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DYNAMICAL SYSTEMS, TURBULENCE AND THE NUMERICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS

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DYNAMICAL SYSTEMS, TURBULENCE AND THE NUMERICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS

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INTRODUCTION

Our aim in this article is to describe new numerical algorithms that are well suited for the solution of the Navier-Stokes equations over large invervals of time. Although we investigate here the case of incompressible flows, the methods can be extended to other flows such as thermohydraulics and reactive flows or, as well, compressible flows.

The understanding and the numerical solution of the Navier-Stokes equations are major problems of fluid mechanics with important implications in engineering and in fundamental research : the study of compressible or incompressible flows with or without reactive, electromagnetic or thermal phenomena necessitates among other things the understanding of the Navier-Stokes equations.

On the other hand the considerable increase of the computing power due to the appearance of supercomputers has made possible the solution of numerical problems that were unthinkable a few years ago. In the case of the Navier-Stokes equations we can now consider ranges of values of the physical parameters that are close to values physically relevant (at least in two space dimensions) and we can consider the onset of turbulence. In particular we can study turbulent flows that are time dependent, the laminar stationary solution being unstable ; and the computation of such flows necessitates clearly a large (theoretically infinite) time integration of the Navier-Stokes equations.

One of the first difficulties encountered in the solution of these equations and which appears even when the flow is laminar, is the treatment of the incompressibility condition

$$\text{div } u = 0. \qquad (0.1)$$

There are now several methods that are available for the treatment of (0.1) and that are well suited for computing stationary solutions and laminar flows. In particular, among the many important contributions of N.N. Yanenko to numerical analysis and computational fluid dynamics, the *fractional step method* (or splitting-up method) that he has introduced and contributed to develop is one of the classical available methods (see [RY][Y1,Y2]).

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The fractional step method has also laid A.J. Chorin [C2] and R. Temam [T3] to introduce the *projection method.* Also although this is far from the preoccupations of this article we should like to mention a recent application of the fractional step method to liquid crystal problems (see R. Cohen *et al.* [CHKL], R. Cohen [C0]) connected to the extension of the fractional step method to constrained optimization (see J.L. Lions and R. Temam [LT1][LT2]).

When we reach turbulent regimes new difficulties arise. In particular an essential aspect of turbulent flows is the relation/interaction between small and large eddies. All the frequencies of the spectrum, up to the Kolmogorov dissipation frequency k_d interact ; large eddies break into small eddies and those, in turn, feed the large eddies. Besides the usual difficulties (incompressibility, nonlinearity, large Reynolds number), a new difficulty occuring in computing high Reynolds fluid flows is the interaction of small and large eddies : small eddies are negligible at each given time but their cumulative effect is not negligible on a large interval of time. Thus *a proper and economical treatment of small eddies is necessary for large time computations.* The algorithms that we propose here, called nonlinear Galerkin methods, stem from recent developments in dynamical systems theory and are motivated by this preoccupation.

This article is organized as follows. In Section 1 we study the interaction of small and large eddies in a turbulent flow. In Section 2 we present the simplest nonlinear Galerkin method while Section 3 contains another (more involved) version of the method. Section 4 gives some theoretical justifications of the methods related to recent developments in the dynamical system approach to turbulence. Finally Section 5 presents the results of some *spectral* numerical computations based on the nonlinear Galerkin method in Section 2 ; these results show for a given accuracy, a significant gain in computing time (of the order of 20 % to 40 %).

CONTENTS.

1. Interaction of small and large eddies.
2. The nonlinear Galerkin method.
3. Another nonlinear Galerkin method.
4. Theoretical justification : attractors and inertial manifolds.
5. Numerical results.

1. INTERACTION OF SMALL AND LARGE EDDIES.

We consider in space dimension $n = 2$ or 3 the incompressible Navier-Stokes equations with density 1 :

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u.\nabla) + \nabla p = f, \qquad (1.1)$$

$$\nabla \cdot u = 0. \qquad (1.2)$$

Here u is the velocity vector, p the pressure ; $\nu > 0$ is the kinematic viscosity and f represents volume forces.

For the sake of simplicity we restrict ourselves to space periodic flows and assume that the average flow vanishes. Assuming that the period is 2π in each direction we expand u in Fourier series

$$u = \sum_{j \in Z^n} u_j e^{ij \cdot x}, \tag{1.3}$$

where $n = 2$ or 3, $j = \{j_1, j_2\}$or $\{j_1, j_2, j_3\}$and $u \in C^n$, $u_{-j} = \bar{u}_j$. Of course $u = u(x,t)$ and $u_j = u_j(t)$.

