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WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

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INTERNAL GRAVITY WAVES

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LECTURE NOTES ON INTERNAL GRAVITY WAVES

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Stratified Flows

Some general reading: C.-S. Yih, *Stratified Flows*, Academic press, 1980.

Equations of motion of an ideal fluid:

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\hat{\underline{k}} \qquad (1) \qquad \underline{u} = (u,v,w) = \text{velocity}$$

$$\frac{D\rho}{Dt} + \rho\nabla.\underline{u} = 0 \qquad (2) \qquad p = \text{pressure}, \quad \rho = \text{density}$$

$$\frac{Dp}{Dt} = c_s^2\frac{D\rho}{Dt} \qquad (3) \qquad c_s = \text{speed of sound} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S}$$

(1) are Euler's equations of motion, (2) is the continuity equation and (3) is the thermodynamic energy equation. (3) arises because for in ideal fluid the entropy, S is conserved moving with the flow, i.e. $DS/Dt = 0$ and so

$$\frac{Dp}{Dt} = \left(\frac{\partial p}{\partial \rho}\right)_S\frac{D\rho}{Dt} + \left(\frac{\partial p}{\partial S}\right)_\rho\frac{DS}{Dt} = c_s^2\frac{D\rho}{Dt} \quad \text{since} \quad p = p(\rho,S).$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad = \text{rate of change moving with the flow.}$$

Stratified flows are ones in which the density varies even in the absence of any flow. Usually most of the variation takes place in the vertical direction. There are two approaches to modelling: one approximates the full equations above but retains an assumption of continuous variation of ρ with z. The other approach assumes distinct layers with constant ρ within layers.

Boussinesq Approximation

For continuously stratified flows analysis is made more tractable by using the Boussinesq approximation. Amongst other things, this eliminates sound wave solutions from the equations. Assume a *basic state* with horizontal flow: $\underline{u} = (u_o,v_o,0)$. Vertical component of (1) is (subscript o denotes basic state

$$0 = -\frac{1}{\rho_o}\frac{\partial p_o}{\partial z} - \rho_o g \quad \text{and so} \quad \frac{\partial p_o}{\partial z} = -\rho_o g. \quad \text{This is the } \textit{hydrostatic equation.}$$

In the Boussinesq approximation we assume $p = p_o(z) + p'$, $\rho = \rho_o(z) + \rho'$ and that $|p'| \ll p_o$ and $|\rho'| \ll \rho_o$. Eq. (1) becomes

$$(\rho_o+\rho')\frac{D\underline{u}}{Dt} = -\nabla(p_o+p') - (\rho_o+\rho')g\hat{\underline{k}} = \left(-\frac{dp_o}{dz} - \rho_o g\right)\hat{\underline{k}} - \nabla p' - \rho' g\hat{\underline{k}}.$$

Now let $\rho_o + \rho' \approx \rho_o$ giving

$$\boxed{\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_o}\nabla p' - \frac{\rho'}{\rho_o}g\hat{\underline{k}}}$$

Eq. (2): $\frac{D}{Dt}(\rho_o+\rho') + (\rho_o+\rho')\nabla.\underline{u} = 0$ and thus $w\frac{d\rho_o}{dz} + \frac{D\rho'}{Dt} + (\rho_o+\rho')\nabla.\underline{u} = 0.$

Again neglecting ρ' compared to ρ_o we get

$$\boxed{w\frac{d\rho_o}{dz} + \rho_o\nabla.\underline{u} = 0}$$

We often write $-\frac{1}{\rho_o}\frac{d\rho_o}{dz} = \frac{1}{H_\rho}$ where H_ρ is the *density scale height*. Thus the continuity equation can be written as $\nabla.\underline{u} - \frac{w}{H_\rho} = 0$. If we assume horizontal and vertical length scales L and H and horizontal and vertical velocity scales U and UH/L, the non-dimensional version of this equation is $\nabla.\tilde{\underline{u}} - \frac{H}{H_\rho}\tilde{w} = 0$. For many flows $H \ll H_\rho$ (e.g. bucket of water has $H_\rho \sim 100$km) and so the continuity equation may be approximated by $\boxed{\nabla.\underline{u} = 0}$

