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WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

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MATHEMATICAL THEORY OF SHOCK WAVES

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Abstract

We plan to survey shock wave theory for hyperbolic conservation laws and viscou conservation laws. The theory is applicable to compressible Euler and Navier-Stokes equa tions. We also discuss models for elasticity models and multiphase flows which take th form of nonstrictly hyperbolic systems or conservation laws with relaxation. Basic ideas o time-invariants, nonlinear superpositions, compressibility and expansion, energy metho characteristic method, and time-asymptotic expansion are explained for simple models.

1. Hyperbolic Conservation Laws.

The simplest mathematical models which carry shock waves are hyperbolic conservation laws:

$$u_t + f(u)_x = 0, \quad t \geq 0, \quad -\infty < x < \infty, \tag{1.1}$$

where $u = u(t,x) \in \mathbb{R}^n$ and $f(u) \in \mathbb{R}^n$ represents density of physical quantities and the flux. An important example is the compressible Euler equations

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \quad \text{(conservation of mass)} \\ (\rho v)_t + (\rho v^2 + p)_x &= 0 \quad \text{(conservation of momentum)} \\ (\rho(e + \frac{v^2}{2}))_t + (\rho(e + \frac{v^2}{2})v + pv)_x &= 0 \quad \text{(conservation of energy)} \end{aligned} \tag{1.2}$$

where ρ, v, p and e are density, velocity, pressure and internal energy of the gas. The gas is described by the constitutive relation $p = p(\rho, e)$. For polytropic gases $p = (\gamma - 1)\rho e$, $\gamma > 1$ the adiabatic constant, [1]. In being consistent with polytropic gases we assume that (1.1) is strictly hyperbolic, that is, $\partial f/\partial u$ has real and distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)$:

$$\frac{\partial f(u)}{\partial u} r_i(u) = \lambda_i(u) r_i(u), \quad i = 1, 2, \ldots, n. \tag{1.3}$$

To gain basic understanding we study first the scalar equation, $u \in \mathbb{R}^1$. Formula (1.1) is equivalent to

$$\frac{du}{dt} = 0, \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \lambda(u)\frac{\partial}{\partial x}, \quad \lambda(u) \equiv f'(u)$$

That is, u is constant along the *characteristic* curves $\frac{dx}{dt} = \lambda(u(x,t))$. Since $\lambda(u)$ depends on u only, each characteristic curve has constant speed and thereby is a line, Figure 1.1.

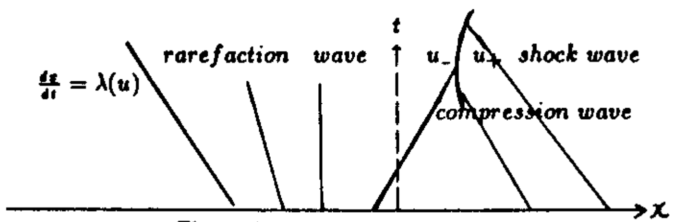

Figure 1.1

If initial data $u(x,0)$ is such that $\lambda(u(x,0))$ is increasing in x then we obtain a *rarefaction wave*, otherwise a *compression wave*. Clearly a compression wave will produce a multivalued solution in finite time. A single-valued solution is possible only if discontinuities, *shock waves*, are admitted in the solution. To consider discontinuity waves for (1.1) one can either resort to theory of distribution, or the integral version of (1.1),

$$\frac{d}{dt}\int_a^b u(x,t)dx = f(u(a,t)) - f(u(b,t)).$$

In either case across a discontinuity $x = x(t)$ the following jump (Rankine-Hugoniot) condition must be satisfied

$$x'(t)(u_+ - u_-) = f(u_+) - f(u_-), \quad u_\pm \equiv u(x(t) \pm 0, t). \tag{R-H}$$

Since shock waves arise out of compression we require that

$$\lambda(u_+) < x'(t) < \lambda(u_-) \tag{E}$$

which is usually called entropy condition as an analogous condition for the gas dynamics equations (1.2) comes from the second law of thermodynamics. If the flux function is convex

$$f''(u) > 0$$

then (E) is equivalent to $u_- > u_+$. We now study the consequence of the nonlinearity and entropy condition. Two shock waves must combine by simple geometric reasoning, Figure 1.2.

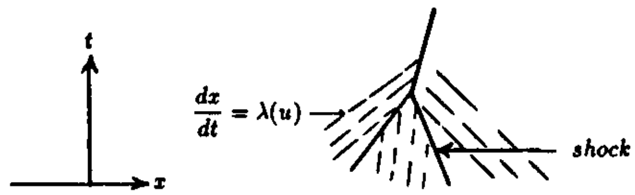

Figure 1.2

A rarefaction wave and a shock wave cancel, Figure 1.3.

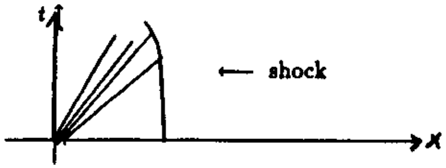

Figure 1.3

In other words, the solution becomes more regular, except for the new shock waves generated through compression. For instance consider periodic initial data $u(x,0) = u(x+L,0)$, $-\infty < x < \infty$. It is clear that $u(x,t)$ should be periodic and with the same mean. Moreover, both rarefaction waves and compression waves exist and are of the same strength. Thus they will cancel and the solution decays. To see this we draw characteristic lines backward in time. These lines do not meet shock waves because of (E). Thus they reach $t = 0$, Figure 1.4.