At each time t the kinetic energy of the flow $u(t)$ is

$$\frac{1}{2} \| u(t) \|^2 = \frac{1}{2} \int_\Omega | u(x,t) |^2 \, dx = \frac{(2\pi)^n}{2} \sum_{j \in Z^n} | u_j(t) |^2, \tag{1.4}$$

where Ω is the cube $(0, 2\pi)^n$. The enstrophy is

$$\| \nabla u(t) \|^2 = \int_\Omega | \operatorname{grad} u(x,t) |^2 \, dx = (2\pi)^n \sum_{j \in Z^n} | j |^2 | u_j(t) |^2, \tag{1.5}$$

$| j |^2 = | j_1 |^2 + | j_2 |^2$ or $| j_1 |^2 + | j_2 |^2 + | j_3 |^2$.

We denote by λ_r, $r \in N$, the eigenvalues of the Stokes operator, i.e. $| j |^2, j \in Z^n$, ordered in a sequential increasing order. We select a cut-off value m and decompose u into a finite part

$$y_m = P_m u = \sum_{|j|^2 \leq \lambda_m} u_j(t) e^{ijx}, \tag{1.6}$$

and a tail

$$z_m = Q_m u = \sum_{|j|^2 \geq \lambda_{m+1}} u_j(t) e^{ijx}. \tag{1.7}$$

Of course y_m corresponding to the small eigenvalues represent the *large eddies* of the flow, while z_m corresponding to the large eigenvalues of the flow represent the *small structures.*

By projection of (1.1) onto the space of divergence free functions, the pressure gradient disappears and we obtain the functional form of the equation, namely .

$$\frac{\partial u}{\partial t} - \nu \Delta u + B(u,u) = f \tag{1.8}$$

where for two vector fields v, w,

$$B(v,w) = \Pi\{(v.\nabla)w\}, \tag{1.9}$$

$\Pi\varphi$ corresponding to the divergence free component of φ in the Helmotz decomposition of a vector field φ.

Now we write

$$u = y_m + z_m$$

and project equation (1.8) (or (1.1)) onto the first m modes (small wavelengths) and onto the other modes (large wavelengths). This yields a coupled system of equations for y_m and z_m :

$$\frac{\partial y_m}{\partial t} - \nu \Delta y_m + P_m B(y_m + z_m, y_m + z_m) = P_m f \tag{1.10}$$

$$\frac{\partial z_m}{\partial t} - \nu \Delta z_m + Q_m B(y_m + z_m, y_m + z_m) = Q_m f. \tag{1.11}$$

Here P_m and Q_m are the Fourier series truncations corresponding to the modes $| j |^2 \leq \lambda_m$ and $| j |^2 \geq \lambda_{m+1}$ (see (1.6),(1.7)).

Behavior of small eddies.

Some computations which will not be reproduced here show that if m is sufficiently large, the small eddies component z_m carries only a small part of the kinetic energy (and of the enstrophy). More precisely after a transient time of the order of $\kappa(\nu\lambda_{m+1})^{-1}$, we have

$$\| z_m(t) \| \leq \begin{cases} K\delta L^{1/2} & \text{in dimension 2} \\ K\delta^{3/2} & \text{in dimension 3,} \end{cases} \tag{1.12}$$

where

$$\delta = \frac{\lambda_1}{\lambda_{m+1}}, \quad L = 1 + \log \frac{1}{\delta} \tag{1.13}$$

and K is a nondimensional number depending only on the Reynolds number Re, and of upper bounds M_0, M_1 of the kinetic energy and the enstrophy :

$$\begin{aligned} &\frac{1}{2} \| u(t) \|^2 \leq M_0, \ \forall t, \\ &\| \nabla u(t) \|^2 \leq M_1, \ \forall t. \end{aligned} \tag{1.14}$$

The Reynolds number is that based on the norm of f, $\| f \|$, which has the dimension of the square of the velocity, and the typical length 1 (or 2Π) :

$$Re = \frac{\| f \|^{1/2}}{\nu} \tag{1.15}$$

In fact, in the space-periodic case considered here the norm of $\| z_m(t) \|$ decays much faster than indicated in (1.12), namely at an exponential rate

$$\| z_m(t) \| \leq K_1 \exp\left(-K_2 m^{1/n}\right), \quad n = 2 \text{ or } 3, \tag{1.16}$$

This follows readily from the properties of Fourier series coefficient of an analytic function. Explicit values of K_1 and K_2 can be derived from [FT2].

A simplified interaction law.

A simplified interaction law of small and large eddies is obtained by taking into account the following facts :

(i) As recalled before z_m is small compared to y_m (which is of the same order as u itself) :

$$\| z_m \| << \| y_m \| \simeq \| u \| = \| y_m + z_m \| .$$

(ii) The relaxation time for the small eddies of the order of $(\nu\lambda_{m+1})^{-1}$, is much smaller than that of the large eddies (of the order of $(\nu\lambda_1)^{-1}$).