Eq. (3) is $\frac{D}{Dt}(p_o+p') = c_s^2\frac{D}{Dt}(\rho_o+\rho')$ which, after using the hydrostatic equation becomes $-\rho_o g w + \frac{Dp'}{Dt} = c_s^2 w\frac{d\rho_o}{dz} + c_s^2\frac{D\rho'}{Dt}$. Rather surprisingly, it is consistent to neglect p' but not ρ'. This is because, from the vertical equation of motion, we expect $p'/H \sim \rho' g$ and so $p'/c_s^2\rho' \sim Hg/c_s^2 \ll 1$ for most flows. Then $\frac{D\rho'}{Dt} + w\frac{d\rho_o}{dz} = -\frac{\rho_o g w}{c_s^2}$. Using the hydrostatic equation, this is more usefully written as

$$\boxed{\frac{D}{Dt}\left[-\frac{\rho' g}{\rho_o}\right] = -\left[-\frac{g}{\rho_o}\frac{d\rho_o}{dz} - \frac{g^2}{c_s^2}\right]w.}$$

$b = -\frac{\rho' g}{\rho_o}$ is the *buoyancy acceleration*. $N^2 = -\frac{g}{\rho_o}\frac{d\rho_o}{dz} - \frac{g^2}{c_s^2}$ where N is the Brunt-Väisälä frequency. This is a natural frequency of oscillation in a stratified flow (more later). Thus our approximate equations are

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho_o}\nabla p' + b\hat{\underline{k}} \qquad (4)$$

$$\nabla.\underline{u} = 0 \qquad (5)$$

$$\frac{Db}{Dt} + N^2 w = 0 \qquad (6)$$

As we saw above, there is more than one level of approximation possible. Some appropriate references are:

1) E.A. Spiegel & G. Veronis, On the Boussinesq approximation for a compressible fluid, *Astrophysical Journal*, 131, 442-447 (1960). This describes a quite basic version of the Boussinesq approximation.
2) J.A. Dutton & G.H. Fichtl, Approximate equations of motion for gases and liquids, *Journal of the Atmospheric Sciences*, 26, 241-254 (1969). This describes several levels of approximation.
3) L. Mahrt, On the shallow water approximations, *Journal of the Atmospheric Sciences*, 43, 1036-1044 (1986). This continues the discussion begun by Dutton & Fichtl.

We note in particular a set of equation derived by Dutton & Fichtl which allow $H \sim H_\rho$. In this approximation (which Dutton & Fichtl call the *deep convection approximation*) Eq. (4)-(6) are replaced by

$$\frac{D\underline{u}}{Dt} = -\nabla\left(\frac{p'}{\rho_o}\right) + b\hat{\underline{k}} \qquad (7)$$

$$\nabla.\underline{u} + \frac{1}{\rho_o}\frac{d\rho_o}{dz}w = 0 \qquad (8)$$

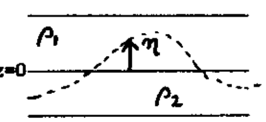
$$\frac{Db}{Dt} + N^2 w = 0 \qquad (9) \qquad \text{where } b = -\frac{\rho' g}{\rho_o} + \frac{p'}{\rho_o^2}\frac{d\rho_o}{dz}$$

A quite different way of applying a "Boussinesq" type approximation is

described later under the heading Taylor-Goldstein Equation.

Layer Models

Here we have a number of discrete layers in which the density is constant. No Boussinesq approximation is necessary since each layer is homogeneous but boundary conditions must be applied on each interface. There are 2 boundary conditions:



Kinematic Boundary Condition: $\underline{u}_1.\underline{n} = \underline{u}_2.\underline{n}$ on $z=\eta$. It is often more useful to express this as $\eta_1 = \eta_2$ where $D\eta/Dt = w$.

Dynamic Boundary Condition: $p_1 = p_2$ on $z=\eta$.

Within each layer there is no buoyancy and so we solve $\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p'$.

A good reference giving many different variations of the stratified layer model is E.E. Gossard & W.H. Hooke, *Waves in the Atmosphere*, Elsevier, 1975.

