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WORKSHOP ON THEORETICAL FLUID MECHANICS AND APPLICATIONS

(9 - 27 January 1989)

TRIPLE-DECK THEORY

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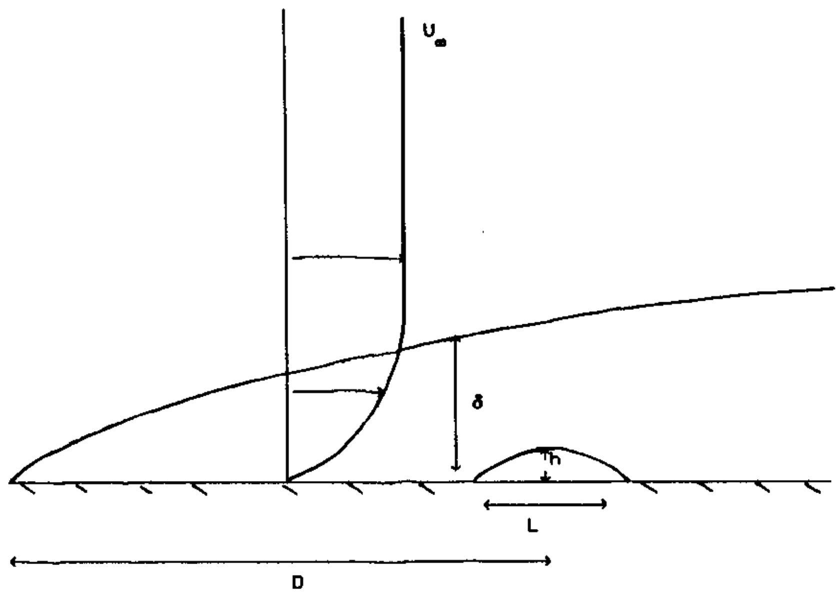
Triple-Deck Theory

Look at a model problem (many other examples).

Rivet on a flat plate (Smith 1973):

Use dimensional variables:



Many length scales.

- D = length from leading edge of plate
- δ = width of boundary-layer at rivet
- h = height of rivet
- L = length of rivet

U_∞ = free stream velocity, ν = viscosity, $\rho = 1$.

Coordinates: Fix at centre of rivet:

$$\text{Rivet:} \qquad \hat{y} = hF\left(\frac{\hat{x}}{L}\right) .$$

Sub-boundary layer:

Rivet will create a perturbation to the developing boundary-layer. Assume rivet small, i.e.

$$L \ll D, \qquad h \ll \delta.$$

(i) $L \ll D$ ensures that the oncoming flow is independent of $\hat{x}$ on a

(i) $L \ll D$ ensures that the oncoming flow is independent of $\hat{x}$ on a lengthscale L, i.e.

$$\hat{u} \sim U_\infty U_0\left[\frac{\hat{x}}{D}, \frac{\hat{y}}{\delta}\right] \approx U_\infty U_0\left[0, \frac{\hat{y}}{\delta}\right] \qquad \text{for } |\hat{x}| \ll D.$$

(ii) $h \ll \delta$ means that a small perturbation to the Blasius boundary-layer solution is expected.

Reynolds Number

Let $\qquad Re = \dfrac{U_\infty D}{\nu} =$ Reynolds number based on distance from the leading edge

Balance inertia terms and viscous terms in x-momentum equation

$$\hat{u}\hat{u}_{\hat{x}} \sim \nu \hat{u}_{\hat{y}\hat{y}}$$

$$\Rightarrow \quad \frac{U_\infty^2}{D} \sim \frac{\nu U_\infty}{\delta^2}$$

$$\Rightarrow \quad \delta = O\left(\frac{\nu D}{U_\infty}\right)^{1/2} = \frac{D}{Re^{1/2}} \ll D .$$

Let $\qquad R = \dfrac{U_\infty \delta}{\nu} = \dfrac{U_\infty D}{\nu Re^{1/2}} = Re^{1/2} =$ Reynolds number based on width of boundary-layer at rivet.