Taking into account these observations, we can simplify (1.10),(1.11). Due to (ii) it seems legitimate to neglect $\partial z_m/\partial t$ in (1.11) and, due to (i), to approximate $B(y_m + z_m, y_m + z_m)$ by $B(y_m, y_m)$. Hence (1.1) yields

$$-\nu\Delta z_m + Q_m B(y_m, y_m) \simeq Q_m f. \tag{1.17}$$

Relation (1.17) appears as an adiabatic (or quasistatic) law of interaction between small and large eddies.

Returning then to (1.10), we observe that

$$B(y_m + z_m, y_m + z_m) = B(y_m, y_m) + B(y_m, z_m) + B(z_m, y_m) + B(z_m, z_m),$$

and

$$\| B(y_m + z_m, y_m + z_m) \| \simeq$$
$$\simeq \| B(y_m, y_m) \| >> \| B(y_m, z_m) \| + \| B(z_m, y_m)) \| >> \| B(z_m, z_m) \| .$$

We could of course approximate again $B(y_m + z_m, y_m + z_m)$ by $B(y_m, y_m)$, but this has the undesirable effect of destroying the interaction of small eddies on the large one. Hence we can either leave (1.10) unchanged or replace it by

$$\frac{\partial y_m}{\partial t} - \nu\Delta y_m + P_m\{B(y_m, z_m) + B(z_m, y_m)\} + P_m B(y_m, y_m) \simeq P_m f. \tag{1.18}$$

More rigorous justifications of (1.17),(1.18) (or (1.18),(1.10)) were derived in [FMT]; the arguments will be sketched in Section 4. Also Section 5 contains some numerical evidences supporting this model. For the moment we shall show how one can infer from (1.17), (1.18) a new type of Galerkin approximation of the Navier-Stokes equations.

2. THE NONLINEAR GALERKIN METHOD.

The eigenfunctions for the Stokes operator are just the exponential functions

$$\alpha_j e^{ijx}, \quad j \in Z^n, \ \alpha_j \in R^n, \quad | \alpha_j | = 1.$$

In space dimension 2 we simply have

$$\alpha_j = \frac{\tilde{j}}{|j|}, \quad \tilde{j} = \{j_2, -j_1\}, \ j = \{j_1, j_2\}.$$

The corresponding eigenvalue is $| j |^2$ and, as indicated before, we number in sequential order the eigenvalues and the eigenfunctions ; for r integer we denote by w_r the eigenfunction corresponding to the eigenvalue λ_r.

For m fixed the Galerkin method based on $w_1, ..., w_m$, reduces to one of the spectral methods advocated in [GO][CHQZ]. With the notations above the approximate functions u_m is of the form

$$u_m(x,t) = \sum_{k=1}^{m} g_{km}(t) w_k(x) \tag{2.1}$$

and satisfies the approximate equation

$$\frac{\partial u_m}{\partial t} - \nu\Delta u_m + P_m B(u_m, u_m) = P_m f. \tag{2.2}$$

The nonlinear Galerkin method based on the simplified model (1.10),(1.12) proceeds as follows : we consider $2m$ eigenfunctions w_k and write (see (2.1)) :

$$u_{2m} = y_m + z_m \text{ (1)} \tag{2.3}$$

$$y_m(x,t) = \sum_{k=1}^{m} g_{km}(t) w_k(x), \quad z_m(x,t) = \sum_{k=m+1}^{2m} g_{km}(t) w_k(x). \tag{2.4}$$

The equations satisfied by y_m, z_m are (compare to (1.10),(1.11)) :

$$\frac{\partial y_m}{\partial t} - \nu\Delta y_m + P_m\{B(y_m) + B(y_m, z_m) + B(z_m, y_m)\} = P_m f \tag{2.5}$$

---

(1) Note that the quantities denoted here y_m, z_m are not the same as in Section 1.

$$-\nu\Delta z_m + (P_{2m} - P_m)B(y_m, y_m) = (P_{2m} - P_m)f. \tag{2.6}$$

At this point we emphasize the fact that (2.3),(2.4) is by no mean a modeling of turbulence : we are working with the full Navier-Stokes equations ; and *the point of view is that some terms have been indentified as (very) small and we just set them equal to 0 instead of ordering the computer to handle very small quantities.* The situation is reminescent of the uncomplete Cholesky factorization in linear algebra.

With an estimate similar to (1.12) one can show that at each given time, after a transient period, z_m is small compared to y_m. Hence $u_m(t)$ and $y_m(t)$ are very close at each time t. However, on a large interval of time, the effects of the z_m add up and produce significant changes ; this is one of the aspects of sensitive dependence to initial data which is well known in nonlinear dynamics.

We call the algorithm (2.5),(2.6) a nonlinear Galerkin method : if we set $z_m = 0$ in (2.5), then $u_m = y_m$ and (2.5) reduces to (2.2). Thus (2.5) appears as a modification of (2.2) with some nonlinear correction terms $z_m = z_m(y_m)$ which are "explicitly" determined in terms of y_m by resolution of the Stokes problem (2.6). Hence the terminology.