The Taylor-Goldstein Equation

Suppose we take a basic state $\underline{u}_o = (u_o(z),0,0)$ and linearise about this for two dimensional disturbances, i.e. perturbations only in the x and z directions. Then equations (4), (5) and (6) become

$$\frac{\partial u'}{\partial t} + u_o\frac{\partial u'}{\partial x} + w'\frac{du_o}{dz} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial x} \qquad (10)$$

$$\frac{\partial w'}{\partial t} + u_o\frac{\partial w'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b' \qquad (11)$$

$$\frac{\partial b'}{\partial t} + u_o\frac{\partial b'}{\partial x} + N^2 w' = 0 \qquad (12)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad (13)$$

The aim is to obtain a solution which is periodic in x and t. It turns out to be convenient to look for solutions of the form

$$u' = \rho_o^{-1/2}\hat{u}(z)e^{i(kx-\omega t)}$$

$$w' = \rho_o^{-1/2}\hat{w}(z)e^{i(kx-\omega t)}$$

$$b' = \rho_o^{-1/2}\hat{b}(z)e^{i(kx-\omega t)} \quad \text{and} \quad p' = \rho_o^{1/2}\hat{p}(z)e^{i(kx-\omega t)}$$

where in fact the real part is implied. k is the horizontal wavenumber and ω is the angular frequency (which may be complex). Then Eq. (10)-(13) become

$$-i\hat{\omega}\hat{u} + \hat{w}\frac{du_o}{dz} = -ik\hat{p} \qquad (14) \qquad\qquad -i\hat{\omega}\hat{b} + N^2\hat{w} = 0 \qquad (16)$$

$$-i\hat{\omega}\hat{w} = -\frac{d\hat{p}}{dz} - \frac{1}{2\rho_o}\frac{d\rho_o}{dz}\hat{p} + \hat{b} \qquad (15) \qquad\qquad ik\hat{u} + \frac{d\hat{w}}{dz} - \frac{1}{2\rho_o}\frac{d\rho_o}{dz}\hat{w} = 0 \qquad (17)$$

In these equations $\hat{\omega} = \omega - u_o k$ which is the angular frequency measured by an observer moving with the mean flow. This is usually called the *intrinsic frequency*. We reduce Eq. (14)-(17) to a single equation for $\hat{w}$ as follows. Substitute for $\hat{u}$ from (17) into (14) and hence from (14) get an expression for $\hat{p}$ in terms of $\hat{w}$. From (16) obtain an expression for $\hat{b}$ in terms of $\hat{w}$. Hence substitute for $\hat{p}$ and $\hat{b}$ in Eq. (15) giving

$$\frac{d^2\hat{w}}{dz^2} + \lambda(z)\hat{w} = 0 \qquad (18)$$

$$\text{where } \lambda = \frac{N^2k^2}{\hat{\omega}^2} - k^2 + \frac{k}{\hat{\omega}}\frac{d^2u_o}{dz^2} - \frac{1}{2\rho_o}\frac{d^2\rho_o}{dz^2} + \frac{1}{4\rho_o^2}\left(\frac{d\rho_o}{dz}\right)^2 + \frac{k}{\hat{\omega}}\frac{1}{\rho_o}\frac{d\rho_o}{dz}\frac{du_o}{dz} \qquad (19)$$

Eq. (18) is the Taylor-Goldstein equation. We will use this later, but it is interesting to note how the result would differ if we were to use different versions of the Boussinesq approximation. Eq. (18) came from the "shallow" version which assumes $H \ll H_\rho$. If instead we use Dutton & Fichtl's "deep" version (Eq. 7-9), we get

$$\frac{d^2\hat{w}}{dz^2} + \lambda(z)\hat{w} = 0 \tag{20}$$

where now $\lambda = \dfrac{N^2k^2}{\hat{\omega}^2} - k^2 + \dfrac{k}{\hat{\omega}}\dfrac{d^2u_o}{dz^2} + \dfrac{1}{2\rho_o}\dfrac{d^2\rho_o}{dz^2} - \dfrac{3}{4\rho_o^2}\left(\dfrac{d\rho_o}{dz}\right)^2 - \dfrac{k}{\hat{\omega}}\dfrac{1}{\rho_o}\dfrac{d\rho_o}{dz}\dfrac{du_o}{dz}$ (21)