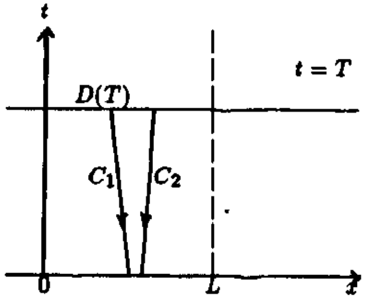

Figure 1.4

Consider two such lines $C_i = \frac{dx}{dt} = \lambda(u_i)$, $i = 1,2$, and $D(T)$ the distance between them at time T. Then

$$D(T) = D(0) + (\lambda(u_2) - \lambda(u_1))T \quad \text{and so}$$

$$\lambda(u_2) - \lambda(u_1) \le \frac{D(T)}{T}.$$

Apply this to any intervals between $x = 0$ and $x = L$ at time T along which $\lambda(u)(x,T)$ is nondecreasing; we have the increasing variation of $u(x,T)$. $IV(T)$ over $0 \le x \le L$ is bounded by L/T. Since $\lambda(u)(x,T)$ is periodic we have the total variation $TV(T)$ over

$0 \le x \le L$ is bounded by $2L/T$. For convex $f(u)$, $f'(u) > 0$, we conclude that the tota variation of $u(x,t)$ over each period decays at the rate $O(1)Lt^{-1}$. Note that the rate i always t^{-1} and the constant $O(1)L$ is independent of size of the data.

When the initial value $u(x,0)$ has compact support, $u(x,0) = 0$ for $|x| \ge M$, the: $u(x,t)$ decays at the rate $t^{-1/2}$. The asymptotic slope is an N*-wave.*

$$N(p,q;x,t) = \begin{cases} x/t & \text{for } -\sqrt{2pt} < x < \sqrt{2qt} \\ 0 & \text{otherwise} . \end{cases}$$

where p and q are two time-invariants of the solutions

$$p = \min_x \int_{-\infty}^{x} u(y,t)\,dy, \quad q = \max_x \int_x^{\infty} u(y,t)\,dy.$$

For a simple proof see [5]. Notice that the nonlinearity $f''(u) \ne 0$ induces decay of th solution at the same rate as the heat kernel for linear heat equations.

A shockwave (u_-, u_+) is stable because of its compressibility, which causes all infor mation traveling along characteristics to be absorbed into the shockwave. To make thi precise consider initial value bounded and with $u(x,0) = u_-$ for $x < -M$, $u(x,0) = u_-$ for $x > M$, $u_- > u_+$. Recall that from entropy condition (E) a characteristic line ma hit a shock wave in forward time. If so, we continue it with the shock wave and call i *generalized characteristic*, Figure 1.5.

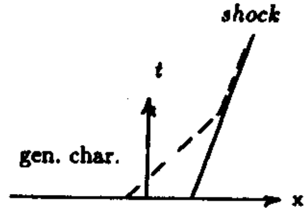

Figure 1.5

Through $(-M,0)$ and $(M,0)$ draw generalized characteristics χ_ℓ and χ_r, Figure 1.6.

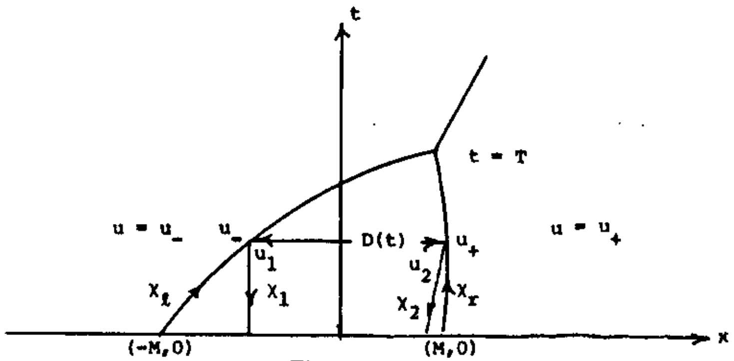

Figure 1.6

Outside of χ_ℓ and χ_r, $u \equiv 0$. Across χ_ℓ and the χ_r there may be shock waves. Let $_1(t)$ and $u_2(t)$ be the limiting values of $u(x,t)$ to the right of χ_ℓ and to the left of χ_r, spectively. Let $D(t)$ be the distance between χ_ℓ and χ_r at time t. From jump condition

$$D'(t) = \frac{f(u_+) - f(u_2(t))}{u_+ - u_2(t)} - \frac{f(u_1(t)) - f(u_-)}{u_1(t) - u_-} \qquad (R-H)$$

n the other hand, the slopes of χ_1 and χ_2 are $f'(u_1(t))$ and $f'(u_2(t))$, respectively, and

$$D(t) = \Big(f'(u_2(t)) - f'(u_1(t))\Big)t + O(1)$$

r some $0 \le O(1) \le 2M$. Since $f''(u) > 0$ and $u(x,t)$ is bounded, there exists $\theta = \theta(t)$, $< \alpha < \theta < \beta < 1$ such that

$$D'(t) = \theta(f'(u_2(t)) - f'(u_1(t)) + (1-\theta)(f'(u_+) - f'(u_-))$$
$$= \frac{\theta}{t}D(t) - \frac{O(1)}{t} + (1-\theta)(f'(u_+) - f'(u_-))$$

oticing that $f'(u_+) - f'(u_-) < 0$, the above can be solved to yield $D(T) = 0$ for some ite T. Thus we conclude that χ_ℓ and χ_r coalesce and the solution $u(x,t)$ becomes a ngle shock wave after time T. This shows that a shock wave is stable even with large erturbation.