The convergence of the nonlinear spectral-Galerkin method (2.5),(2.6) is proved in [MT]. As far as computing time is concerned, for a spectral method with m modes, it is known that the computing time is proportional to m^2 (in space dimension 2). Hence the computing time for a usual Galerkin method with $2m$ modes is 4 times that of the computing time for m modes. A significant part of this computing time (about 1/3) is spent in the FFT computations related to the nonlinear terms. Since the nonlinear terms in (2.5),(2.6) involve only m modes, it is about 3/4 of the computing time on the nonlinear terms which is saved with the utilization of the nonlinear Galerkin method.

The results of numerical computations performed with this nonlinear Galerkin method are reported in Section 5.

3. ANOTHER NONLINEAR GALERKIN METHOD.

The nonlinear Galerkin method (2.5),(2.6) is based on the simplified form (1.17),(1.18) of the Navier-Stokes equations. This simplified form has been derived by observing that $\| z_m(t) \|$ is small compared to $\| y_m(t) \|$ like a power of

$$\delta = \frac{\lambda_1}{\lambda_{m+1}}$$

(see (1.12)) and, a sort of small perturbation method has been used ; (2.5),(2.6) correspond to the first order perturbation terms. Now, using asymptotic expansion technics we can envisage higher order perturbation terms leading to higher order perturbations ; this program has been carried out in [T5] and our aim now is to describe another simplified interaction law between small and large eddies and the corresponding nonlinear Galerkin method.

A second order perturbation equation consists in leaving (1.10) unchanged and replacing (1.11) by

$$-\nu\Delta z_m + Q_m\{B(y_m, y_m) + B(y_m, z_m) + B(z_m, y_m)\} \cong Q_m f. \tag{3.1}$$

By comparison with (1.11), the term $\partial z_m/\partial t$ has been dropped and, as well, $Q_m B(z_m, z_m)$.

For a nonlinear Galerkin (spectral) method, we change the notations and consider $u_{2m} = y_m + z_m$ as in (2.3),(2.4). The function y_m, z_m satisfy

$$\frac{\partial y_m}{\partial t} - \nu\Delta y_m + P_m B(y_m + z_m, y_m + z_m) = P_m f, \tag{3.2}$$

$$\begin{aligned} -\nu\Delta z_m + (P_{2m} - P_m)\{B(y_m, y_m) + B(y_m, z_m) + \\ + B(z_m, y_m)\} = (P_{2m} - P_m)f. \end{aligned} \tag{3.3}$$

Note that (3.3) is a linear equation for z_m, which gives z_m in terms of y_m by resolution of a linearized steady Navier-Stokes equation. We then insert $z_m = z_m(y_m)$ in (3.2) and obtain a nonlinear perturbation of the $m-$modes spectral method.

The convergence of this method as $m \to \infty$, has been proved in [MT]. In theory this method is more accurate than that of Section 3 (see [T5] and Section 4) ; it would be interesting to test the method and see if computational efficiency confirms the theoretical result.

Remark 3.1.

(i) Higher order methods corresponding to higher order perturbations can be also derived; they necessitate a proper approximation of $\partial z_m/\partial t$ (and $B(z_m, z_m)$) in (1.11).

(ii) For (3.2),(3.3) as well as for (2.5),(2.6) we have chosen to approximate z_m with as many modes (m) as y_m. However we could approximate z_m with $(d-1)m$ modes, $d > 2$, so that the approximation involves a total of dm modes. The choice $d = 2$ is convenient in FFTs and seems also appropriate.

4. THEORETICAL JUSTIFICATION : ATTRACTORS AND INERTIAL MANIFOLDS.

Although the practical value of a numerical method is decided by the quality of the actual computational tests, we would like here to give some indications on the theoretical aspects and motivations of the nonlinear Galerkin methods.

We are considering the Navier-Stokes equations (1.1),(1.2) with space periodic boundary condition and a driving force f which maintains the motion ; the force f is time independent. For simplicity we restrict ourselves to space dimension 2. If the Reynolds number Re defined in Section 1, based on f, is small then there exists a unique stationary (laminar) solution u_S to (1.1),(1.2) and each solution of these equations converges to the steady state as $t \to \infty$. It is also believed that if the Reynolds number is larger than a critical Reynolds number, then the stationary solution u_S loses its stability. In this case the solutions of (1.1),(1.2) never converge to a steady state and the flow remains constantly time-dependent (and turbulent).

The natural phase space for the problem is an infinite dimensional Hilbert space H, spanned by the eigenfunction w_k. A stationary solution is a point in H, while a time dependent soluton of (1.1),(1.2) is represented by a curve in H. In the laminar situation the orbit $u(t)$ converges to the unique stable stationary solution u_S. In the turbulent regime *the flow becomes time dependent* : even if the data are time independent, the flow never converges to a stationary state and remains time dependent. It was shown by Foias-Temam [FT1] that there exists a compact attractor $\mathcal{A}$ to which all the orbits converge. The attractor $\mathcal{A}$ appears as a substitute of u_S when u_S loses its stability; it is the mathematical object that represents the turbulent permanent regime.