Note that the 4th, 5th and 6th terms have changed. The most important difference is the 6th term, since the 4th and 5th are $O(1/H_\rho)^2$ whereas the 6th is $O(1/H_\rho)$. We might expect that (21) is more accurate that (19), since the deep version of the Boussinesq approximation retains terms of $O(1/H_\rho)$ in the continuity equation whereas the shallow version does not. However, there is an alternative method of obtaining the Taylor-Goldstein equation, which is to start from the full equations (1)-(3) and linearise about the basic state density ρ_o. Thus we don't initially need to use the Boussinesq approximation at all. Remarkably, we find that Eq.(18) still holds but with a much more complicated λ. A form of Boussinesq approximation can be applied to λ by neglecting terms proportional to $\hat{\omega}^2/c_s^2k^2$ and higher powers of this quantity. This merely says that we are concerned with disturbances travelling much more slowly than sound. After applying this approximation we get

$$\lambda = \frac{N^2k^2}{\hat{\omega}^2} - k^2 + \frac{k}{\hat{\omega}}\frac{d^2u_o}{dz^2} - \frac{1}{2\rho_o}\frac{d^2\rho_o}{dz^2} + \frac{1}{4\rho_o^2}\left(\frac{d\rho_o}{dz}\right)^2 + \frac{k}{\hat{\omega}}\frac{du_o}{dz}\left(\frac{1}{\rho_o}\frac{d\rho_o}{dz} + \frac{2g}{c_s^2}\right) \tag{22}$$

This must be more accurate than (19) or (21) but strangely, it is closer to (19) than (21). *This suggests that there is some problem with the "deep convection" version of the Boussinesq approximation.* This is an important unresolved problem.

One may well ask why we should be concerned with the Boussinesq approximation of the full equations if we can obtain Eq. (22) merely by using $\hat{\omega}^2/c_s^2k^2 \ll 1$. The answer is of course that Eq. (18) with λ given by Eq. (22) is valid only for linearised, periodic disturbances. For large amplitude, non-periodic flows we want a method for applying the Boussinesq approximation to the full equations.

Internal Gravity Waves

In order to simplify matters, take Eq.(18) and (19) neglecting all terms of $O(1/H_\rho)$ and also neglecting variations of u_o with z. Then

$$\frac{d^2\hat{w}}{dz^2} + \left[\frac{N^2k^2}{\hat{\omega}^2} - k^2\right]\hat{w} = 0 \tag{23}$$

Look for solutions of form $w' = \rho_o^{-1/2}We^{i(kx+mz-\omega t)}$ so that $\hat{w} = We^{imz}$. Then

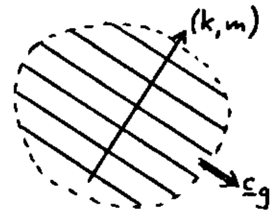
$-m^2 - k^2 + \dfrac{N^2k^2}{\hat{\omega}^2} = 0$ so that $\boxed{\omega - u_ok = \pm\dfrac{Nk}{\sqrt{(k^2+m^2)}}}$ Dispersion relation for internal gravity waves.

Note that $|\hat{\omega}| \le N$ so that the Brunt-Väisälä frequency is the maximum possible frequency for internal gravity waves.

Phase Speeds: $c_x = \dfrac{\omega}{k} = u_o \pm \dfrac{Nk}{\sqrt{(k^2+m^2)}}$ $\quad c_z = \dfrac{\omega}{m} = \dfrac{u_ok}{m} \pm \dfrac{Nk}{m\sqrt{(k^2+m^2)}}$

Group Velocity: $\underline{c}_g = \left[\dfrac{\partial\omega}{\partial k}, \dfrac{\partial\omega}{\partial m}\right] = \left[u_o \pm \dfrac{Nm^2}{(k^2+m^2)^{3/2}}, \mp \dfrac{Nkm}{(k^2+m^2)^{3/2}}\right]$