The above analysis is for scalar equations. We now turn to the system. For the linear system, $\partial f(u)/\partial u$, a $n \times n$ constant matrix, it can be shown by elementary linear algebra that

$$u(x,t) = \sum_{i=1}^{n} \alpha_i(x,t)r_i$$
$$(\alpha_i)_t + \lambda_i(\alpha_i)_x = 0, \quad \alpha_i(x,t) = \alpha_i(x - \lambda_i t, 0).$$

Note that in the linear case λ_i and r_i are constant. Thus a solution can be decomposed into n modes, each taking values in the r_i direction, and travels with *characteristic* speed λ_i. For nonlinear systems, λ_i and r_i depend on u. Nevertheless a class of solutions, called *simple waves* can be found easily as follows. Suppose that $u(x,t)$ is a smooth solution of (1.1) depending only on one parameter

$$u(x,t) = \phi(\xi(x,t)).$$

Then it follows from (1.1) and (1.3) that

$$(\xi_t + \xi_x f'(\phi))\phi' = 0,$$
$$-\xi_t/\xi_x = \lambda_i(\phi), \quad \phi' = \text{scalar } r_i(\phi)$$

for some i, $1 \le i \le n$. Conversely, suppose that initial value $u(x,0)$ is on an integral curve R_i of the vector field $r_i(u)$ (in the u-space). Then an i-*simple wave* $u(x,t)$ can be constructed so as to be constant along the i-characteristic curves (lines in this case) $dx/dt = \lambda_i(u)$, much as in the scalar case, cf. Figure 1.1. We have i-compression waves and i-rarefaction waves when the i-characteristic lines are converging or diverging. Thus it becomes clear that the basic effect of nonlinearity is the behavior of $\lambda_i(u)$ in the $r_i(u)$ direction. Following the terminology of [6] we say that an i-characteristic field is *genuinely nonlinear* if $\lambda_i(u)$ is strictly monotone, and *linearly degenerate* if it is constant in the $r_i(u)$ direction.

$$(g.n.l.) \qquad \nabla\lambda_i(u)\cdot r_i(u) \neq 0$$

$$(l.d.g.) \qquad \nabla\lambda_i(u)\cdot r_i(u) = 0$$

For gas dynamics equations (1.2), λ_1 and λ_3 are g.n.l. and λ_2 l.d.g. A g.n.l. field produces compression waves and rarefaction waves, a l.d.g. field possesses linear waves. Unlike the scalar equation, a compression wave does not form only shock waves of the same family. The reason is that, although a shock waves (u_-, u_+) satyisfies the same jump condition (R-H), unlike simple waves u_+ is not on the R_i curve through u_-. To study this consider the *Hugoniot* curves

$$S(u_0) = \{u | \sigma(u - u_0) = f(u) - f(u_0) \text{ for some scalar } \sigma = \sigma(u_0, u)\}$$

We claim that in a small neighborhood of u_0,

$(1.4)_1$ $\quad S(u_0)$ is the union of n smooth curves$S_i(u_0)$, $i = 1, 2, \ldots, n$, through u_0,

$(1.4)_2$ $\quad S_i(u_0)$ and $R_i(u_0)$ have second order contact at u_0

$(1.4)_3$ $$\sigma(u_0, u) = \frac{1}{2}(\lambda_i(u_0) + \lambda_i(u)) + O(1)|u_0 - u|^2 \text{ for } u \in S_i(u_0).$$

We now discuss this. Since

$$\begin{aligned} f(u) - f(u_0) &= \int_0^1 df(u_0 + \alpha(u - u_0))d\alpha \\ &= \int_0^1 \partial f(u_0 + \alpha(u - u_0))(u - u_0)d\alpha \\ &\equiv G(u)(u - u_0), \end{aligned}$$

the jump condition (R-H) becomes

$$[G(u) - \sigma](u - u_0) = 0$$

It is clear that $G(u_0) = \partial f/\partial u(u_0)$. Thus $G(u)$ has real and distinct eigenvalues $\mu_i(u) < \mu_2(u) < \cdots < \mu_n(u)$ for u close to u_0, and $\mu_i(u) \to \lambda_i(u_0)$ as $u \to u_0$. Let $L_1(u), \ldots, L_n(u)$ be the left eigenvectors. The above equation has a solution $u \neq u_0$ if and only if $\sigma = \mu_i(u)$ for some i, and

$$L_j(u)(u - u_0) = 0, \quad j = 1, 2, \ldots, n, \quad j \neq i.$$

This represents $(n-1)$ equations in $n-$unknowns u. Moreover, since $G(u)$ is close to $\partial f/\partial u(u_0)$, which has a complete set of left eigenvectors, the above system has rank $n-1$. Thus from the implicit function theorem it can be solved as a smooth curve $S_i(u_0)$ through u_0. This shows (1.4). Differentiate the jump condition (R-H) $\sigma(u - u_0) = f(u) - f(u_0)$ along $S_i(u_0)$:

$$\sigma'(u - u_0) = \Big(f'(u) - \sigma\Big)u',$$

$$\sigma''(u - u_0) = (f''(u)u' - 2\sigma')u'|(f'(u) - \sigma)u''$$

Evaluate the first identity at $u = u_0$; we have

$$u' = r_i(u_0), \quad \sigma = \lambda_i(u_0) \quad \text{at} \quad u = u_0,$$

where we normalized r_i by $|r_i| \equiv 1$ and parametrize $S_i(u_0)$ by the arc length, $|u'| = 1$. In particular u'' is perpendicular to u'. With these we have from the second identity

$$(f''(u_0)\, r_i(u_0) - 2\sigma')\, r_i(u_0) + -(f'(u_0) - \sigma)u''\Big|_{u=u_0} = 0$$