So $\qquad D = \delta R$.

Governing Equations

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} = 0 \qquad \Rightarrow \qquad \hat{v} = O\left[\frac{\hat{y}\hat{u}}{\hat{x}}\right] .$$

$$\hat{u}\hat{u}_{\hat{x}} + \hat{v}\hat{u}_{\hat{y}} = -\hat{p}_{\hat{x}} + \nu\hat{u}_{\hat{y}\hat{y}} + \cancel{\nu\hat{u}_{\hat{x}\hat{x}}}$$

$$\hat{u}\hat{v}_{\hat{x}} + \hat{v}\hat{v}_{\hat{y}} = -\hat{p}_{\hat{y}} + \nu\hat{v}_{\hat{y}\hat{y}} + \cancel{\nu\hat{v}_{\hat{x}\hat{x}}}$$

Seek *perturbation* to the Blasius boundary-layer profile. Since rivet small, assume perturbation small:

$$\hat{u} = U_\infty\left[U_0\left(\frac{\hat{x}}{D}, \frac{\hat{y}}{\delta}\right) + \epsilon U_1\left(\frac{\hat{x}}{L}, \frac{\hat{y}}{\delta}\right) + \dots\right]$$

↑ small perturbation ↓

$$\hat{v} = U_\infty\left[\frac{\delta}{D}V_0\left(\frac{\hat{x}}{D}, \frac{\hat{y}}{\delta}\right) + \epsilon\frac{\delta}{L}V_1\left(\frac{\hat{x}}{L}, \frac{\hat{y}}{\delta}\right) + \dots\right]$$

U_0 is oncoming velocity profile and satisfies

$$U_\infty^2(U_0 U_{0\hat{x}} + V_0 U_{0\hat{y}}) = \nu U_\infty U_{0\hat{y}\hat{y}} .$$

Hence leading order perturbation satisfies

$$\epsilon U_\infty^2\left[U_0 U_{1\hat{x}} + U_1 U_{0\hat{x}} + \frac{\delta V_1}{L}U_{0\hat{y}} + \frac{\delta V_0}{D}U_{1\hat{y}}\right] = -\hat{p}_{\hat{x}} + \nu U_\infty \epsilon U_{1\hat{y}\hat{y}}$$

$$\epsilon U_\infty^2\left[\frac{1}{L}, \quad \frac{1}{D}, \quad \frac{1}{L}, \quad \frac{1}{D}\right] : \frac{P}{L} : \frac{\nu U_\infty \epsilon}{\delta^2}$$

$$\frac{1}{D} \ll \frac{1}{L}, \qquad \frac{\nu U_\infty \epsilon}{\delta^2} = \frac{U_\infty^2 \epsilon}{R\delta} = \frac{\epsilon U_\infty^2}{L}\left(\frac{L}{D}\right) \ll \frac{\epsilon U_\infty^2}{L}$$

Assume that $\qquad \dfrac{P}{L} \ll \epsilon\dfrac{U_\infty^2}{L}, \qquad$ i.e. $\qquad P \ll \epsilon U_\infty^2$

– this will be verified subsequently.

Then with

$$\hat{y} = \delta Y , \qquad \hat{x} = LX$$

at leading order

$$U_0(0,Y)U_{1X} + V_1 U_{0Y}(0,Y) = 0, \qquad U_{1X} + V_{1Y} = 0 .$$

with general solution

$$V_1 = B(X)U_0(0,Y) , \qquad U_1 = -U_{0Y}\int B \, dX .$$

Note that expected inviscid condition

$$V_1 = 0 \quad \text{on } Y = 0 \qquad (h \ll \delta)$$

is automatically satisfied since $\qquad U_0(0,0) = 0.$

B(X) is undetermined. It is like an eigenvalue since it is the solution to a homogeneous equation with homogeneous boundary conditions. The freedom to choose B(X) is the key to triple-deck analysis.

