 \quad : \quad \text{uniform stream at infinity}$$

$$\phi_0 = (r + \frac{1}{r})\cos\theta$$

$$\frac{1}{r}\frac{\partial \phi_0}{\partial \theta}\bigg|_{r=1} = -2\sin\theta \qquad : \qquad \underline{\text{slip}} \text{ velocity at surface}$$

↑ arises from missing highest derivative

Solution looks nothing like experiment (due to instability or something else?). For a viscous fluid thin 'boundary layers' exist close to the body, where a transition is made from the slip velocity to zero velocity.



Prandtl-limit

Introduce local coordinates where X axis is along boundary
Y axis is normal to boundary

Let $h = 1 + K(X)Y$, where K is curvature of boundary.

Let U, V be velocities in X, Y directions.

$$\nabla.\underline{u} = 0 \quad \Rightarrow \quad U = \psi_Y, \quad V = -\frac{1}{h}\psi_X \qquad : \quad \psi \text{ streamfunction}$$

Vorticity:

$$\underline{\omega} = (0,\ 0,\ \omega), \quad \omega = -\nabla^2\psi = -\frac{1}{h}\left[\frac{\partial}{\partial X}\left(\frac{1}{h}\frac{\partial\psi}{\partial X}\right) + \frac{\partial}{\partial Y}\left(h\frac{\partial\psi}{\partial Y}\right)\right]$$

2D Vorticity equation:

$$\frac{1}{h}(\psi_Y\omega_X - \psi_X\omega_Y) - \frac{1}{R}\nabla^2\omega = 0$$

By definition, viscous forces are important in boundary layers.

Let lengthscale in the X-direction be 1
Y-direction be $\delta \ll 1$ (boundary layers are thin)

In boundary layer

$$h = O(1)$$

$$U = \psi_Y = O(\frac{\psi}{\delta}) = O(1) \qquad \Rightarrow \qquad \psi = O(\delta)$$

$$\omega = O(\nabla^2\psi) = O(\psi,\ \frac{\psi}{\delta^2}) = O(\frac{1}{\delta})$$

$$\frac{1}{h}\psi_Y\omega_X = O(\frac{\psi\omega}{\delta}) = O(\frac{1}{\delta})$$

$$\frac{1}{h}\psi_X\omega_Y = O(\frac{\psi\omega}{\delta}) = O(\frac{1}{\delta})$$

$$\frac{1}{R}\nabla^2\omega = O\left[\frac{1}{R}\omega,\ \frac{1}{R}\frac{\omega}{\delta^2}\right] = O\left[\frac{1}{R\delta^3}\right]$$

Viscous forces comparable with inertia forces if

$$\frac{1}{\delta} = O\left[\frac{1}{R\delta^3}\right], \qquad \text{i.e.} \qquad \delta = O(R^{-1/2}).$$

Then $\quad V = O(\psi) = O(R^{-1/2})$.

Prandtl scaling:

$$Y = R^{-1/2}\bar{Y}$$

$$U = \bar{U}_0 + R^{-1/2}\bar{U}_1 + \dots$$

$$V = R^{-1/2}(\bar{V}_0 + R^{-1/2}\bar{V}_1 + \dots)$$

$$\psi = R^{-1/2}(\bar{\Psi}_0 + R^{-1/2}\bar{\Psi}_1 + \dots)$$

$$\omega = R^{-1/2}(\bar{\Omega}_0 + R^{-1/2}\bar{\Omega}_1 + \dots)$$

$$p = R^{-1/2}(\bar{P}_0 + R^{-1/2}\bar{P}_1 + \dots)$$

Substitute into Navier-Stokes equations

$$\bar{U}_{0X} + \bar{V}_{0\bar{Y}} = 0$$

$$\bar{U}_0\bar{U}_{0X} + \bar{V}_0\bar{U}_{0\bar{Y}} = -\bar{P}_{0X} + \bar{U}_{0\bar{Y}\bar{Y}}$$

$$0 = -\bar{P}_{0\bar{Y}}$$

Boundary conditions

$$\bar{U}_0 = \bar{V}_0 = 0 \qquad \text{on } \bar{Y} = 0$$

$$\bar{U}_0 \to u_0(X,\ 0) \qquad \text{as } \bar{Y} \to \infty \qquad (1)$$

↑ inviscid slip velocity

(1) comes from 'matching' the 'outer' Euler solution to the 'inner' Prandtl solution using either Van Dyke's matching principle or an intermediate variable approach.