On the other hand differentiate (1.3) in the direction of $r_i(u_0)$ to get

$$\begin{aligned} &f''(u_0)r_i(u_0)r_i(u_0) + f'(u_0)(\nabla r_i(u_0) \cdot r_i(u_0)) \\ &\quad = (\nabla\lambda_i(u_0) \cdot r_i(u_0))r_i(u_0) + \lambda_i(u_0)\nabla r_i(u_0) \cdot r_i(u_0), \end{aligned}$$

$$f''(u_0)r_i(u_0)r_i(u_0) + f'(u_0(r_i(u)'\Big|_{u=u_0} r_i(u_0) = \lambda_i(u)'\Big|_{u=u_0} + \lambda_i(u_0)\, r_i(u)'\Big|_{u=u_0}$$

From the above two identities we conclude

$$(f'(u_0) - \lambda_i(u_0)u'' + (\lambda_i(u_0) - f'(u_0))r_i(u)' = (2\sigma' - \lambda_i(u)')\, r_i(u_0) \quad \text{at} \quad u = u_0.$$

The right hand side is a multiple of $r_i(u_0)$. But $(\lambda_i(u_0) - f'(u_0))v$ is a combination of $r_j(u_0)$, $j \neq i$, for any vector v and so the left hand side is a combination of $r_j(u_0)$, $j \neq i$. Thus both sides are zero. In particular $\sigma' = \frac{1}{2}\lambda_i(u)'$ at $u = u_0$ which shows $(1.4)_3$. Also, from

$$(f'(u_0) - \lambda_i(u_0))(u'' - r_i(u)') = 0 \quad \text{at} \quad u = u_0$$

we see that $u'' - r_i(u)'$ has only $r_i(u_0)$ component. But since both u' and $r_i(u)$ are unit vectors by our normalization, $u'' - r_i(u)'$ is perpendicular to $r_i(u)$ and so $u'' - r_i(u)' = 0$, which shows $(1.4)_2$.

With $(1.4)_2$ we see that $R_i(u_0)$ and $S_i(u_0)$ are close for u near u_0. But in general they are not identical. Thus a i−compression wave takes values along an R_i curve, when it forms a shock wave it takes values along an S_i curve and since $S_i \not\equiv R_i$, other waves are also formed. This is so for the gas dynamics equation (1.2). Indeed a very complicated wave pattern arises out of a compression wave. So far the qualitative theory of shock waves for (1.1) is based on the random choice method of [3]. The method uses shock waves and rarefaction waves for each g.n.l. field and linear waves for each l.d.g. field as building blocks. A form of nonlinear superposition is introduced, [4], [7], to study the wave behavior [8]. In these studies two main mechanisms of nonlinearity are identified and investigated. The first is the compression and expansion of nonlinear waves pertaining to the given characteristic field. This is already present for the scalar equation and is briefly discussed above. The second is the coupling of waves of different characteristic fields. This is measured in part by the bifurcation of S_i curves from R_i curves. We now describe briefly the ideas. The first step is to solve the *Riemann problem* for (1.1) with initial data

$$u(x,0) = \begin{cases} u_\ell & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases} \tag{1.5}$$

for two constant states u_ℓ and u_r. Since both (1.1) and (1.5) are invariant under the transformation $x \to cx$, $t \to ct$, $c > 0$, the solution is a function of x/t, and consists of shock waves, rarefaction waves and linear waves issued from (0.0), Figure 1.7, see [6]. To

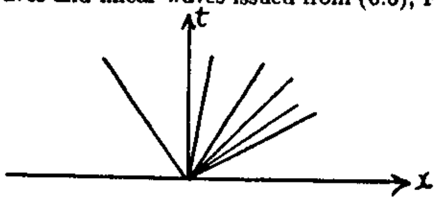

Figure 1.7

construct a general solution to the initial value problems of (1.1) approximate the initial

data by a step function and resolve the discontinuities by solving the Riemann problem. Before the waves issued from each discontinuity at $t = 0$ interact approximate the solution by a step function by random sampling, and so on, [3]. When each wave is partitioned and traced, the sampling can be made to be any equidistributed sequence, [7]. The idea of generalized characteristics for scalar equations can be generalized to the system through the wave tracing method to study the behavior of solutions, [8].

2. Viscous Conservation Laws

Consider conservation laws with dissipative effects of rate type:

$$u_t + f(u)_x = (B(u)u_x)_x \tag{2.1}$$

where $B(u)$, the *viscosity matrix* is a $n \times n$ matrix. An important example is the compressible Navier-Stokes equations

$$\begin{aligned} &\rho_t + (\rho v)_x = 0 \\ &(\rho \nu)_t + (\rho v_x^2)_x = (\mu v_x)_x \\ &(\rho e + \rho\frac{u^2}{2})_t + ((\rho e + \frac{\rho u^2}{2})u + pu)_x = (kT_x + \mu v v_x)_x \end{aligned} \tag{2.2}$$

where μ is the viscosity and k the heat conductivity coefficient, and T the temperature of the gas. The dissipative effects of the right hand side smooth shock waves, which becomes smooth traveling waves, the viscous shock waves. Since most physical dissipations are not uniform, for instance the continuity equation in (2.1) does not have dissipation, the system is not parabolic but hyperbolic-parabolic. A consequence is that discontinuities in the initial data propagate into the solution. Nevertheless the system remains essentially parabolic in its local behavior. However, for intermediate and large time behaviors, the nonlinear hyperbolic nature of the sysstem becomes important. This is because of the nonlinearity of the flux function $f(u)$ as described in the last section. To study nonlinear waves for (2.1) one needs an approach which incorporates this dual nature of nonlinear hyperbolic and parabolic of the system. We present below stability analysis for three different types of nonlinear waves.

