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WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

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INCOMPRESSIBLE EULER EQUATIONS, PARTICLE TRAJECTORIES AND RELATED QUESTIONS

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(1)

Course Trieste (P. L. Lions)

The main subject of this course is: Incompressible fluid mechanics

2D, 3D Euler eqs, inhomogeneous Navier-Stokes eqs.

joint work(s) with R. J. DiPerna

Summary

(1) 10' on kinetic models, statistical cp and fluid mechanics

(2) incompressible models: presentation and basic facts.

(3) Ordinary differential equations and transport theory

(4) 2D Euler eqs. (and 3D NS): a few new facts

(5) 3D Euler eqs: blowup of weak solutions and instability of smooth solutions

(6) Inhom. incomp. NS (2D-3D, vacuum, ...)

(1) 10': names and subjects

N body Quantum Mech. (Liouville) $\xrightarrow[(2)]{N\to\infty}$ Mean-field = Self-consistent = Hartree, Hartree-Fock (eqs)

N body Quantum Mech. $\xrightarrow[(1)]{\hbar\to 0}$ N body Classical Mech. (Liouville)

Hartree, Hartree-Fock $\xrightarrow[(4)]{\hbar\to 0}$ Vlasov

N body Classical Mech. (Liouville) $\xrightarrow[N\to\infty]{(3)}$ Vlasov < Poisson ± / Maxwell (5)

Boltzmann (6) $\xrightarrow[(8)]{\varepsilon\to 0}$ / $\xrightarrow[(7)]{t\to\infty}$ Fluid Mechanics
a) gaz dynamics
b) incompr. models

Rks

(1), (4) semi classical limits

(2), (3) propagation of chaos $\xrightarrow[N\to\infty]{}$: Hartree - Quantum Hierarchy

Hartree - Quantum Hierarchy ⟷ Hartree eqs ⟷ statistical solutions of H eqs

Vlasov - Boltzmann hierarchies

Vlasov-Boltzmann hierarchies ⟷ Vlasov-Boltzmann eqs ⟷ statistical solut. of V-B eqs

Open Pb 1 (4)

(5), (6) study of these nonlinear pbs
- existence, stability
- uniqueness, regularity
- approximation

Open Pb 2

(7)
(8)
B eq. ⟷ eqs of F.M.

hierarchy stat. sols ⟶ stat sols of F.M.

Millions of open pbs and questions: $\varepsilon\to 0$ Open Pb3

except for incompr. limits: some partial infos

joint work with R. J. DiPerna: includes (1), (2), (3), (5), (6), (7)

→ techniques described in this course play a role in all these problems.

→ (8) Bardos-Golse-Levermore (incompr. limits: Stokes, Navier Stokes)

→ (6) BVP Hamdache
BGK model Perthame.

(2) Incompressible models

Navier Stokes eqs: Temam, Euler eqs: survey paper Majda 1986
notes Princeton 85 Majda

Euler eqs (E) $\frac{\partial u}{\partial t} + u.\nabla u + \nabla p = 0$; $\operatorname{div} u = 0$

Navier-Stokes eqs (NS) $\frac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u + \nabla p = 0$, $\operatorname{div} u = 0$

hom. NS eqs (${}^{i}NS_\rho$) $\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(u\rho) = 0 \ , \ \operatorname{div} u = 0 \\ \frac{\partial (\rho u)}{\partial t} + \operatorname{div}(\rho u \otimes u) - \nu \Delta u + \nabla p = 0 . \end{cases}$

$\partial_j(\rho u_j u_i)$ i^{th} eq.

p pressure, u velocity, ρ density

I will concentrate now on Euler eqs (E)

Basic facts :

(1) energy $\int |u(t)|^2 \, dx$ ind^t of t.

(2) vorticity $\omega = \operatorname{curl} u$ • scalar 2D • vector 3D

(V) $\frac{\partial \omega}{\partial t} + \operatorname{div}(u\omega) = 0$ 2D, $u = \nabla_\perp \Psi$, $-\Delta\Psi = \omega$

in 3D only the total mass of ω $\int |\omega| dx$ is bounded.

but in 2D since $\operatorname{div} u = 0$ (Liouville thm)

$$\int F(\omega) dx \quad \text{ind}^t \text{ of } t \quad \forall F$$

In particular if $\|\omega_0\|_{L^p} < \infty \Rightarrow \|\omega(t)\|_{L^p} \le C$ ind^t of t

for all $1 \le p \le \infty$.