Note (a) that the Reynolds number does not occur in the equations or the boundary conditions.

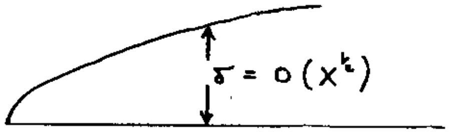
(b) from the continuity equation

$$\bar{V}_0 \to -u_{0X}(X,0)Y + \bar{V}_{0\infty}(X) + \ldots \quad \text{as } Y \to \infty .$$

Inviscid Contribution — Viscous Contribution

Flow past a semi-infinite flat plate (Blasius solution)

Boundary-layer solution. Drop $^-$ and $_0$.

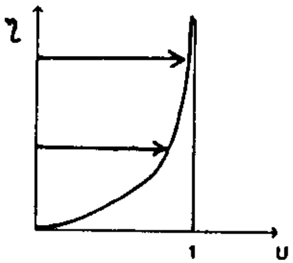


$$U = f'(\eta) , \qquad \eta = \frac{Y}{X^{1/2}}$$

$$V = \frac{1}{2X^{1/2}} (\eta f' - f)$$

$$2f''' + ff'' = 0$$

$$f(0) = f'(0) = 0 , \qquad f'(\eta) \to 1 \quad \text{as} \quad \eta \to \infty .$$



Blasius Boundary-Layer Solution.

Flow past a circular cylinder

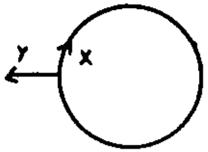
$$U_X + V_Y = 0 , \qquad UU_X + VU_Y = -P_X + U_{YY} \qquad (2)$$

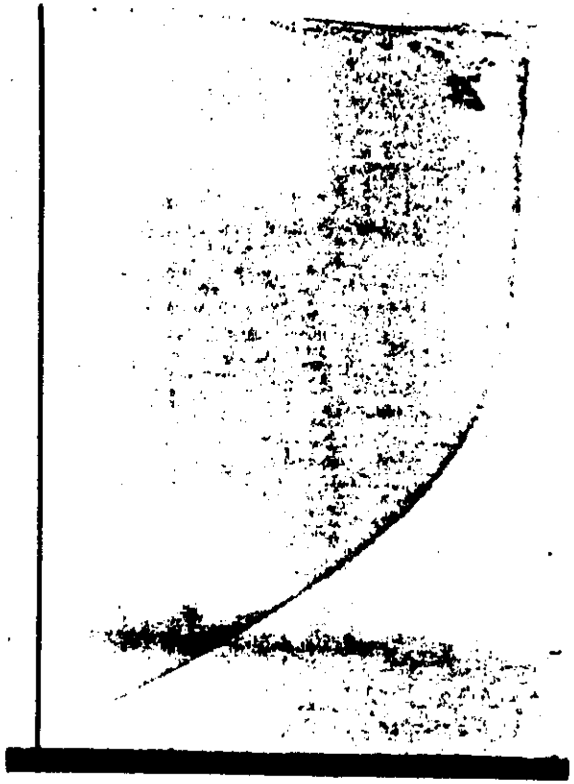
$$U = V = 0 \qquad \text{on } Y = 0$$

$$U \to u = 2\sin X \qquad \text{as } Y \to \infty .$$

From (2) evaluated as $Y \to \infty$, $-P_X = uu_X = 2\sin 2X$.

(2) is a parabolic equation: if solution known at X_1 then the solution can be calculated at





30. Blasius boundary-layer profile on a flat plate. The tangential velocity profile in the laminar boundary layer on a flat plate, discovered by Prandtl and calculated accurately by Blasius, is made visible by tellurium. Water is flowing at 9 cm/s. The Reynolds number is 500 based on distance from the leading edge, and the displacement thickness is about 5 mm. A fine tellurium wire perpendicular to the plate at the left is subjected to an electrical impulse of a few milliseconds duration. A chemical reaction produces a slender colloidal cloud, which drifts with the stream and is photographed a moment later to define the velocity profile. *Photograph by F. X. Wortmann*

$X_2 > X_1$ if $U > 0\ \forall\ Y$.