Consider initial data $u(x,0)$ a perturbation of a constant state $u = 0$ for system (2.1). For simplicity we assume $B = I$,

$$\begin{aligned} &u_t + f(u)_x = u_{xx} \\ &u(x,0) = 0 \text{ as } |x| \to \infty \end{aligned} \tag{2.3}$$

The problem has been studied by comparing (2.3) with the linearized system

$$w_t + f'(0)w_x = \omega_{xx}$$

whose solution decays like a heat kernel in $L_\infty(x)$ and $L_x(x)$. This is a linear result. Since (2.3) is the conservation laws,

$$\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dt, \quad t \geq 0, \tag{2.4}$$

the $L_1(x)$ norm of the solution does not decay. In the study of $L_1(x)$ behavior of the solution of (2.3) one finds certain nonlinear behavior of the solution. The first step is to identify plausible large-time states for the solution by a certain asymptotic expansion. The solution $w(x,t)$ of the linear equations is decomposed into scalar modes taking values in the $r_i(0)$, $i = 1, 2, \ldots, n$, directions:

$$\begin{aligned} &\omega(x,t) = \sum_{i=1}^{n} a_i(x,t)\, r_i(0) \\ &(\alpha_i)_t + \lambda_i(0)(\alpha_i)_x = (\alpha - i)_{xx} \end{aligned} \tag{2.5}$$

The nonlinear solution $u(x,t)$ should eventually decay into n models, each taking values along the $r_i(0)$ direction. This observation prompts us to seek an approximate solution $\psi_i(x,t)$ of (2.3) which takes values in the $r_i(0)$ direction:

$$\begin{aligned} \frac{\partial \psi_i(x,t)}{\partial t} &= a_i(x,t)\, r_i(0), \\ \frac{\partial \psi_i(x,t)}{\partial x} &= b_i(x,t)\, r_i(0). \end{aligned}$$

for some scalar $a_i(x,t)$ and $b_i(x,t)$. There are two cases, either the i-characteristic field is g.n.l. or l.d.g. Suppose it is g.n.l., we normalize $r_i(u)$ so that $\nabla\lambda_i(u) \cdot r_i(u) = 1$, and so

$$\frac{\partial \lambda_i(\psi_i(x,t))}{\partial t} = \nabla\lambda_i(\psi_i)(a_i r_i(\psi_i)) \simeq a_i \nabla\lambda_i(0) \cdot r_i(0) = a_i$$

where we have used the fact that $\psi_i(x,t)$ should tend to zero as t becomes large. Similarly $\lambda_{ix} \simeq b_i$

We want $\psi_i(x,t)$ to be an approximate solution of (2.3)

$$\begin{aligned} &\frac{\partial \psi_i}{\partial t} + f'(\psi_i)\frac{\partial \psi_i}{\partial t} \simeq \frac{\partial^2 \psi_i}{\partial x^2}, \\ &(\lambda_i(\psi_i)_t + \lambda_i(\psi_i)\lambda(\psi_i)_x)r_i(0) \simeq (\lambda_i(\psi_i)_x r_i(0))_x. \end{aligned}$$

Thus we define $\psi_i(x,t)$ by first choosing a solution $\lambda(x,t)$ of the Burgers equation

$$\begin{aligned} &\lambda_t + \lambda\lambda_x = \lambda_{xx} \\ &\lambda(x,0) \to 0 \text{ as } |x| \to \infty \end{aligned} \tag{2.6}$$

and then set

$$\psi_i(x,t) = \lambda(x - \lambda_i(0)t, t) r_i(0).$$

For the l.d.g. field, instead of Burgers' equation we have the heat equation. For simplicity we assume that the system is g.n.l. Solutions of Burgers' equation with given

$$m = \int_{-\infty}^{\infty} \lambda(x,t)\, dx,$$

all converge to the self-similar solution with $\lambda(x,0) = m\delta(x)$:

$$\lambda(x,t) \equiv \theta(m;x,t) == \frac{(\exp(m/2)-1)t^{-1/2}\exp(-x^2/4t)}{(2\sqrt{\pi} + \exp(m/2) - 1)\int -x/2\sqrt{t}^{\infty} \exp(-\xi^2)d\xi} \tag{2.7}$$

Thus, time-asymptotically, $\psi_i(x,t)$ depends only on one parameter. We decompose (2.4) in the coordinate $\{V_i(0)\}$:

$$\int_{-\infty}^{\infty} u(x,0)dx = \sum_{i=1}^{n} m_i r_i(0)$$

and set

$$\psi_i(x,t) = \theta(m_i; x - \lambda_i(0)(t+1), t+1)\, r_i(0) \equiv \theta_i(x,t)\, r_i(0)$$

where θ is evaluated at $t+1$ to avoid the singularity at $t = 0$. The above analysis is in anticipation of the fact that

$$u(x,t) - \sum_{i=1}^{n} \psi_i(x,t) \to 0 \text{ as } t \to \infty$$

in the $L_p(x)$ norm for $\infty \geq p \geq 1$, under appropriate assumptions. Note that since Burgers' equation is nonlinear, such a result is nonlinear. We now indicate the proof and ident fy the rates of convergence. First we assess the accuracy of ψ_i:

$$\begin{aligned} &(\psi_i)_t + f(\psi_i)_x - \psi_{xx} \\ &= (\theta_{it} - \lambda_i(0)\theta_{ix} + f'(\psi_i)\theta_{ix})\, r_i(0) - \theta_{ixx}\, r_i(0) \\ &= (\theta_{it} + (f'(\theta_i r_i(0)) - f'(0))\theta_{ix})\, r_i(0) - \theta_{ixx} r_i(0). \end{aligned}$$