(3) (E) well posed for small time in H^s s large, $C^{1,\alpha}$ $\alpha > 0$.
$s > 2$ 2D
$s > \frac{5}{2}$ 3D

(4) If ω_0 smooth: a smooth solution exists up to time $T+\varepsilon$ for some $\varepsilon > 0$ if $\int_0^T \|\omega(s)\|_{L^\infty} ds < \infty$ .. Beale-Kato-Majda

(3) • in particular in 2D smooth sols. exist ($\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$)
• $\omega_0 \in L^\infty$ 2D "almost Lipschitz" solution Yudovich

(5) weak solution : (3D no chances, vortex sheets, vortex methods)

To simplify the exposition, in all the following lectures
[either periodic BC, or unbounded domain
(E) $u.n = 0$ on $\partial\Omega$
(NS) $u = 0$ on $\partial\Omega$]

i) $\omega_0 \in L^p$ $p > 1$ $\Rightarrow$ $\omega(t) \in L^p$ $\Rightarrow$ $u \in W^{1,p}$ elliptic est.

ii) $W^{1,p} \hookrightarrow L^2$ compact embedding ($W^{1,1} \hookrightarrow L^2$ but non compact)

iii) $\Rightarrow$ (E) solved in distribution sense $\underset{\in L^2}{\operatorname{div}(u\otimes u)}$ if $\omega_0 \in L^p$ $p > 1$

(V) solved ——— if $\omega_0 \in L^p$ $p > 4/3$

ωu: $\in L^p$, $\in W^{1,p} \hookrightarrow L^{\frac{2p}{2-p}}$

not enough $\omega_0 \in BM$, $u_0 \in L^2$ (unstable ... but?) Greengard

Difficulty compactness

if $u \in \underset{BV}{W^{1,1}} \Rightarrow u \in L^2$ Sobolev embeddings

losses of compactness are described

Th (P.L. conc. compactness) u_n bdd in $\overset{BV}{W^{1,1}} \underset{w, L^2}{\rightharpoonup} u$, $|Du_n| \rightharpoonup \mu \in BM$, $u_n^2 \rightharpoonup \nu \in BM$

then $\exists$ almost countable set J, distinct points $(\bar{x}_j)_{j\in J}$, reals $\nu_j > 0$

$$\nu = u^2 + \sum_{j\in J} \nu_j \delta_{\bar{x}_j} \qquad \mu \ge |Du| + c_0 \sum_{j\in J} \nu_j^{1/2} \delta_{\bar{x}_j} \qquad \sum \nu_j^{1/2} < \infty$$

(optimal, given u, ν_j, $\bar{x}_j$ one can build u_n!)

2D Euler even (possibly) worse since $\omega(t)$ bdd in BM $\not\Rightarrow$ Du bdded in BV

plausible : concentrations occur on large sets
however
Th (DiPerna-Majda) "concentration occurs on sets with $(1+\varepsilon)$ Hausdorff measure 0 $\forall \varepsilon>0$"
(precise statement is much more technical)

Final observation on 2D Euler eqs: using symmetrization results (Talenti ..)
one can refine the existence of weak sols to

Th (PLL) If $\omega_0 \in$ "$L^{1,2}$", $\exists$ global weak solution in L^2 of (E)

"$L^{1,2}$" $= \{\omega_0 \in L^1 / \int_0 \frac{1}{t}\left(\sup_{|A|\le t} \int_A |\omega_0|\,dx\right)^2 dt < \infty\}$

$\omega_0 \in L^{1,2} \Rightarrow \nabla_\perp(\Delta)^{-1}\omega_0 \in L^2$ (compact)

Final remark : $\cdot\ \frac{D}{Dt}$, $\frac{\partial}{\partial t} + u\cdot\nabla$, $\operatorname{div} u = 0$

$\cdot\ X(x,t)$ $\quad \frac{dX}{dt} = u(X,t)$, $X|_{t=0} = x$

particle trajectories (Lagr. form:)

2D "vorticity cst along the particle trajectories".