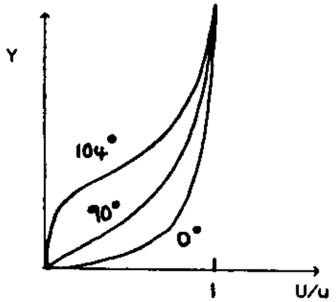
At $X = 0$, solution is stagnation point solution, i.e.

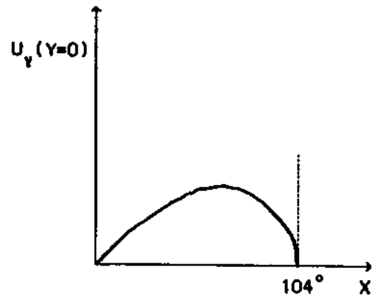
$$U = Xf'(Y) + \dots$$
$$V = -f(Y) + \dots$$

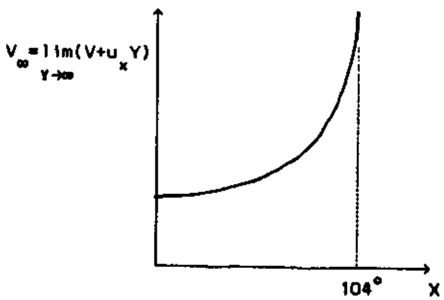
$$f'^2 - ff'' = -4 + f''', \qquad f = f' = 0 \text{ on } Y = 0, \qquad f' \to 2 \text{ as } Y \to \infty.$$

Numerical solution can be obtained by standard finite-difference techniques (e.g. Keller box, Crank-Nicholson, etc..)

Schematic results:







For $X < X_s \approx 104.5°$ a numerical solution can be found. For $0 < X_s - X \ll 1$,

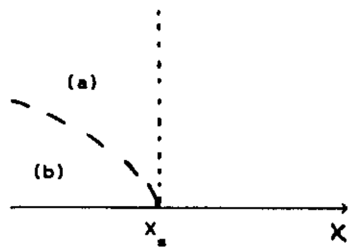
$$\tau = U_Y(Y=0) \sim c_1(X_s - X)^{1/2}, \qquad V_\infty \sim c_2(X_s - X)^{-1/2}.$$

Hence at $X = X_s$ fluid is being ejected from the boundary layer at ∞ velocity. Unrealistic. No solution for $X > X_s$ (Stewartson 1970). Hypothesised irrotational flow is incorrect.

Goldstein singularity (Goldstein 1948, Stewartson 1958)

Singularity seems to develop at point X_s where $U_Y(X_s, 0) = 0$, i.e.

$$U_Y(X_s, Y) \sim \tfrac{1}{2}U_{YY}(X_s, 0)Y^2 + \dots \qquad \text{for } 0 \le Y \ll 1.$$



Without loss of generality let

$$X_s = 0$$
$$U_{YY}(0,0) = 1$$

Hypothesise that boundary layer splits into two regions:

(a) an inviscid one

(b) a viscous one valid for $Y \ll 1$.

In viscous region

$$UU_X = O\left[\frac{Y^2}{X}U\right], \qquad U_{YY} = O\left[\frac{U}{Y^2}\right].$$

Hence terms balance where $\quad Y = O(|X|^{1/4})$.

Viscous Region $\quad Y = O(|X|^{1/4})$

Let $\xi = (-X)^{1/4}, \quad Y = \xi\eta$.

Substitute into the boundary-layer equations, expand in powers of ξ, find

$$U = \xi^2\left[\tfrac{1}{2}\eta^2 + \xi(2a_1\eta) + \xi^2\left(2a_2\eta - \frac{a_1^2}{12}\eta^4\right) + \dots\right]$$

where a_1 is arbitrary, a_2 is a function a_1, ...

$$U_Y(Y=0) \sim 2a_1\xi^2 = 2a_1(-X)^{1/2} \quad : \quad a_1 > 0, \text{ singular at } X = 0.$$

Inviscid Region $Y = O(1)$

$$U = U_0(Y) + 0 + 2\xi^2 a_1 U_0' + \dots$$

$$V = \qquad + \frac{a_1}{2\xi^2}U_0 \quad + \dots$$

$$V_\infty \sim \frac{a_1 U_0(\infty)}{2\xi^2} = \frac{a_1 U_0(\infty)}{2(-X)^{1/2}} \quad : \quad \text{singular at } X = 0$$

Summary

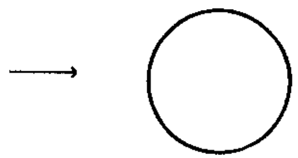
(a) Boundary-layer solutions OK if $U > 0$.