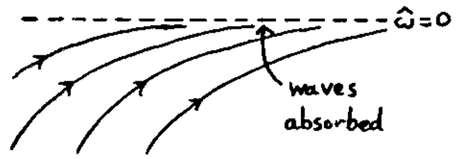
The wave crests advance in the direction (k,m) with speed c given by $\dfrac{1}{c^2} = \dfrac{1}{c_x^2} + \dfrac{1}{c_z^2}$ (and speeds c_x and c_z in the x and z directions). However, the wave energy propagates with velocity $\underline{c}_g$ (and so an isolated "packet" of waves will move with this velocity). Consider the special case $u_o=0$. Then it is easily seen that $\underline{c}_g\cdot(k,m) = 0$. In other words, the wave energy moves in a direction perpendicular to the wave crests.



Critical levels

Using the dispersion relation the group velocity can be written as

$\hat{\underline{c}}_g = \left[\dfrac{\hat{\omega}}{N^2k}(N^2 - \hat{\omega}^2), \dfrac{\hat{\omega}^2}{N^2k}\sqrt{(N^2 - \hat{\omega}^2)}\right]$ where $\hat{\underline{c}}_g = \underline{c}_g - u_o\underline{i}$ is the group velocity relative to the mean flow. This has some special properties at levels where $\hat{\omega}=0$, i.e. $\omega = u_ok$. This could happen if u_o were to vary slowly with z, so that changes in u_o are allowed even though d^2u_o/dz^2 has been neglected in Eq. (23). As $\hat{\omega}\to 0$, $\underline{c}_g \to 0$ and $\underline{c}_{gz}/\underline{c}_{gx} \to 0$. Thus the group velocity tends to zero and becomes horizontal. Energy approaching a level where $\hat{\omega}=0$ from below will travel ever more slowly and is absorbed into the mean flow. The level $\hat{\omega}=0$ is call a critical level: these were first analysed by J.R. Booker & F.P. Bretherton, The critical layer for internal gravity waves in a shear flow, *Journal of Fluid Mechanics*, 27, 513-539, 1967.



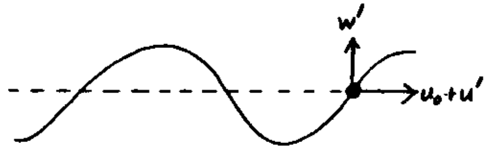
Relative Phases. Let us return to Eq. (14)-(17) but now neglecting $1/H_\rho$ and mean shear. We also note that $d\hat{w}/dz = im\hat{w}$ with a similar expression for $\hat{p}$.

$-i\hat{\omega}\hat{u} = -ik\hat{p}$ (24) $\qquad -i\hat{\omega}\hat{b} + N^2\hat{w} = 0$ (26)

$-i\hat{\omega}\hat{w} = -im\hat{p} + \hat{b}$ (25) $\qquad ik\hat{u} + im\hat{w} = 0$ (27)

From (27) we see that $\hat{u} = -m\hat{w}/k$ and since $w' = \mathrm{Re}\{\hat{w}e^{i(kx+mz-\omega t)}\}$, etc., it follows that u′ and w′ are either in phase or 180° out of phase. Similarly, (24) shows that u′ and p′ are either in phase or 180° out of phase (unless $\hat{\omega}=0$). Thus w′ and p′ are also either in phase or 180° out of phase. Finally, from (26) we see that b′ (and hence ρ′) is ±90° out of phase with the other quantities.

Effect on Mean Flow



Consider a fluid element being carried along in the wavy flow. Its horizontal momentum is $\rho_o(u_o+u')$ and its vertical velocity is w′. Thus the instantaneous upward flux of horizontal momentum is $\rho_o(u_o+u')w'$. Averaging over one wavelength we get a mean upward flux of horizontal momentum of $\rho_o\overline{u'w'} = \frac{1}{2}\rho_o\hat{u}\hat{w} = -\frac{1}{2}\rho_o\frac{m}{k}\hat{w}^2 \neq 0$. However, the corresponding mean upward flux of density is $\overline{\rho'w'} = 0$ since w′

and ρ' are $\pm 90^{\circ}$ out of phase. Thus internal gravity waves tend to change the mean flow but not the mean density. Note that from Eq. (10) and (13) we get $\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial z}(\overline{u'w'})$ which shows that the mean flow is only changed if $\overline{u'w'}$ changes with height.