From (1.3) and $\nabla\lambda_i \cdot r_i = 1$ we have

$$\begin{aligned} f''(0)(r_i(0), r_i(0)) &= \nabla\lambda_i r_i(0) r_i(0) + (\lambda_i(0) - f'(0))\nabla r_i \cdot r_i(0) \\ &= r_i(0) + (\lambda_i(0) - f'(0))\nabla r_i \cdot r_i(0), \text{ and so} \\ k(f'(\theta_i r_i(0)) - f'(0))r_i(0) &= f''(0)(\theta_i r_i(0), r_i(0)) + O(1)\theta_i^2 \\ &= \theta_i r_i(0) + (\lambda_i(0) - f'(0))\theta_i \nabla r_i \cdot r_i(0) + O(1)\theta_i^2 \\ (\psi_i)_t + f(\psi_i)_x - \psi_{ixx} &= (\theta_{it} + \theta_i\theta_{xxx} - \theta_{xxx})r_i(0) + (\tfrac{1}{2}\lambda_i(0)\theta_i^2\nabla r_i \cdot r_i(0) + O(1)\theta_i^3)_x \\ &= (\tfrac{1}{2}(\lambda_i(0) - f'(0))\theta_i^2\nabla r_i \cdot r_i(0) + O(1)\theta_i^3)_x \end{aligned}$$

Write

$$u(x,t) = v(x,t) + \sum_{i=1}^{n} \psi_i(x,t) \equiv v(x,t) + \psi(x,t)$$

By the choice of m_i we have

$$\int_{-\infty}^{\infty} v(x,t)dx = 0, \quad t \geq 0$$

Since u satisfies (2.3) and ψ_i satisfies the above equation we have

$$\begin{aligned} v_t + &[f(\psi + v) - f(\psi)]_x = v_{xx} - E \\ E = &\psi_t + f(\psi)_x - \psi_{xx} \\ = &\sum_{i=1}^{n}(\psi_{it} + f(\psi_i)_x - \psi_{ixx}) + (\sum_{i\neq j} O(1)\theta_i\theta_j)_x \\ = &\sum_{i=1}^{n}(\frac{1}{2}(\lambda_i(0) - f'(0))\theta_i^2\nabla r_i \cdot r_i(0) + \sum_{i=1}^{n} O(1)\theta_i^3 + \sum_{i\neq j} O(1)\theta_i\theta_j)_x \end{aligned}$$

Thus by Taylor expansion we have

$$v_t + f'(0)v_x = v_{xx} - (f''(0)\psi v + f''(0)v^2 + O(1)\psi v^2 + O(1)v^2)_x - E.$$

Since $(\lambda_i(0) - f'(0))\alpha$ does not contain the $r_i(0)$ component for any vector α, the above identity yields

$$v(x,t) = \sum_{i=1}^{n} \beta_i(x,t)\, r_i(0),$$

$$\int_{-\infty}^{\infty} \beta_i(x,t)\, dx = 0, \quad t \geq 0,$$

$$\begin{aligned}\beta_{it} + \lambda_i(0)\beta_{ix} &= \beta_{ixx} + [\sum_{j,k} a_{ijk}\theta_j\beta_k + \sum_{j\neq i} a_{ijj}\theta_j^2 + \sum_{j\neq i} a_{ijk}\theta_j\theta_k \\ &\quad + O(1)\beta^2 + O(1)\theta^3]_x \\ &\equiv \beta_{ixx} + F_{ix}, \quad i = 1,2,\ldots,n,\end{aligned}$$

$$\beta^2 = \sum_{i=1}^{n} \beta_i^2, \quad O(1)\theta^3 = \sum_{i=1}^{n} O(1)\theta_i^3$$

for some scalar a_{ijk} and bounded functions $O(1)$. Since the nonlinear waves θ_i have been put in, the rest of the analysis may follow the weakly nonlinear approach of viewing the above equations as the perturbation of the linear heat equations:

$$\beta_i(x,t) = \int_{-\infty}^{\infty} G_i(x,t;y,0)\beta_i(y,0)\, dy + \int_0^t \int_{-\infty}^{\infty} G_i(x,t;y,s)F_{iy}(y,s)\, dy\, ds$$

$$G_i(x,t;y,s) \equiv \frac{1}{\sqrt{4\pi(t-s)}} \exp\left(-\frac{(x-y-\lambda_i(0)(t-s))^2}{4(t-s)}\right).$$

The rate of decay of $v(x,t)$, or equivalently of $\beta_i(x,t)$, $i = 1,2,\ldots,n$, depends on the largest contributions of the right hand side of the above equation. Since

$$\int_{-\infty}^{\infty} \beta_i(x,0)\, dx = 0,$$

the first term decays like the derivative of the heat kernel G, which is $O(1)t^{-1+1/2p}$ in $L_p(x)$. Since β is presumed to decay faster than the solutions of the heat equation and Burgers' equation, among the terms in F_i the one with least decay rate is the second term:

$$\int_0^t \int_{-\infty}^{\infty} G_i(x,t);y,s)(\sum_{j\neq i} a_{ijj}\theta_j^2(y,s))_y\, dy\, ds.$$

Each term in this integral is of the form

$$O(1)\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-s)}}(\exp -\frac{(x-y-\lambda_i(0)(t-s))^2}{4(t-s)})_y \frac{1}{4\pi s}\exp\left(-\frac{(y-\lambda_i(0)s)^2}{2s}\right) dy\, ds$$

Direct calculations yield that it decays in $L_p(x)$ at the rate $t^{\frac{-1}{4}+\frac{1}{2p}}$, which can be show to be the rate of decay of $v(x,t)$.