(b) 'Classical' boundary-layer solutions terminate in a singularity if $U = 0$ for some Y.

(c) This is a true singularity, and indicates that the assumed irrotational flow is wrong. Reassuring since irrotational solution does not look like experiment.

Unsteady boundary-layer separation

Is the irrotational solution for flow past a cylinder ever valid?
Yes at small times for impulsively started flow:



$\underline{u} = 0$ for $t < 0$

$\underline{u} \to (1,0)$ as $r \to \infty$, for $t > 0+$.

Unsteady boundary-layer problem:

$$U_t + UU_X + VU_Y = -P_X + U_{YY}, \qquad U_X + V_Y = 0$$

↑ new unsteady term

$U = V = 0$ for $t < 0$

$U = V = 0$ on $Y = 0$,

$U \to u = 2\sin X$ as $Y \to \infty$ for $t > 0$.

$P_X = -2\sin 2X$

Finite-difference solutions, series solutions, etc. can be found until the

cylinder has moved about 3/4 diameter. Is breakdown due to singularity?

Lagrangian Coordinates. Shen (1978), van Dommelen & Shen (1980)

Introduce Lagrangian coordinates (ξ, η) to label fluid particles.

Let $X = \xi, \quad Y = \eta \quad$ at $t = 0$.

$$U = \left.\frac{\partial X}{\partial t}\right|_{\xi,\eta}, \qquad V = \left.\frac{\partial Y}{\partial t}\right|_{\xi,\eta}$$

$$\frac{DU}{Dt} = U_t + UU_X + VU_Y = \left.\frac{\partial U}{\partial t}\right|_{\xi,\eta}$$

Eulerian derivatives — Lagrangian derivative

Conservation of Mass (e.g. see Lamb 1945)

$$J = \frac{\partial(X, Y)}{\partial(\xi, \eta)} = X_{,\xi}Y_{,\eta} - X_{,\eta}Y_{,\xi} = 1$$

↑ comma indicates a Lagrangian derivative

Transformation Formulae

$$Y_{,\eta} = J\xi_X = \xi_X, \qquad Y_{,\xi} = -\eta_X$$

$$X_{,\xi} = \eta_Y, \qquad X_{,\eta} = -\xi_Y$$

Momentum Equation

$$U_{,t} = -P_X(X) + \left[\eta_Y\frac{\partial}{\partial\eta} + \xi_Y\frac{\partial}{\partial\xi}\right]\left[\eta_Y\frac{\partial}{\partial\eta} + \xi_Y\frac{\partial}{\partial\xi}\right]U$$

Formulation

$$X_{,t} = U$$

$$U_{,t} = -P_X(X) + \left[X_{,\xi}\frac{\partial}{\partial\eta} - X_{,\eta}\frac{\partial}{\partial\xi}\right]\left[X_{,\xi}\frac{\partial}{\partial\eta} - X_{,\eta}\frac{\partial}{\partial\xi}\right]U$$

$U = 0, \quad X = \xi \quad$ on $\eta = 0$.

$U = 0, \quad X = \xi \quad$ on $\xi = 0, \pi$.

$U \to 2\sin X, \quad X_{,\eta} \to 0 \quad$ as $\eta \to \infty$.

This part of formulation is self contained and does not include Y. Can solve for X, and then obtain Y by solving

$J = X_{,\xi}Y_{,\eta} - X_{,\eta}Y_{,\xi} = 1$: hyperbolic first order equation.

This has a solution if $X_{,\xi} \neq 0$ and $X_{,\eta} \neq 0$.

Van Dommelen & Shen's Hypothesis

Solution for $X(\xi, \eta, t)$ is regular, but a singularity develops at the first time, t_s, at which

$$X_{,\xi} = X_{,\eta} = 0 \qquad \text{for some } \xi = \xi_s,\ \eta = \eta_s,$$

i.e. a singularity develops if at some point

$$\text{grad } X = 0$$

Numerical integration can accurately pinpoint x_s and t_s because $x(\xi, \eta, t)$ remains regular. Calculation of Y for $0 < t_s - t \ll 1$ is harder since Y is singular at $t = t_s$.