It was shown in [FT1] that $\mathcal{A}$ has finite dimension. An explicit bound on this dimension was derived in [CFT1] for space dimension 3 and in [CFT2] for space dimension 2 ; in the latter case it reads :

$$\dim \mathcal{A} \leq Re^{4/3} \log Re. \tag{4.1}$$

Furthermore the estimate (4.1) is essentially optimal, since Babin-Vishik [BV] proved that the dimension can be $\geq Re^{4/3}$. Hence *the number of degrees of freedom of such a turbulent flow can be as large as* $Re^{4/3}$ and this agrees with Batchelor and Kraichnan results on two-dimensional turbulence [B][K].

For the computation of a permanent turbulent regime we may need a number of parameters of the same order as the number of degrees of freedom, i.e. the dimension of the corresponding attractor.

Since the attractor can be a complicated set (of fractal type), the idea in [FST] was to imbed it in a smooth finite dimensional manifold called an inertial manifold. This manifold which is positively invariant for the semigroup associated to the equation attracts all the orbits at an exponential rate. Using a decomposition of the space H similar to that used in the previous sections, namely $H = P_m H \oplus Q_m H$, $u = y + z$, $y = P_m u$, $z = Q_m u$, the inertial manifold $\mathcal{M}$ is a manifold of equation

$$z = \Phi(y). \tag{4.2}$$

When the manifold $\mathcal{M}$ exists, its equation (4.2) provides a quasistatic interaction law between small and large eddies. And this law applies in the permanent regime i.e. as soon as the orbits are close enough from the manifold $\mathcal{M}$.

Unfortunately the existence of inertial manifolds for the Navier-Stokes equations is not proved. A substitute and perhaps computationally more convenient concept is that

of approximate inertial manifolds (AIM) : these are manifolds that contains the attractor and attract all the orbits in a thin neighborhod.

For example a restatement of (1.12) is that the linear space $P_m H$ is an approximate inertial manifold for m sufficiently large. Less trivial AIMs were found in [FMT] and [T5]. The approximate inertial manifold $\mathcal{M}_0$ of [FMT] is precisely that appearing in (1.13) and of equation

$$-\nu\Delta z + Q_m B(y,y) = Q_m f. \tag{4.3}$$

The approximate inertial manifold $\mathcal{M}_1$ of [T] is that appearing in (3.1) and of equation

$$-\nu\Delta z + Q_m\{B(y,y) + B(y,z) + B(z,y)\} = Q_m f. \tag{4.4}$$

The thickness of the neighborhod of $\mathcal{M}_0$ and $\mathcal{M}_1$ attracting the orbits are determined in [T5] and are respectively in space dimension 2

$$\delta^{3/2} L \text{ and } \delta^2 L^{3/2}, \quad L = 1 + \log \frac{1}{\delta}.$$

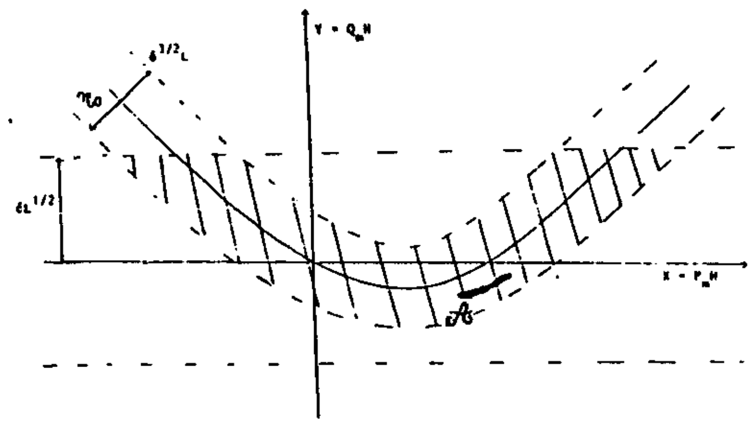

Figure 4.1.

Localization of the attractor $\mathcal{A}$ in the phase space : $\mathcal{A}$ lies in the dashed region

In conclusion, the attractor $\mathcal{A}$ is closer to $\mathcal{M}_1$ than it is to $\mathcal{M}_0$ and it is closer to $\mathcal{M}_0$ than it is to $P_m H$. Projecting the Navier-Stokes equations onto $P_m H$ we obtain the usual Galerkin method. Projecting the equations onto $\mathcal{M}_0$ or $\mathcal{M}_1$, we obtain the nonlinear Galerkin methods of Section 2 or 3.