To avoid integral signs, define

$$A(X) = -\int B\,dX \quad , \quad \text{so that}$$

$$U_1 = A(X)U_{0Y} \quad , \quad V_1 = -A_X U_0.$$

Properties of Solution

From scaling $U_0(\infty) = 1$, and write $\lambda = U_{0Y}(0,0)$.

Then

$$U_1 = \lambda A \quad , \quad V_1 = 0 \quad \text{on} \quad Y = 0.$$

(λA: slip velocity)

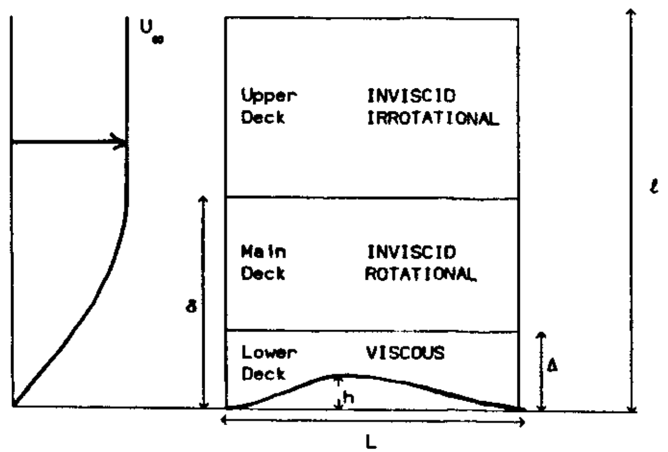
$$U \to 0 \quad , \quad V_1 \to -A_X \quad \text{as} \quad Y \to \infty.$$

Physically this solution represents a displacement of the Blasius boundary-layer solution downwards by εA - since

$$U_0(0, Y+\varepsilon A) = U_0(0,Y) + \varepsilon A U_{0Y}(0,Y) + \ldots .$$

Expect A to be *generally* negative if $F > 0$.

Scaling Arguments



Assume $h \le \Delta$, i.e. hump is in lower deck. Choose scales to give interesting interaction.

Lower Deck

1. Basic $\hat{u}$ velocity has magnitude:

$$\hat{u} = O\left[U_\infty U_0\left[0, \frac{\Delta}{\delta}\right]\right] = O\left(\frac{U_\infty \Delta}{\delta}\right).$$

(shear: $\frac{U_\infty}{\delta}$; height: Δ)

2. Assume perturbation velocity, $\tilde{u}$, is no greater than this basic velocity.

3. Motion of basic velocity over hump produces a vertical perturbation velocity

$$\tilde{v} = O\left(\left(\frac{U_\infty \Delta}{\delta}\right)\frac{h}{L}\right) \qquad \theta = \frac{h}{L}.$$

4. From continuity

$$\tilde{u} = O\left(\frac{\hat{x}\tilde{v}}{\hat{y}}\right) = O\left(\frac{L}{\Delta}\cdot\frac{U_\infty \Delta h}{\delta L}\right)$$

$$= O\left(\frac{U_\infty h}{\delta}\right)$$

↑ This is the magnitude of the basic velocity profile at the hill height.

5. Estimate b.l. thickness:

$$\hat{U}\hat{u}_{\hat{x}} \sim \nu \hat{u}_{\hat{y}\hat{y}}$$

$$\Rightarrow \quad \frac{U_\infty \Delta}{\delta}\cdot\frac{\tilde{u}}{L} \sim \frac{\nu\tilde{u}}{\Delta^2}$$

Hence

$$\Delta^3 = O\left(\frac{\nu}{U_\infty}L\delta\right)$$

$$\frac{\Delta^3}{\delta^3} = O\left(\frac{\nu L}{U_\infty \delta^2}\right) = O\left(\frac{1}{R}\frac{L}{\delta}\right).$$

So

$$\frac{\Delta}{\delta} = O\left(\frac{L}{R\delta}\right)^{1/3}.$$

6. Boundary-layer flows can be viewed as driven by pressure gradients. Hence expect pressure-gradient to be important in lower deck, i.e.

$$\hat p_{\hat x} \sim \hat u \hat u_{\hat x}$$

$$\frac{\tilde p}{L} = O\left(\left[\frac{U_\infty \Delta}{\delta}\right]\left[\frac{U_\infty h}{\delta}\right]\frac{1}{L}\right)$$

$$\tilde p = O\left(\frac{U_\infty^2 h\Delta}{\delta^2}\right).$$

$\tilde p$ does not vary in $\hat y$-direction across this layer if $\Delta \ll L$.