Numerical Results for Circular Cylinder Problem (van Dommelen & Shen 1980, Cowley 1983)

At $t_s \approx 3.0$

$\xi_s \approx 1.57$, $\eta_s \approx 0.51$

$x_s \approx 1.98$ radians $\approx 111^\circ$

Displacement thickness

$$\delta = \int_0^\infty \left(1 - \frac{U}{u}\right) dY$$

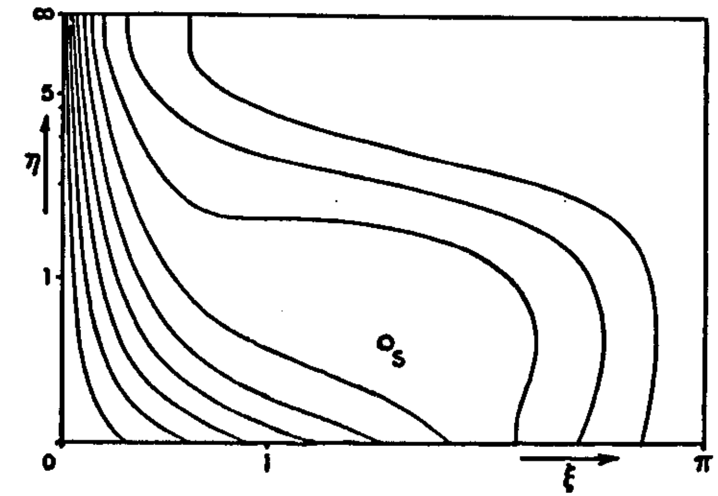
found to go infinite.

Physical Interpretation

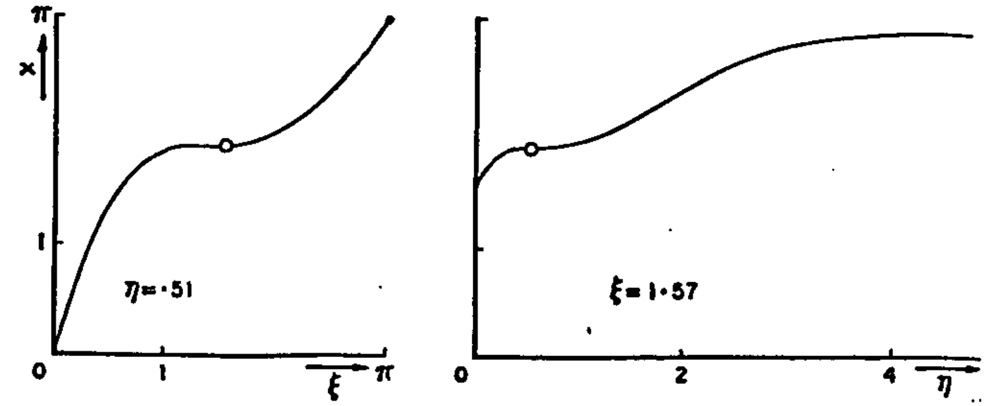
At t = 0 mark out a small 'cube' round separation particle $\xi = \xi_s$, $\eta = \eta_s$.

At $t = t_s$, $X_{,\xi} = X_{,\eta} = 0$ at $\xi = \xi_s$, $\eta = \eta_s$

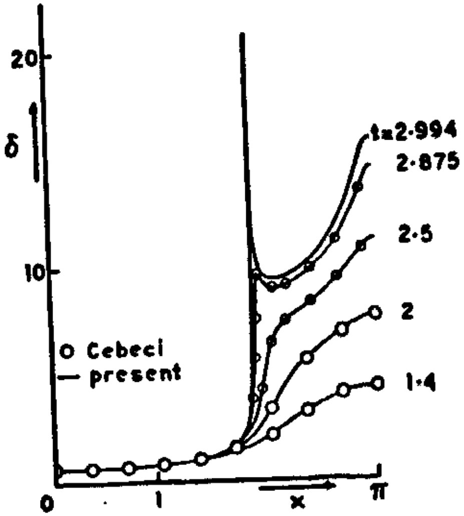
- i.e. rate of change of position X with particles ξ, η is zero
- i.e. more than one fluid particle is trying to occupy the same x position.