5. NUMERICAL COMPUTATIONS.

We describe here the results of computational tests performed with the nonlinear Galerkin method (2.5),(2.6). Comparisons are also made with the usual Galerkin method (i.e. (2.2)). In both cases the two-dimensional Navier-Stokes equations with space periodicity boundary conditions are considered. The numerical results are due to C. Rosier [R].

The time discretization of the equations is performed in both cases with a predictor-corrector scheme. The predictor step is done by solving :

$$\tilde{y}^{n+1} = e^{-A\Delta t}y^n + \Delta t\sum_{j=0}^{p-1}\alpha_0^j P[f^{n-j} - b(u^{n-j},u^{n-j})] - \nabla\tilde{\Pi}_y^{n+1}$$

$$\nu A\tilde{z}^{n+1} = Q[b(y^n,y^n) - f^n] - \nabla\tilde{\Pi}_z^{n+1}$$

$$\tilde{u}^{n+1} = \tilde{y}^{n+1} + \tilde{z}^{n+1}.$$

The corrector step is the following

$$y^{n+1} = e^{-A\Delta t}y^n + \Delta t\alpha_1^0 P_m[f^{n+1} - b(\tilde{u}^{n+1},\tilde{u}^{n+1})]$$

$$+\sum_{j=1}^{p-1}\alpha_1^j P[f^{n+1-j} - b(u^{n+1-j},u^{n+1-j})] - \nabla\Pi_y^{n+1}$$

$$\nu A z^{n+1} = Q[b(y^{n+1},y^{n+1}) - f^{n+1}] - \nabla\Pi_z^{n+1}$$

$$u^{n+1} = y^{n+1} + z^{n+1}$$

where n is the time step, p is the order of the scheme, $\alpha_r^j = \int_0^1 e^{A(f-1)\Delta t}\Phi_r^j(f)df$, and Φ_r^j is a polynomial of degree $p-1$ such that $\Phi_r^j(1-k) = \delta_{jk}$, $j,k = 0,\dots,p-1$, $r = 0,1$.

The nonlinear terms are treated as usual by collocation and FFT.

Implicit in (2.5),(2.6) (and in the notation B) is the existence of a pressure like function associated to y_m and z_m; we denote by p_m and q_m the two pressure like functions, $p_m + q_m$ approximating the actual pressure function. Hence (2.5) and (2.6) are equivalent to

$$\frac{\partial y_m}{\partial t} - \nu\Delta y_m + P_m\{(y_m\cdot\nabla)(y_m+z_m) + (z_m\cdot\nabla)y_m + \nabla p_m\} = P_m f \qquad (5.1)$$

$$\nabla\cdot y_m = 0 \qquad (5.2)$$

$$-\nu\Delta z_m + (P_{2m} - P_m)\{(y_m\cdot\nabla)y_m + \nabla q_m\} = (P_{2m} - P_m)f \qquad (5.3)$$

$$\nabla\cdot z_m = 0 \qquad (5.4)$$

Two examples have been considered. In both cases the solution u is a priori chosen and $p = 0$, and f is determined from (1.1). Hence the exact solution of the equation is known and it is easy to test safely the accuracy.

In the first example (see Figures 5.1a,5.1b,5.1c), $m = 8$ and $u = \{u_1,u_2\}$,

$$u_1(x_1,x_2,t) = (\cos t)\,\varphi(x_2)$$

$$u_2(x_1,x_2,t) = (\cos t)\,\varphi(x_1)$$

$$\varphi(x_1) = 10^{-2}\left(x_1^2(x_1^2 - 4\Pi x_1 + 4\Pi^2) - \frac{8\Pi^4}{15}\right)$$

$$Re = 48.30$$

Figure 5.1a shows that the errors for both schemes are about the same as t evolves. Figures 5.1b and 5.1c show the absolute and relative gain in computing time between the usual and the present nonlinear Galerkin method ; here the relative gain of computing time is approximately 18% .

In the second example (see Figures 5.2a, 5.2b, 5.2c), $m = 32$, $Re = 1,704$ and $u = \{u_1,u_2\}$ is as in the first example with now $\varphi(x_1) = 10^{-1}(x_1^2(x_1^2 - 4\pi x_1 + 4\pi^2) - \frac{8\pi^4}{15})$. Figure 5.2a shows that the errors are about the same for both schemes as t evolves. Figures 5.2b and 5.2c show the absolute and relative gain in computing time between the usual and the present nonlinear Galerkin method ; here the relative gain in computing time is approximately 40% .