$\cdot$ standard understanding requires $u \in W^{1,\infty}$ (up to any technical refinements)
too much in general . Euler eqs
. Vlasov eqs
. NS eqs 3D

Leray $u \in L^\infty(0,T; L^2) \cap L^2(0,T; H^1)$
(far from $L^1(0,T; W^{1,\infty})$)
uniq. and regularity : major open pb of 3D NS .

③ Ordinary differential eqs and transport theory (DiPerna-PLL) ..

"bounded domain or bdd domain : vector field tangent on the bdry"

(ODE) $\dot X = B(X)$, $X(0,t) = x$

Cauchy-Lipschitz : $B \in W^{1,\infty}$

1) $\exists!$ flow $C_t(C_x)$ "L^∞_x" s.t., group $X(t+s,x) = X(t, X(s,x))$

2) stable under uniform perturbations of B

3) $\frac{\partial X}{\partial t} = B(x)\cdot\nabla_x X$ (Euler description)

now we want to relax the regularity $W^{1,\infty}$ to Sobolev classes : Lebesgue measure λ plays a role (other measures)

4) $e^{-C_0|t|}\lambda \le \lambda \circ X(t) \le e^{C_0|t|}\lambda$

$e^{-C_0|t|}\int \varphi(x)\,dx \le \int \varphi(X(x,t))\,dx \le e^{C_0|t|}\int \varphi(x)\,dx$

$C_0 = \|\operatorname{div} b\|_\infty$

Thm Let $B \in W^{1,p}$ $p \ge 1$, $\operatorname{div} B \in L^\infty$, $\exists!$ flow $C_t(L^p_x)$ satisfying (ODE)
and $X \in L^p_x(W^{1,p}_t)$, a.e. x $X \in C^1_t$, $B(X) \in C_t$ and $\dot X = B(X)$

2) B_n "smooth", X_n solves B_n ODE , $B_n \xrightarrow{L^p} B \in W^{1,1}$, $\operatorname{div} B \in L^\infty$

$X_n \underset{C_t(L^p_x)}{\longrightarrow} X$

3) $\frac{\partial X}{\partial t} = B(x)\cdot\nabla_x X$ (in renormalized form)

Same result holds if $B(x,t)$: $\operatorname{div} B \in L^1_t(L^\infty_x)$, $B \in L^1_t(W^{1,1}_x)$
Counterexample below $W^{1,1}$ (later on)
complete proof is much too long but main ideas:

one analyses the associated transport eq.

(T) $\frac{\partial w}{\partial t} = B \cdot \nabla_x w$

existence, uniqueness, Liouville ptes → (*), stability.

→ ODE's : $w(X(x,t),t) = w(x,0)$ a.e.

Three ingredients

(A) Renormalized solutions (B, V, h., FPB ...)

(B) Regularization trick

(C) Stability analysis

(A) " w cst along characteristics = particle paths "

" ⇕ "

" $F(w)$ cst ———————— V maps F. "

in other words " w sol of (T) " ⟺ " $F(w)$ sol of (T) "

Def Let $B \in L^1$, $w \in L^\infty_t(L^1)$ is a RN sol of T if $\forall F \in C^1_b$(bounded)

$$\iint F(w) \left\{ -\frac{\partial \varphi}{\partial t} - \mathrm{div}(B\varphi) \right\} dx\, dt = 0 \quad \forall \varphi \in C_0^\infty$$

Rk 1) If $Bw \in L^1_{t,x}$ ex. $w \in L^\infty_{t,x}$ $B \in L^1_{t,x}$ ...

then RN sol. ⇒ $\mathcal{D}'$ solutions

2) we will see in (B) that if $\mathrm{div}\, B \in L^1_t(L^\infty)$, $B \in L^1_t(W^{1,1})$, $w \in L^\infty_{t,x}$ then

$\mathcal{D}'$ sol ⇒ RN sol.

3) in general diff. notions.

(B) regularization trick

Lemma $B \in L^1_t(W^{1,1}_x)$, $u \in L^\infty_{t,x}$, $\frac{\partial u}{\partial t} = B \cdot \nabla_x u$ in $\mathcal{D}'$

Then, $u_\varepsilon = u * \rho_\varepsilon$ solves $\frac{\partial u_\varepsilon}{\partial t} = B \cdot \nabla_x(u_\varepsilon) + r_\varepsilon$ and $r_\varepsilon \xrightarrow[\varepsilon]{L^1} 0$.