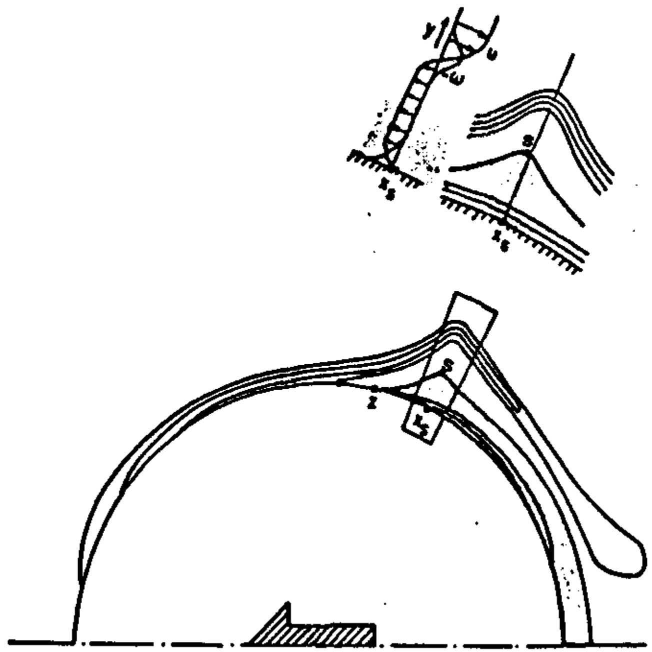
Lines of constant streamwise position in the Lagrangian plane at time t = 3



Lagrangian x-profiles through the stationary point at time t = 3



Evolution of the displacement thickness of the boundary layer about the impulsively started circular cylinder into a singularity at time t_S.

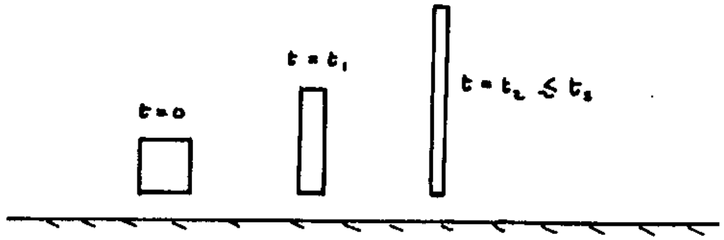


Lines of constant vorticity (Schematic)

ω = Vorticity

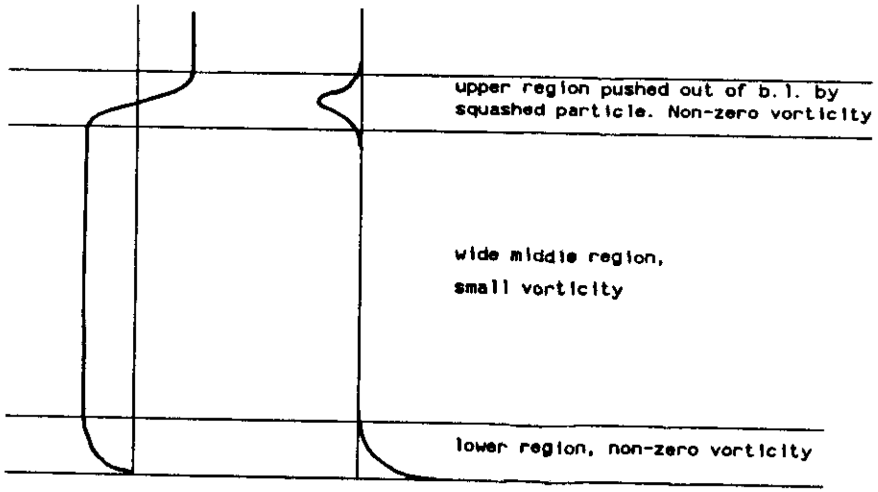
x_S = Separation Point.

Fluid particle has got squashed parallel to boundary. Since incompressible must expand away from boundary:



At $t = t_s$, particle has infinite extent, and has pushed flow above it out of the boundary layer: UNSTEADY SEPARATION.

Structure of boundary layer:



velocity profile vorticity

Boundary-layer theory based on $R \gg 1$. Experiments performed at finite R. For $R = 9500$, after 3/4 diameter movement the boundary layer shows a rapid thickening, which is in qualitative agreement with theory.

Singularity Structure

Without loss of generality assume $x_s = \xi_s = \eta_s = t_s = 0.$

Journal of Fluid Mechanics, Vol. 101, part 3

FIGURE 6 (*a*). For legend see plate 8.

Expand X as Taylor series about separation time and position (0, 0, 0).