Main Deck

7. Velocity perturbation in lower deck matches slip velocity in middle deck. Hence

$$\tilde u = O\left(\frac{U_\infty h}{\delta}\right) \quad , \quad \text{i.e.} \quad \varepsilon = \frac{h}{\delta}$$

From continuity

$$\tilde v = O\left(\frac{\delta}{L}\frac{U_\infty h}{\delta}\right) = \frac{U_\infty h}{L}.$$

8.
$$\frac{\hat u\hat u_{\hat x}}{\hat p_{\hat x}} = O\left[U_\infty \cdot \frac{U_\infty h}{\delta}\cdot\frac{1}{L}\cdot\frac{1}{U_\infty^2 h\Delta/\delta^2 L}\right]$$

$$= O\left(\frac{\delta}{\Delta}\right).$$

Since $\Delta \ll \delta$ deduce that pressure gradient negligible in main deck. $\tilde p$ does not vary in $\hat y$-direction across this layer if $\delta^3 \ll \Delta L^2$.

Upper Deck

9. Vertical velocity perturbation out of main deck derives motion in this layer, hence

$$\tilde v = O\left(\frac{U_\infty h}{L}\right).$$

From continuity

$$\tilde u = O\left(\frac{U_\infty h}{\ell}\right).$$

10. What is the *new* effect in the upper deck?

Lower deck has viscosity

Main deck has 'eigenfunction' displacement

Upper deck has elimination of displacement by pressure effect

⇒ y dependence.

$$\hat u\tilde v_x \sim -\tilde p_{\hat y} \quad \Rightarrow \quad \hat p = O\left(\frac{U_\infty^2 h\ell}{L^2}\right)$$

$$\hat u\tilde u_x \sim -\hat p_{\hat x} \quad \Rightarrow \quad \hat p = O\left(\frac{U_\infty^2 h}{\ell}\right)$$

For the two pressure perturbations to be consistent require

$$\ell = O(1); \qquad p = O\left(\frac{U_\infty^2 h}{L}\right).$$

11. Assume pressure perturbation in upper deck is same as pressure perturbation in lower deck. INTERACTION ASSUMPTION

Then

$$\frac{U_\infty^2 h\Delta}{\delta^2} = O\left(\frac{U_\infty^2 h}{L}\right)$$

i.e. $\Delta L = O(\delta^2)$.

12. But $\Delta^3 = O\left(\frac{L\delta^2}{R}\right)$

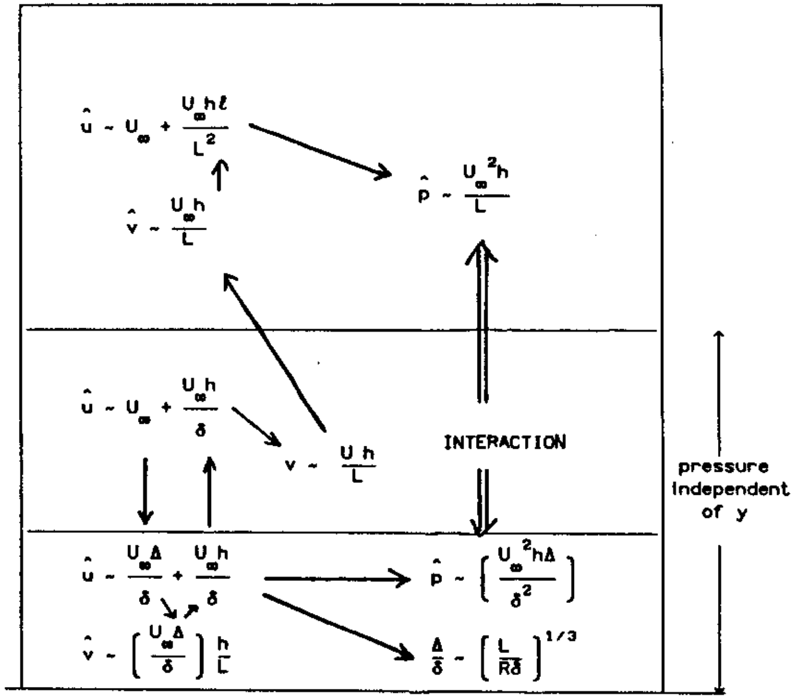
Hence $L = O(\delta R^{1/4})$, $\Delta = O(\delta R^{-1/4})$