A Conservation Law For Wave Activity

Start from Eq. (10)-(13):

$$\frac{\partial u'}{\partial t} + u_o\frac{\partial u'}{\partial x} + w'\frac{du_o}{dz} = -\frac{1}{\rho_o}\frac{\partial p'}{\partial x} \qquad (10) \qquad\qquad \frac{\partial b'}{\partial t} + u_o\frac{\partial b'}{\partial x} + N^2 w' = 0 \qquad (12)$$

$$\frac{\partial w'}{\partial t} + u_o\frac{\partial w'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + b' \qquad (11) \qquad\qquad \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad (13)$$

Define Lagrangian particle displacements (ξ,ζ) by

$$\frac{\partial \xi}{\partial t} + u_o\frac{\partial \xi}{\partial x} = u' + \zeta\frac{du_o}{dz} \qquad (28) \qquad\qquad \frac{\partial \zeta}{\partial t} + u_o\frac{\partial \zeta}{\partial x} = w' \qquad (29)$$

We now take $\frac{\partial \xi}{\partial x}\times(10) + \frac{\partial \zeta}{\partial x}\times(11)$ and average over x. The details will be given in the lectures. The result is

$$\rho_o\left[\frac{\partial}{\partial t} + u_o\frac{\partial}{\partial x}\right]\overline{\left[-\left[u' + \zeta\frac{du_o}{dz}\right]\frac{\partial \xi}{\partial x} - w'\frac{\partial \zeta}{\partial x}\right]} + \frac{\partial}{\partial x}\overline{\left[-p'\frac{\partial \xi}{\partial x}\right]} + \frac{\partial}{\partial z}\overline{\left[-p'\frac{\partial \zeta}{\partial x}\right]} = 0 \qquad (30)$$

This is a *wave action equation*. A number of consequences follow:

1. It can be shown using Eq.(10)-(13) that for internal gravity waves $\overline{-p'\frac{\partial \zeta}{\partial x}} = \rho_o\overline{u'w'}$ and so, since mean quantities cannot depend on x, Eq.(30) is of the form $\frac{\partial \mathcal{A}}{\partial t} + \frac{\partial}{\partial z}(\rho_o\overline{u'w'}) = 0$. Since we have neglected $1/H_\rho$ in (13) we must do so here also, giving $\frac{\partial \mathcal{A}}{\partial t} + \rho_o\frac{\partial}{\partial z}(\overline{u'w'}) = 0$. $\mathcal{A}$ is the wave action density (actually the pseudo-momentum) and is a measure of the amount of wave activity. The equation shows that if the waves are steady (i.e. the amplitude is not changing), then the momentum flux does not change with height and therefore that there is no change to the mean flow. It follows that waves can only change the mean flow if they are unsteady or if dissipation acts.

   Mean Flow Acceleration = Transience + Dissipation

   The result that $\partial(\overline{u'w'})/\partial z = 0$ for steady waves was first obtained using other methods by A. Eliassen & E. Palm, On the transfer of energy in stationary mountain waves, *Geofysiske Publikasjoner Geophysica Norvegica*, XXII, 1-23, 1961.

2. For the linearised internal gravity wave solution to (24)-(27) we can evaluate each of the terms in Eq. (30). We need to assume $\xi = \Re e\{\hat{\xi}e^{i(kx+mz-\omega t)}\}$ and $\zeta = \Re e\{\hat{\zeta}e^{i(kx+mz-\omega t)}\}$ also. The total wave energy density $\mathcal{E}$ is given by

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \tfrac{1}{2}\overline{(u'^2+v'^2)} + \tfrac{1}{2}N^2\overline{\zeta^2} = \frac{1}{2}\,\frac{(k^2+m^2)}{k^2}\hat{w}^2$$

   It turns out that Eq.(30) becomes

$$\frac{\partial}{\partial t}\left[\frac{\mathcal{E}k}{\hat{\omega}}\right] + \frac{\partial}{\partial z}\left[c_{gz}\frac{\mathcal{E}k}{\hat{\omega}}\right] = 0 \qquad (31)$$

   This is the so-called Bretherton and Garrett equation, first derived using alternative methods by F.P. Bretherton & C.J.R. Garrett, Wavetrains in inhomogeneous moving media, *Proceedings of th Royal Society*, A302, 529, 1969.