Since $X_{,\xi}(0,0,0) = X_{,\eta}(0,0,0) = 0$

$$X = \frac{1}{2}\xi^2 X_{,\xi\xi}(0,0,0) + \xi\eta X_{,\xi\eta}(0,0,0) + \frac{1}{2}\eta^2 X_{,\eta\eta}(0,0,0)$$
$$+ t\left[X_{,t}(0,0,0) + \xi X_{,t\xi}(0,0,0) + \eta X_{,t\eta}(0,0,0) \right]$$
$$+ \dots$$

Write $X_{,\xi\xi}(0,0,0) = X_{0\xi\xi}$, etc., and rotate ξ, η coordinates so that $X_{0\xi\eta} = 0$. Then $X_{,\xi} = X_{,\eta} = 0$ at

$$\xi \approx tX_{0t\xi}/X_{0\xi\xi}$$

$$\eta \approx tX_{0t\eta}/X_{0\eta\eta}$$

BUT, $t = 0$ must be the first time that $X_{,\xi} = X_{,\eta} = 0$, hence

$$X_{0\xi\xi} = 0 \qquad \text{or} \qquad X_{0\eta\eta} = 0 .$$

Without loss of generality, $X_{0\eta\eta} = 0$. So

$$X = \frac{1}{2}\xi^2 X_{0\xi\xi} + \frac{1}{6}\eta^3 X_{0\eta\eta\eta} + \frac{1}{2}\eta^2\xi X_{0\eta\eta\xi} + \frac{1}{2}\eta\xi^2 X_{0\eta\xi\xi} + \frac{1}{6}\xi^3 X_{0\xi\xi} + \dots$$
$$+ t\left[X_{0t} + \xi X_{0t\xi} + \eta X_{0t\eta} + \dots \right] + \dots \tag{3}$$

Can now show that $t = 0$ is the first time that a singularity forms if

$$X_{0\eta t}X_{0\eta\eta\eta} < 0 .$$

To examine structure of singularity, note that the characteristics of

$$X_{,\xi}Y_{,\eta} - X_{,\eta}Y_{,\xi} = 1 \tag{4}$$

are lines of constant X. To obtain full structure of singularity at any fixed time require both ξ and η to vary on lines of constant X. From (3) this suggests

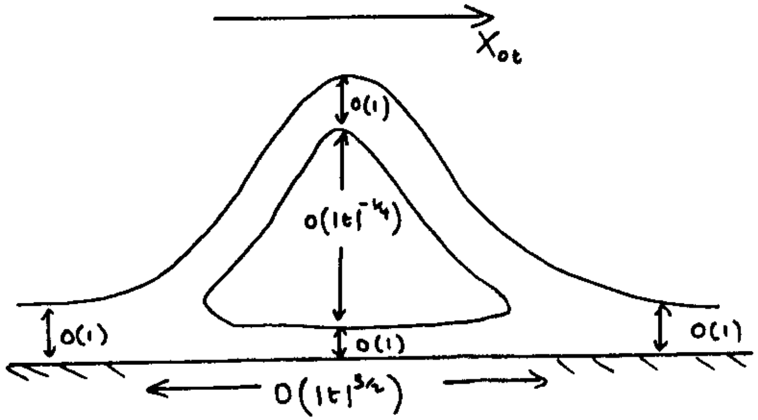
$$X^* = (X - tX_{0t}) = O(\xi^2) = O(\eta^3) = O(t\eta)$$

i.e. $\eta = O(|t|^{1/2})$, $\xi = O(|t|^{3/4})$ $X^* = O(|t|^{3/2})$.

Hence from (4)

$$Y = O(|t|^{-1/4})$$

Precise analytic structure can be deduced in terms of elliptic functions, by integrating (4). A schematic of the singularity is:

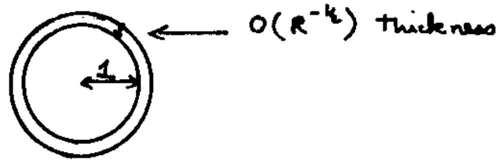


Interactive Effect

As $t \to t_s = 0$, the thickness of the boundary layer becomes infinite. This is physically unrealistic. Clearly the asymptotic scaling must become invalid when

$$\delta = O(R^{1/2}),$$

if not before, since the boundary layer is then the same size as the cylinder.



However, the classical boundary-layer solution becomes invalid before that, due to a pressure effect often known as a 'triple-deck' interaction. This interaction effectively allows the pressure to vary in y, and in some respects acts like a 'lid'.