⇒ i) $L^\infty_{x,t}$ $\mathcal{D}'$ sol. ⇒ RN sols

ii) uniq. of $L^\infty_{x,t}$ $\mathcal{D}'$ sol. , existence is easy for $\mathcal{D}'$ sols

iii) uniq. of RN sols

But existence of RN sols in L^1 requires some (little) extra work and stability

(C)

Thm: Let B_n smooth, assume that $B_n \xrightarrow[w\text{-}L^1_{x,t}]{n} B \in L^1_t(W^{1,1}_x)$ + (C), $\mathrm{div}\, B \in L^1_t(L^\infty_x)$

Let u_n solve $\frac{\partial u_n}{\partial t} = B_n \cdot \nabla u_n$, $u_n|_{t=0} = u_n^0$, $u_n^0 \xrightarrow{L^p} u^0$ $p \geq 1$

Then, $u_n \xrightarrow[n]{} u$ RN sol. of T , $u|_{t=0} = u^0$

in $C([0,T]; L^p)$

where (C) $\int_0^T \int |B_n(t, x+h) - B_n(t,x)|\, dx \xrightarrow[h \to 0]{} 0$ uniformly in n.

Rk (C) holds if B_n bdd in $L^q_t(W^{s,p}_x)$ $s>0$, $p \geq 1$, $q > 1$.

Sketch

$$\left\{ \begin{array}{ll} \frac{\partial}{\partial t} F_1(u_n) = B_n \cdot \nabla_x F_1(u_n) , & F_1(u_n) \to v_1 \\ \frac{\partial}{\partial t} F_2(u_n) = B_n \cdot \nabla_x F_2(u_n) , & F_2(u_n) \to v_2 \end{array} \right.$$

$$\left\{ \begin{array}{lll} \frac{\partial v_1}{\partial t} = B \cdot \nabla_x v_1 & \mathcal{D}' \to \text{RN sol.} & v_1|_{t=0} = F_1(u^0) \\ \frac{\partial v_2}{\partial t} = B \cdot \nabla_x v_2 & \text{————} & v_2|_{t=0} = F_2(u^0) \end{array} \right.$$

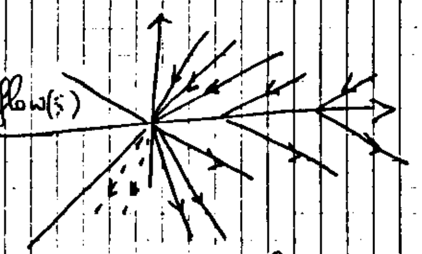
Hence if $F_2 = \beta \circ F_1$ β strictly convex , v_1, $\beta(v_2)$ solve the same eq (RN sol)

$v_2 = \beta(v_1)$ ⇒ $F_1(u_n) \xrightarrow[L^q\ \forall q<\infty]{} v_1$ F_1 one to one ... ∎

Counterexample : $\exists\, B \in W^{s,1}(\mathbb{R}^2)$ $\forall s<1$, $\operatorname{div} B = 0$
$(\cap L^\infty_{loc})$

$\nexists$ measure preserving L^1 flows (locally bounded in x, t) ...

trajectories of the flow(s)



blob transported by the flows

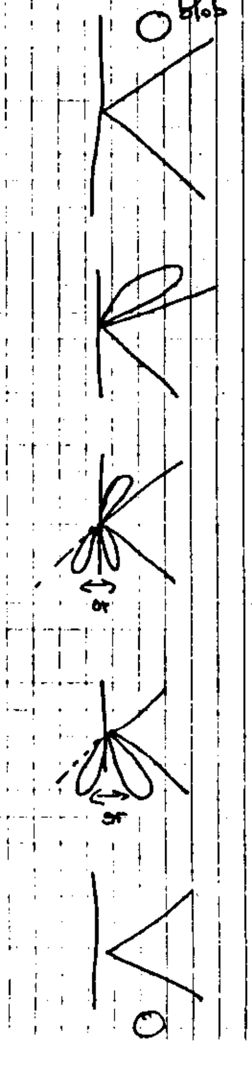
time 0

later

later

later

later



at all times same area

Rk "pbly" we can build another counterexample in BV

Open pb (Fundamental)
2D Euler flow converges with
$\to B \in BV$ (or L^1) ?
$\Rightarrow u \in L^2$ $(\nabla^\perp(-\Delta)^{-1}B)$
$B = \omega$