3. The method used above to derive Eq. (30) is a small amplitude simplification of a very general and powerful method call the Generalised Lagrangian Mean theory. This was developed by Andrews & McIntyre:

   D.G. Andrews & M.E. McIntyre, An exact theory of nonlinear waves on a Lagrangian-mean flow, *Journal of Fluid Mechanics*, 89, 609-646, 1978.

   D.G. Andrews & M.E. McIntyre, On wave action and its relatives, *Journal of Fluid Mechanics*, 89, 647-664, 1978.

   Before trying to read these difficult papers it is recommended that the following should be consulted:

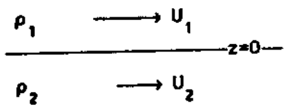
   M.E. McIntyre, An introduction to the Generalised Lagrangian-Mean description of wave, mean-flow interaction, *Pure & Applied Geophysics*, 118, 152-176, 1980.

   Using the Generalised Lagrangian-Mean theory, an equation equivalent to (30) can be derived for *arbitrary amplitude waves without any approximation*.

Instability of Stratified Flows

Kelvin-Helmholtz Instability

This is the simplest layer model in which each of the two layers extends from z=0 to $\pm\infty$. In each layer, N=0 and so Eq.(23) becomes



$$\frac{d^2\hat{w}}{dz^2} - k^2\hat{w} = 0 \qquad (32)$$

Thus $\hat{w}_1 = A_1e^{-kz}$ and $\hat{w}_2 = A_2e^{kz}$ since the disturbance must decay at $\pm\infty$. The total pressure is $p = -\rho gz + \hat{p}e^{i(kx-\omega t)}$ and from Eq. (24) and (27), $\hat{p}_1 = -\frac{i(\omega-U_1k)}{k}A_1e^{-kz}$ and $\hat{p}_2 = \frac{i(\omega-U_2k)}{k}A_2e^{kz}$. Thus the dynamic boundary condition $p_1=p_2$ on z=0 (valid for small amplitude) gives

$$-\rho_1 gh - i\rho_1\frac{(\omega-U_1k)}{k}A_1 = -\rho_2 gh + i\rho_2\frac{(\omega-U_1k)}{k}A_2$$

where it is assumed that the displacement of the interface is $\eta = he^{i(kx-\omega t)}$. The kinematic boundary condition gives $\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = w$ on $z=\eta$ in each layer which reduces to

$$-i(\omega-U_1k)h = A_1 \text{ and } -i(\omega-U_2k)h = A_2$$

Combining the results from the two boundary conditions gives

$$(\rho_1 - \rho_2) - \rho_1\frac{(\omega-U_1k)^2}{k} = \rho_2\frac{(\omega-U_2k)^2}{k} \text{ with solution}$$

$$\omega = \frac{(\rho_1U_1 + \rho_2U_2)k}{\rho_1+\rho_2} \pm \frac{k}{\rho_1+\rho_2}\left[\frac{(\rho_1+\rho_2)(\rho_2-\rho_1)}{k}g - \rho_1\rho_2(U_1-U_2)^2\right]^{1/2} \qquad (33)$$

Thus for $\frac{(\rho_1+\rho_2)(\rho_2-\rho_1)}{k}g \geq \rho_1\rho_2(U_1-U_2)^2$, ω is real and the flow is stable. The disturbances are internal gravity waves trapped near the interface. For $\frac{(\rho_1+\rho_2)(\rho_2-\rho_1)}{k}g < \rho_1\rho_2(U_1-U_2)^2$, ω is complex and the root with positive imaginary part corresponds to an exponentially growing (i.e. unstable) mode. The condition for instability can be written in terms of $\Delta U = U_1-U_2$,