Finally, Figures 5.3a,b,c, are numerical evidences supporting the small-large eddies model described in Section 1. They give also a posteriori justifications of the nonlinear Galerking method proposed here. The computations were made for the first example above. Figure 5.3a shows the evolution of the ratio

$$|\,B(y^{n+1},y^{n+1})\,|_{L^2} \,/\, |\,(z^{n+1} - z^n/\Delta t)\,|_{L^2}$$

along the iterations ; the ratio is of the order of 10^5. Figure 5.3b shows the evolution of the ratio

$$|\,B(y^n,y^n)\,|_{L^2} \,/\, |\,B(y^n,y^n) + B(z^n,y^n) + B(z^n,z^n)\,|_{L^2}$$

Finally Figure 5.3c shows the evolution along the iterations of the ratio

$$|\,(y^{n+1} - y^n)/\Delta t\,|_{L^2} \,/\, |\,(z^{n+1} - z^n)/\Delta t\,|_{L^2}$$

The ratio is of order 10^4 ($\Delta t = 10^{-3}$ in the three figures).

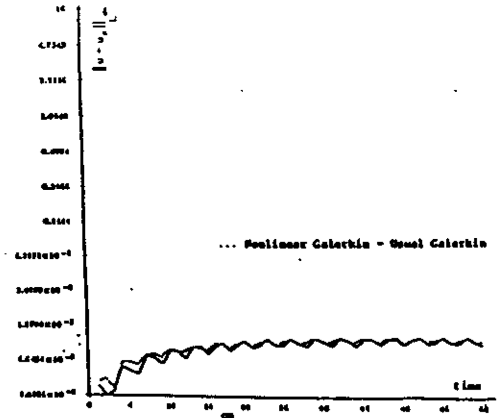

Figure 5.1a. Comparison of L^∞ norm of the error for the nonlinear Galerkin scheme and the usual Galerkin scheme

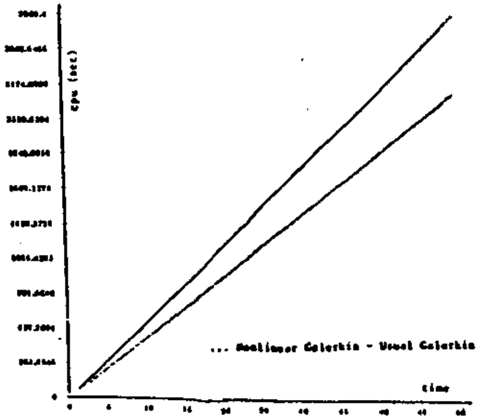

Figure 5.1b. Comparison of CPU time for the nonlinear Galerkin scheme and the usual Galerkin scheme

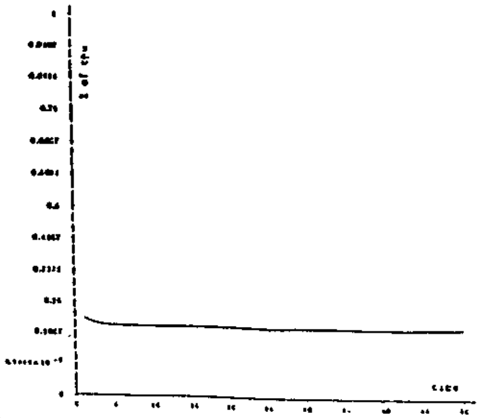

Figure 5.1c. Gain of CPU time for the nonlinear Galerkin scheme over the usual Galerkin scheme.

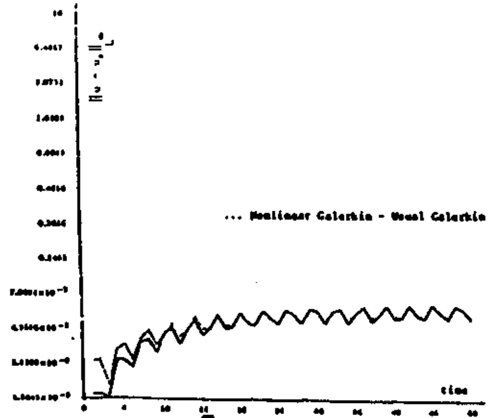

Figure 5.2a. Comparison of L^∞ norm of the error for the nonlinear Galerkin scheme and the usual Galerkin scheme.

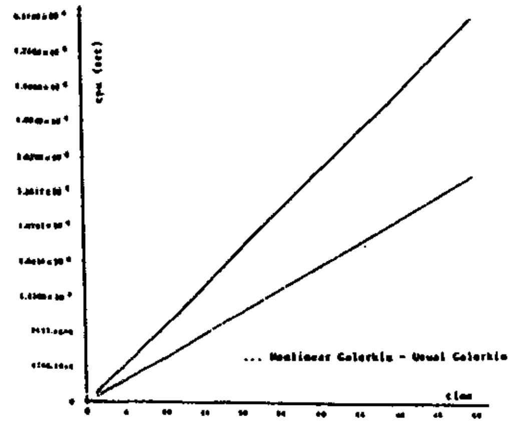

Figure 5.2b. Comparison of CPU time for the nonlinear Galerkin scheme and the usual Galerkin scheme.