(4) 2D Euler eqs (and 2D NS) : a few new facts

2D Euler $\quad u \in L^\infty(0,\infty; W^{1,p})$ if $\omega_0 \in L^p$
3D NS $\quad u \in L^2(0,\infty; H^1)$ if $u_0 \in H^1$

Cor. 1 For each weak solution u, particle paths are uniquely defined in the above sen[se]
2D E : 1) $\exists$ weak solution such that
2) vorticity cst along particle trajectories
3) $u \in C([0,\infty); W^{1,p})$, $\omega \in C([0,\infty); L^p)$, u solves (E) in $\mathcal{D}'$
4) 2D NS $\xrightarrow[\nu\to 0]{}$ 2D Euler $\omega_0^\varepsilon \in L^p \xrightarrow[L^p]{} \omega_0 \Rightarrow u^\varepsilon \xrightarrow[C_t(W^{1,p})]{} u$, $\omega^\varepsilon \xrightarrow[C_t(L^p)]{} \omega$
5) vorticity eq. holds in Rd form
And 6) If $\omega_0 \in L^2$, every weak ($\mathcal{D}'$) sol. of (E) satisfies 2), 3), 5) and vorticity eq.

" "

Rk $\omega_0 \in L^1 \log L^1$, everything holds with L^p replaced by $L^1 \log L^1$ (harmonic anal[ysis])
Open pb : use these informations to prove convergence of vortex methods
methods
i) for weak solutions
ii) without cut-off ...

(5) 3D Euler eqs : blowup of weak solutions and instability of smooth solutio[ns]

periodic BC to simplify (more complicated in other cases but still true)

Thm 1 $\forall \varepsilon>0$, $\forall s \in (0,1]$, $\forall p \in [1,\infty)$, $\forall t_0 > 0$
$\exists$ smooth solution of 3D Euler equation : $\|u(0)\|_{W^{s,p}\cap L^2} \le \varepsilon$, $\|u(t_0)\|_{W^{s,p}} \geq \frac{1}{\varepsilon}$
$u(t)$
: $\|\omega(0)\|_{L^\infty} \le \varepsilon$, $\|\omega(t_0)\|_{L^\infty} \geq \frac{1}{\varepsilon}$
$\|u_0\|_{L^2} \le \varepsilon$
fact follows from

Thm 2 $\exists$ weak sol. of 3D Euler eq. $u \in C([0,\infty); L^2)$ limit of smooth solutions
such that $\bullet$ $\|u(0)\|_{W^{s,p}\cap L^2} \le \varepsilon$, $\|u(t)\|_{W^{s,p}} \uparrow +\infty$ $t \uparrow t_0$.
or
$\bullet$ $\|\omega(0)\|_{L^\infty}$, $\|u_0\|_{L^2} \le \varepsilon$, $\|\omega(t)\|_{L^\infty} \uparrow +\infty$ $t \uparrow t_0$.

(11)

1/2 D flows

→ u_1, u_2 (x_1, x_2) solve 2D Euler eq. $\bar{u} = (u_1, u_2)$

→ $u_3(x_1, x_2)$ $\quad \frac{\partial u_3}{\partial t} + \bar{u}.\nabla u_3 = 0$

pure transport : well posed even for weak solutions of 2D Euler eqs provided vorticity lies in L^p $p>1$

limit of smooth solutions

u_3 cst along particle trajectories i.e.

$$u_3(X(x,t),t) = u_3^0(x)$$

so whenever $x \longmapsto X(x,t_0)^{-1}$ is not Lipschitz

we have

$$u_3(x,t_0) = u_3^0 \circ X(t_0)^{-1}\ (x)$$

and one can show this produces one example as in Th 2

non Lipschitz characteristics are not rare : simplest example

rest flow $\quad u_1 \equiv 0,\ u_2^0(x_1)$

$$X_1(x,t) = x_1, \qquad X_2(x,t) = x_2 + t\,u_2^0(x_1)$$

$$u_3(x,t) = u_3^0(x_1, x_2 - t\,u_2^0(x_1))$$

$u_2^0 \in W^{1,p}$, $u_3^0 \in W^{1,p}$ $\not\Rightarrow$ $u_3 \in W^{1,p}$ $\forall t \geq 0$

easy examples are possible

(12)