i.e. $L \gg \delta \gg \Delta$

Also $\frac{L}{D} = O\left(\frac{R^{1/4}\delta}{R\delta}\right) = O(R^{-3/4})$

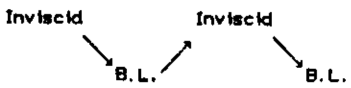
i.e. $L \ll D$.

$L = R^{1/4}\delta$ TRIPLE DECK SCALING



Interaction means that we have to solve upper deck at the same time as the lower deck!

This is different from 'classical' boundary-layer theory in which you solve



Summary

The displacement of the original boundary-layer caused by the presence of the hump produces pressure perturbations in the upper deck which are transmitted to the lower deck where they both interact with and help maintain the flow.

Derivation of the Governing Equations

Assume $h = O(\Delta)$, i.e. replace h by Δ in scalings.

Middle Deck

$$\hat{y} = \delta Y, \qquad \hat{x} = LX = R^{1/4}\delta X$$

$$\hat{u} = U_\infty(U_0(Y) + R^{-1/4}U_1(X,Y) + \dots)$$
$$\hat{v} = U_\infty R^{-1/2}V_1(X,Y) + \dots$$
$$\hat{p} = U_\infty^2 R^{-1/2}P_1(X,Y) + \dots$$

Continuity $\qquad U_{1X} + V_{1Y} = 0$

x-momentum $\qquad U_0 U_{1X} + V_1 U_{0Y} = 0$

y-momentum $\qquad P_{1Y} = 0$

$$U_1 = A(X)U_{0Y}, \qquad V_1 = -A_X(X)U_0, \qquad P_1 = P_1(X).$$

Upper Deck

$$\hat{y} = L\bar{y} = R^{1/4}\delta\bar{y}, \qquad \hat{x} = LX = R^{1/4}\delta X$$

$$\hat{u} = U_\infty(1 + R^{-1/2}\bar{u}_1(X,\bar{y}) + \dots)$$
$$\hat{v} = U_\infty R^{-1/2}\bar{v}_1(X,\bar{y}) + \dots$$
$$\hat{p} = U_\infty^2 R^{-1/2}\bar{p}_1(X,\bar{y}) + \dots$$

$$\left.\begin{aligned} \bar{u}_{1X} &= -\bar{p}_{1X} \\ \bar{v}_{1X} &= -\bar{p}_{1\bar{y}} \\ \bar{u}_{1X} + \bar{v}_{1\bar{y}} &= 0 \end{aligned}\right\} \quad \nabla^2\bar{p}_1 = 0.$$

Matching with middle deck $\qquad \bar{p}_1(X,0) = P_1(X)$

$\qquad \bar{v}_1(X,0) = -A_X(X)$

No disturbance at ∞ : $\quad \bar{p}_1 \to 0, \quad \bar{v}_1 \to 0 \quad$ as $\bar{y} \to \infty$.