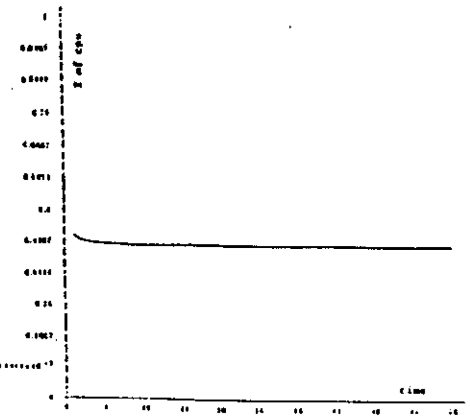

Figure 5.2c. Gain of CPU time for the nonlinear Galerkin scheme and the usual Galerkin scheme

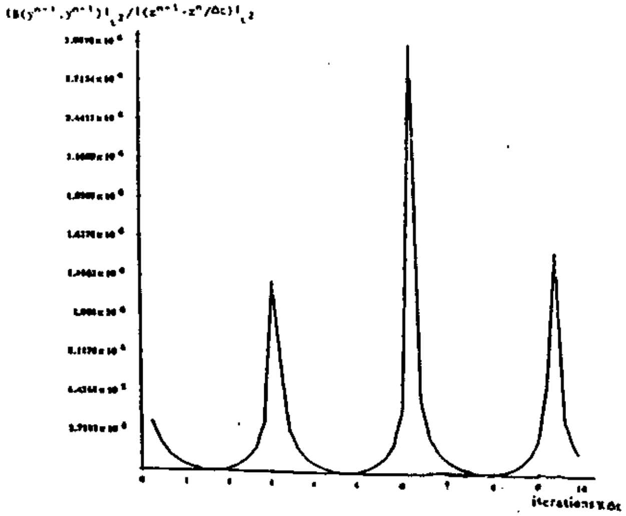

Figure 5.3a
Evolution of the ratio
$|B(y^{n+1},y^{n+1})|_{L^2}/|(z^{n+1}-z^n/\Delta t)|_{L^2}$
along iterations

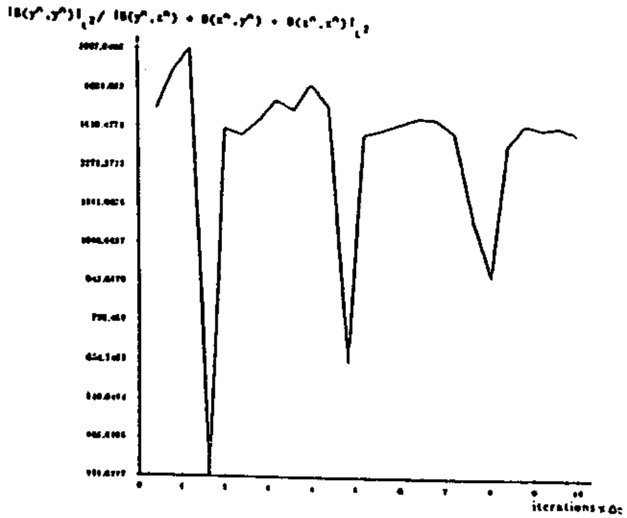

Figure 5.3b

Evolution of the ratio
$|B(y^n,y^n)|_{L^2}/\ |B(y^n,z^n) + B(z^n,y^n) + B(z^n,z^n)|_{L^2}$
along iterations

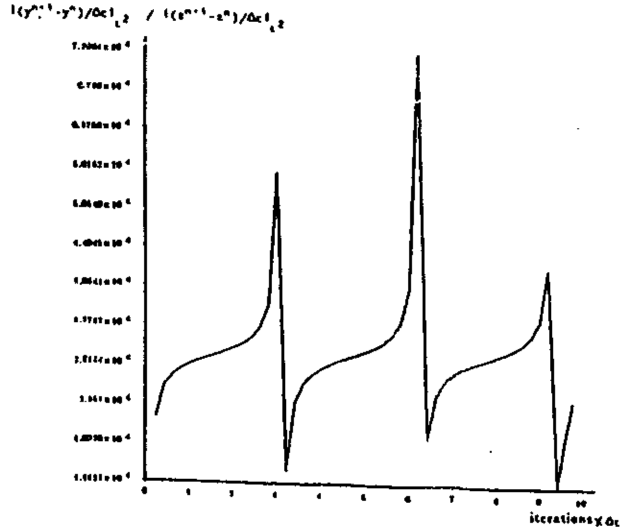

Figure 5.3c
Evolution of the ratio
$|(y^{n+1}-y^n)/\Delta t|_{L^2}\ /\ |(z^{n+1}-z^n)/\Delta t|_{L^2}$
along iterations

CONCLUSIONS.

In this article we have presented a new discretization algorithm called the nonlinear Galerkin method. The algorithm seems robust and well suited for large time integration of the Navier-Stokes equations. The preliminary numerical tests show a substantial gain in computing time. Further numerical experiments and the extension of the method to other equations and to other forms of discretization (finite elements, finite differences...) will be presented elsewhere.

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