Dreaming about 3D Euler eqs

Case 1 : either smooth solutions do not break down

(not really plausible , Open pb)

or

Case 2 : vorticity blows up in L^∞ norm in finite time.

but then our examples show that no estimates remain, so no clear mechanism for compactness in L^2 as we are

Case 2.1 : global L^2 solutions exist (weak solutions!)

or

Case 2.2 : global L^2 solutions disappear in finite time, concentrations occur and the equation is lost.

In fact, there is more : if one can get a uniq. counterex. with 2D flow as described above then it proves the

Conjecture : $\exists\, u_0 \in L^2$, $\operatorname{curl} u_0 \in L^2$ s.t. $\exists$ 2 distinct $C_t(L^2_x)$ solutions of 3D Euler eq.

(nonuniq , possibly non cv of NS to E as $\nu \to 0$ if 3D ("conj. s1")

"s1" for super

(6) Inhomogeneous, incompressible NS equations

$$(NS_\rho)\quad \begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 , \quad \rho \geq 0 & \text{2D or 3D} \\[2mm] \dfrac{\partial \rho u}{\partial t} + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\nu(\rho)\nabla u) + \nabla p = 0 \\[2mm] \operatorname{div} u = 0 \end{cases}$$

$$\rho|_{t=0} = \rho_0 , \qquad u|_{t=0} = u_0$$

$$\begin{cases} \nu \in C([0,\infty)) , \quad \nu(t) \geq \underline{\nu} > 0 \\ \rho_0 \in L^\infty , \ \rho_0 \geq 0 , \quad u_0 \in L^1_{loc} \quad \rho_0 |u_0|^2 \in L^1 \end{cases}$$

Thm $\exists$ solution of $(NS\rho)$ i) $\rho \in C_t(L^q_x)$ $(\forall q < \infty)$, $\int F(\rho(t))\,dx$ indep^t of t $\forall F$

ii) $\sqrt{\rho}\, u \in L^\infty_t(L^2_x)$, $u \in L^2_t(H^1_x)$

iii) $\int \rho |u|^2\,dx(t) + \int_0^t \int \nu(\rho) |\nabla u|^2\,dx \leq \int \rho_0 |u_0|^2\,dx$

Rk 1) Kajikov, Antonsev-Kajikov

$\nu(t) \equiv \nu$, $\rho_0 \geq \underline{\rho} > 0$ no vacuum, $\rho \in L^\infty_{t,x}$ instead of i)

2) If $\rho_0 = cst$, then $\rho(t) \equiv \rho_0$, $u(t)$ is a solution of the standard 2D or 3D NS eqs

3) If $\rho_0 = \rho_0^1 1_A + \rho_0^2 1_B \quad (+ \cdots)$

$\rho_0^1 \neq \rho_0^2 \geq 0$, A, B Borel sets

then $\rho(t) = \rho_0^1 1_{A(t)} + \rho_0^2 1_{B(t)}$ $\quad |A(t)| = |A|, \ |B(t)| = |B|$

$\Rightarrow$ no homogen. in ρ, on the diffusion (viscosity) coefficients

4) If $\rho_0 \geq \underline{\rho} > 0$, $\rho(t) \geq \underline{\rho} > 0$ --- 5) $\rho_0 \in L^q$ for q large OK too.

- approximate: ρ_n, u_n
- $\begin{cases} u_n \text{ bdd in } L^2_t(H^1_x) \\ \operatorname{div} u_n = 0 \end{cases}$ $\quad u_n \xrightarrow[w]{} u$

stability DE's $\Longrightarrow$ ρ_n converge to some ρ in $C_t(L^q_x)$, $\int F(\rho(t))$ indep^t of t.

- compactness in $L^2_{x,t}$ of $\sqrt{\rho}\, u$ (comp. in t is the difficulty, it usually comes from $\frac{\partial}{\partial t}(\cdots)$ but here ρ may vanish ....)

Rk similar applications to incompressible k-ε models (turbulence models)