We next turn to the viscous shock waves of (2.1), which are smooth traveling wav $u(x,t) = \phi(x-ct)$, c the speed of the wave.

$$(2.8)\qquad \begin{aligned} &-c\phi' + f(\phi)' = (b(\phi)\phi')', \\ &A - c\phi + f(\phi) = B(\phi)\phi', \\ &A \equiv cu_\pm - f(u_\pm), \\ &u_\pm \equiv \phi(\pm\infty). \end{aligned}$$

Thus the jump condition relating values at $x = \pm\infty$ is the same as for the hyperbol conservation laws (1.1).

$$(R-H)\qquad c(u_+ - u_-) = f(u_+) - f(u_-).$$

Formula (2.8) is a system of autonomous first-order ordinary differential equations and ϕ a solution connecting the critical points u_- and u_+. This has been studied for any limiti states for the Navier-Stokes equations (1.2) or when u_- and u_+ are close for the gener system (1.1). The assumptions are that the viscosity matrix be nonnegative definite ar with positive diagonal elements when expressed in the coordinates $\{r_1,\ldots,r_n\}$ of rigl eigenvectors of $f'(u)$:

$$L(u)B(u)R(u) \geq 0, \quad \ell_i(u)B(u)r_i(u) > 0, \quad i = 1,2,\ldots,n,$$

$$L(u)f'(u) = \Lambda(u)L(u), \quad f'(u)R(u) = \Lambda(u)R(u),$$

$$\Lambda(u) = \begin{pmatrix} \lambda_1(u) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(u) \end{pmatrix},$$

and that $u_-, u_+)$ satisfies the entropy condition

$$(R)\qquad \lambda_i(u_-) > c > \lambda_i(u_+)$$

for some i, $1 \le i \le n$. Since the jump condition is the same $u_+ \in S_i(u_-)$ the $i-th$ Hugoniot set. We have assumed here that the i-field is g.n.l. and so the viscous shock wave here corresponds to the inviscid shock wave (in Section 1) formed out of compression. To study the stability of viscous shock waves we need to identify first the plausible time-asymptotic states. A perturbation of a shock wave

$$u(x,0) = \phi(x) + \overline{u}(x,0), \qquad \overline{u}(\pm\infty, 0) = 0, \tag{2.9}$$

produces two effects. The first is that the viscous shock is translated by a certain amount x_0 so that we expect

$$u(x,t) \to \phi(x + x_0 - ct),$$

uniformly in x as $t \to \infty$. We expect such a result because ϕ is compressive and thereby should be stable, but only ***orbitally stable***, that is, stable after a proper phase shift. The second effect a perturbation produces is other families of waves. For $j < i$, $\lambda_j < \lambda_i \sim c$ the j-waves move slower than ϕ and so they are waves diffused about the left state $u_- \equiv \phi(-\infty)$. Similarly j-waves, $j > i$, move with λ_j speed and diffuse about the state u_+. There is no i-diffusion wave as it would be absorbed into the shock wave ϕ. The diffusion waves ψ_j, $j \ne i$, conserved the integral

$$\int_{-\infty}^{\infty} (\psi_j(x,t) - u_-)dx = \alpha_j r_j(u_-), \quad j < i,$$
$$\int_{-\infty}^{\infty} (\psi_j(x,t) - u_+)dx = \alpha_j r_j(u_+), \quad j > i,$$

as we have studied earlier in this section. A translation of ϕ produces an integral parallel to $u_+ - u_-$:

$$h(x_0) = \int_{-\infty}^{\infty} (\phi(x + x_0 - ct) - \phi(x - ct))\ dt = x_0(u_+ - u_-).$$

The above is proved by noticing that $h(0) = 0$ and

$$h'(t_0) = \int_{-\infty}^{\infty} \phi'(x + t_0 - ct)dx = \phi(\infty) - \phi(-\infty) = u_+ - u_-.$$

The above analysis is to lcoate the time-asymptotic state for $u(x,t)$ of (2.1), (2.9):

$$u(x,t) \to \phi(x + x_0 - ct) - \sum_{j>i} (\psi_j(x,t) - u_i) - \sum_{j<i} (\psi_j(x,t) - u_-)$$

as $t \to \infty$. The consideration of integrals above indicates that

$$\int_{-\infty}^{\infty} \overline{u}(x,0)dx = \int_{-\infty}^{\infty} \overline{u}(x,t)dx = \sum_{j<i} \alpha_j \gamma_j(u_-) + \sum_{j>i} \alpha_j \gamma_j(u_+) + x_0(u_+ - u_-).$$

The left hand integral is a given vector; the right hand is its unique decomposition identifying the diffusion waves ψ_j, $j \ne i$ as well as the translation of viscous shock wave ϕ due to the perturbation. A stability analysis is introduced in [9], [10]. The analysis makes use of the compressibility of ϕ, the dissipation of ψ_j, $j \ne i$, and that these waves decouple as time increases. Consequently the analysis combines hyperbolic techniques such as wave decomposition and characteristic integration and parabolic techniques such as the energy method. We now give a brief account of the much simplified case of the scalar equation, $u \in \mathbb{R}^1$. In this case there is no diffusion wave and a perturbation produces only a translation of the viscous shock wave

$$\int_{-\infty}^{\infty} (u(x,t) - \phi(x + x_0 - ct))\ dt = 0, \quad t \ge 0$$

for some uniquely chosen x_0. Set

$$v(x,t) = u(x,t) - \phi(x + x_0 - ct),$$
$$w(x,t) = \int_{-\infty}^{x} v(y,t)\ dy$$