$\Delta\rho = \rho_2 - \rho_1$ and $2\bar{\rho} = \rho_2 + \rho_1$. Then $\dfrac{g\Delta\rho}{\bar{\rho}(\Delta U)^2 k} < \dfrac{1}{2}$ for instability. It follows that

(i) If $\Delta\rho$ is sufficiently large the flow is stable, (ii) If ΔU is sufficiently large the flow is unstable, (iii) For any $\Delta\rho$ and ΔU, there is always instability for sufficiently large k (i.e. small wavelength). This latter feature is not typical of other models and can in any case be removed by considering small viscosity.

Flows with Direction Changing With Height We can generalise the Kelvin-Helmholtz problem so that the basic flow in each layer is $(U_1, V_1, 0)$ and $(U_2, V_2, 0)$. We now have to consider in what direction the most unstable waves will propagate. We assume disturbances of the form $e^{i(kx+\ell y-\omega t)}$ and define an angle θ such that $\cos\theta = \dfrac{k}{(k^2+\ell^2)}$ and $\sin\theta = \dfrac{\ell}{(k^2+\ell^2)}$. Thus the component of the mean flow in the direction of wave propagation is $U\cos\theta + V\sin\theta$ and the jump in this component across the interface is $\Delta U = (U_1-U_2)\cos\theta + (V_1-V_2)\sin\theta$. It turns out that for this model Eq. (33) is replaced by

$$\omega = \frac{\sqrt{(k^2+\ell^2)}[(\rho_1U_1+\rho_2U_2)\cos\theta + (\rho_1V_1+\rho_2V_2)\sin\theta]}{\rho_1+\rho_2} \pm \frac{\sqrt{(k^2+\ell^2)}}{\rho_1+\rho_2}\sqrt{\left[\frac{2\bar{\rho}\Delta\rho g}{\sqrt{(k^2+\ell^2)}} - \rho_1\rho_2(\Delta U)^2\right]}$$

It follows immediately that the most unstable waves propagate in the direction of maximum ΔU. This corresponds, in the case of continuous stratification, to the direction of maximum shear. Unfortunately, it does not seem possible to prove this general result for continuously stratified flows. In the lectures, a demonstration of the difficulty occurring for the continuously stratified case will be given.

Continuous Stratification With Unidirectional Flow. Neglecting $1/H_\rho$, Eq. (18) and (19) become $\dfrac{d^2\hat{w}}{dz^2} + \left[\dfrac{N^2k^2}{\hat{\omega}^2} - k^2 + \dfrac{k}{\hat{\omega}}\dfrac{d^2u_o}{dz^2}\right]\hat{w} = 0$. From these, some simple but general results can be proved. (The best reference is L. Howard, Note on a paper of John W. Miles, *Journal of Fluid Mechanics*, 10, 509-512, 1961). In particular:

1) The flow must be stable if $N^2 - \dfrac{1}{4}\left(\dfrac{du_o}{dz}\right)^2 > 0$ everywhere. Thus large N (large stratification) or small shear promote stability.

2) If ω_r and ω_i are the real and imaginary parts of ω, then $[\omega_r/k - \frac{1}{2}(u_{max}+u_{min})]^2 + (\omega_i/k)^2 \le \frac{1}{2}[u_{max}-u_{min}]^2$ if $\omega_i > 0$. (u_{max} and u_{min} are the maximum and minimum values of u_o.) This provides a bound on the phase speed of unstable modes (the semi-circle theorem).

For flows in which the mean flow changes direction with height, many similar results follow from the corresponding equation for $\hat{w}$:

$$\frac{d^2\hat{w}}{dz^2} + \left[\frac{N^2(k^2+\ell^2)}{\hat{\omega}^2} - (k^2+\ell^2) + \frac{1}{\hat{\omega}}\left(k\frac{d^2u_o}{dz^2} + \ell\frac{d^2v_o}{dz^2}\right)\right]\hat{w} = 0 \text{ where } \hat{\omega} = \omega - u_ok - v_o\ell.$$