Solve by Fourier Transform

$$P(X) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A'(\xi)}{X - \xi}\, d\xi$$

Lower Deck

$$\hat{y} = \Delta y = R^{-1/4}\delta y\,, \qquad \hat{x} = R^{1/4}\delta X$$

$$\hat{u} = U_\infty R^{-1/4}u_1 + \ldots$$

$$\hat{v} = U_\infty R^{-3/4}v_1 + \ldots$$

$$\hat{p} = U_\infty^2 R^{-1/2}p_1(X) + \ldots$$

Continuity $\quad u_{1x} + v_{1y} = 0$

x-momentum $\quad u_1u_{1x} + v_1u_{1y} = -p_{1x} + u_{1yy}$

y-momentum $\quad 0 = -p_{1y} \quad \Rightarrow \quad p_1 = p_1(X)$

Boundary conditions

$$u_1 = v_1 = 0 \qquad \text{on } y = \frac{h}{\Delta}F(X) = HF(X)$$

Match with middle deck

$$p_1(X) = P_1(X)$$

Middle deck : $\quad \hat{u} = U_\infty(U_0(Y) + R^{-1/4}U_1(X,Y) + \ldots)$

Lower deck : $\quad \hat{u} = \quad U_\infty R^{-1/4}u_1(X,y) + \ldots$

Let $\quad \hat{y} = d\eta \quad$ with $\quad \Delta \ll d \ll \delta$

then $\quad Y = \frac{\hat{y}}{\delta} = \frac{d}{\delta}\eta \ll 1$

$$y = \frac{\hat{y}}{\Delta} = \frac{d}{\Delta}\eta \gg 1$$

$$\hat{u} = U_\infty(U_0(0) + YU_0'(0) + \ldots + R^{-1/4}U_1(X,0) + \ldots) : \text{ from middle deck}$$

$$= U_\infty(\quad \frac{d\eta}{\delta}U_0'(0) + \ldots + R^{-1/4}U_1(X,0) + \ldots)$$

$$= U_\infty R^{-1/4}u_1(X, \frac{d}{\Delta}\eta) + \ldots \qquad : \text{ from lower deck}$$

For matching conclude

$$u_1(X, \frac{d}{\Delta}\eta) + \ldots = R^{1/4}\frac{d\eta}{\delta}U_0'(0) + \ldots + U_1(X,0) + \ldots$$

$$= \frac{d\eta}{\Delta}U_0'(0) + U_1(X,0) + \ldots$$

i.e.

$$u_1(X, y) \to yU_0'(0) + U_1(X,0) = U_0'(y)(y + A) \qquad \text{as } y \to \infty\,.$$

Far upstream require no disturbance:

$$u(X, y) \to U_0'(0)y \qquad \text{as } X \to -\infty\,.$$

Problem is (drop subscript 1)

$$u_x + v_y = 0\,, \qquad uu_x + vu_y = -P_x + u_{yy}$$
$$u = v = 0 \qquad \text{on } y = HF(X)$$
$$u \to U_0'(0)(y + A) \qquad \text{as } y \to \infty$$
$$u \to U_0'(0)y \qquad \text{as } X \to -\infty$$
$$P = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{A'(\xi)}{X - \xi}\,d\xi\,.$$

This is a nonlinear problem which requires a numerical solution. If $H \ll 1$, then linear solution can be found in terms of Airy functions. Numerical nonlinear results show that solutions with reversed flow can be found; the Hilbert integral effectively changes a parabolic problem into an elliptic one, so allowing upstream influence even when $u > 0$ far upstream.

Related triple-deck problems

Similar problems can be formulated when

(a) the outer flow is compressible, e.g. supersonic, hypersonic,

(b) for flow in a channel,

(c) for free surface flow down an inclined slope, etc..

Often the only difference is in the functional relationship between the pressure and displacement,

i.e.

$$P = P(A),$$

e.g.

$P = -A_x$	supersonic
$P = -A$	hypersonic
$P = -A_{xx}$	jet like flows
$A = 0$	short rivets, flow through symmetrically distorted channels.

Numerical Solution to Interactive (Triple Deck) Boundary-layer Equations with $A = 0$.

No influence upstream of start of hill at $X = 0$.

$$\text{Hill}: \quad F(x) = \begin{cases} 2\sin^3 \pi X & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

<u>Graph of Streamlines</u>.