so that both v, w vanish at $x = \pm\infty$. They satisfy

$$v_t + (f(v + \phi) - f(\phi))_x = v_{xx}$$
$$w_t + f(w_x + \phi) - f(\phi) = w_{xx}$$

where we have assumed that $B(u) = 1$ for simplicity. We also assume that $f''(u) > 0$ so that $\phi(x - ct)$ exists provided $u_- > u_+$. ϕ is compressible, cf. condition (E),

$$\lambda(\phi)_x = f'(\phi)_x < 0.$$

Multiply the above equations by v and w, respectively, and integrate

$$\int_{-\infty}^{\infty}\frac{v^2}{2}(x,T)dx - \int_0^T\int_{-\infty}^{\infty} v_x(f(v+\phi)-f(\phi))dxdt + \int_0^T\int_{-\infty}^{\infty} v_x^2 dx\, dt = \int_{-\infty}^{\infty}\frac{v^2}{2}(x,0)\, dx$$

,

$$\int_{-\infty}^{\infty}\frac{w^2}{2}(x,T)dx - \int_0^T\int_{-\infty}^{\infty} w(f(w_x+\phi)-f(\phi))dxdt + \int_0^T\int_{-\infty}^{\infty} w_x^2 dx\, dt = \int_{-\infty}^{\infty}\frac{w^2}{2}(x,0)\, dx$$

,

or by the Taylor expansion and integration by parts, noting that $v = w_x$,

$$\int_{-\infty}^{\infty}\frac{v^2}{2}(x,T)dx + O(1)\int_0^T\int_{-\infty}^{\infty} v_x w_x dx\, dt + \int_0^T\int_{-\infty}^{\infty} v_x^2 dx\, dt = \int_{-\infty}^{\infty}\frac{v^2}{2}(x,0)dx,$$

$$\int_{-\infty}^{\infty}\frac{w^2}{2}(x,T)dx - \int_0^T\int_{-\infty}^{\infty}\frac{w^2}{2}f'(\phi)_x dx\, dt + \int_0^T\int_{-\infty}^{\infty} w_x^2(1+O(1)w)dx\, dt = \int_{-\infty}^{\infty}\frac{w^2}{2}(x,0)dx,$$

From the second identity, since $f'(\phi)_x < 0$ by compressibility, we have for small w, $|O(1)w| < \frac{1}{2}$,

$$\int_{-\infty}^{\infty} w^2(X,T)dt + \int_0^T\int_{-\infty}^{\infty}(w^2|f'(\phi)_x| + w_x^2)dx\, dt \le \int_{-\infty}^{\infty} w^2(x,0)dx.$$

By Cauchy-Schwartz inequality we have

$$\int_{-\infty}^{\infty} v^2(x,T)dx + \int_0^T\int_{-\infty}^{\infty} v_x^2 dx\, dt = O(1)\left[\int_{-\infty}^{\infty} v^2(x,0)dx + \int_0^T\int_{-\infty}^{\infty} w_x^2 dx\, mdt\right].$$

Thus we have the estimate

$$\int_{-\infty}^{\infty}(w^2+v^2)(x,T)dx + \int_0^T\int_{-\infty}^{\infty}(w_x^2+v_x^2)dx\, dt = O(1)\int_{-\infty}^{\infty}(w^2+v^2)(x,0)dx,$$

for any $T > 0$. For small perturbation the right hand side is small and therefore

$$\frac{w^2}{2}(x,t) = \int_{-\infty}^{\infty} w\, w_x(y,t)dy \le \left(\int_{-\infty}^{\infty} w^2 dx \int_{-\infty}^{\infty} w_x^2 dx\right)^{1/2}$$

is small so that $|O(1)w| < \frac{1}{2}$ and the above estimates hold. Similarly to show that th viscous shock wave ϕ is stable, $v(x,t) \to 0$ as $t \to \infty$, suffices to show that

$$\int_{-\infty}^{\infty} v^2 dt + \int_{-\infty}^{\infty} v_x^2 dt \to 0 \text{ as } t \to \infty.$$

From the above estimate we have

$$\int_T^{T+1}\int_{-\infty}^{\infty}(v^2+v_x^2)dx\, dt \to 0 \text{ as } T \to \infty.$$

It remains to show that

$$\int_{-\infty}^{\infty}(v^2+v_x^2)(x,t)dx$$

is equicontinuous in t. This, however, follows easily from the above derivation of th estimate. Thus a viscous shockwave for a scalar equation is stable because of its compress ibility. For the analysis for general systems see [9] and [10].

The rarefaction waves are also stable for quite a different reason. They are expansiv but are stable without having to have proper translation. Another important differenc is that rarefaction waves are stable in $L_\infty(x)$ and not the $L_1(x)$ norm and therefore th consideration of the conservation of integral and of diffusion waves is not necessary. Fo the stability of rarefaction waves for the Navier-Stokes equations (2.2) see [13].

8. Other Systems.

There are many other dissipative-hyperbolic systems. An important class is conservation laws with relaxation. They occur in many physical situations such as gas dynamics with thermo-nonequilibrium and in kinetic theory. For a simple model see [11], for a kinetic model see [1], and for an elastic model with fading memory see [12]. Another important class is related to hyperbolic conservation laws (1.1) which are not strictly hyperbolic. Some of the noninear waves are believed to be unstable even with the presence of dissipative mechanisms. These are interesting topics for future research. One notices that even with system (2.1) many questions remain to be settled.

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