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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

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Case Study: Independent and Combined Coding and Modulation

Reginaldo Palazzo Jr.

FEE-UNICAMP, Dept. Telematica, Campinas, SP, Brazil

CASE STUDY: INDEPENDENT AND COMBINED CODING AND MODULATION

by

Reginaldo Palazzo Jr. (*)

In a previous lecture on error rates for M-ary systems no coding analysis was done. We intended to leave to this one since a better understanding and evolution of the techniques for efficient modulation for band-limited channels could be seen properly.

We start by considering the case of independent coding and modulation. Fig. 1, shows the model of a communication system where the channel is memoryless. When the channel has memory a proper interleaver/deinterleaver pair is used between the encoder-modulator and the demodulator-decoder, respectively.

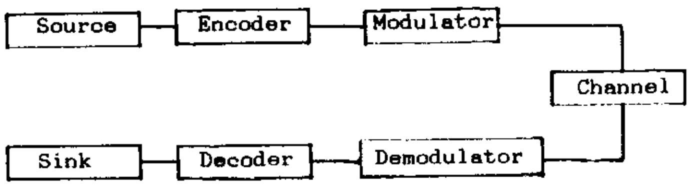

Fig. 1 - Model of a Communication System.

Without any loss of generality, assume the channel is memoryless. Fig. 2, shows this model.

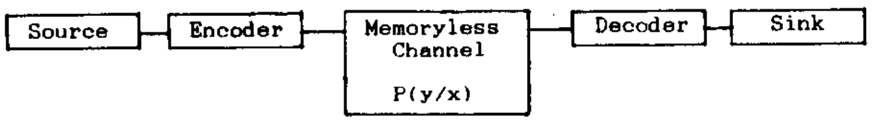

Fig. 2 - Discrete Memoryless Channel.

(*) - FEE-UNICAMP, Dept Telematica, P.O.Box 6101, 13081 Campinas, SP, Brazil.

A coded symbol sequence of length N is denoted by

$$\mathbf{x} = (x1,\ x2,\ \ldots,\ xN)$$

with the corresponding channel output sequence

$$\mathbf{y} = (y1,\ y2,\ \ldots,\ yN)$$

Assuming interleaving/deinterleaving for channels with memory, which render them memoryless, it is possible to have the channel probabilities to satisfy

$$P(\mathbf{y}/\mathbf{x}) = \prod_{i=1}^{N} P(yi/xi)$$

For any coded communication, the decoding process uses a metric of the form $m(\mathbf{y},\mathbf{x})$. When the metric is additive the decoding process is the simplest one. By additive, it is meant that

$$m(\mathbf{y},\mathbf{x}) = \sum_{i=1}^{N} m(yi,xi)$$

As an example, the maximum likelihhod metric is given by

$$m(\mathbf{y},\mathbf{x}) = \log P(\mathbf{y}/\mathbf{x})$$

This is the most important metric used in the decoding process.

Now, let us derive the coded bit error rate bound. For that, assume that there are only two possible coded sequences of length N, that is

$$\mathbf{x} = (x1,\ x2,\ \ldots,\ xN) \quad \text{and} \quad \hat{\mathbf{x}} = (\hat{x}1,\ \hat{x}2,\ \ldots,\ \hat{x}N)$$

When the channel output sequence is $\mathbf{y} = (y1,\ y2,\ \ldots,\ yN)$, the decoder decides that $\hat{\mathbf{x}}$ was the transmitted sequence if

$$m(\underline{y},\hat{\underline{x}}) = \sum_{i=1}^{N} m(yi,\hat{x}i) \geq \sum_{i=1}^{N} m(y,x) = m(\underline{y},\underline{x})$$

otherwise it decides for x.

Assuming x is the actual transmitted coded sequence, the probability that the decoder incorrectly decides $\hat{x}$ is

$$P\ (\ x \rightarrow \hat{x}\ ) = Pr\ \{\ m(\underline{y},\hat{\underline{x}}) \geq m(\underline{y},x)\ /x\ \}$$

This is called the pairwise error probability. By use of the Chernoff bound, we have

$$P(x \rightarrow \hat{x}) \leq \prod_{i=1}^{N} E\{\ \exp(c[m(yi,\hat{x}i) - m(yi,xi)])\ /x\ \}$$

for any $c \geq 0$. When $\hat{x}i = xi$, the average value is 1. When $\hat{x}i \neq xi$

$$D(c) = E\{\ \exp(c[m(yi,\hat{x}i) - m(yi,xi)])\ /x\ \}$$

Thus, the pairwise error probability is bounded by

$$P(x \rightarrow \hat{x}) \leq [\ D(c)\ ]^{\omega(\underline{x},\hat{\underline{x}})}$$

where $w(x,\hat{x})$ is the number of places that $\hat{x}i \neq xi$. Note that the objective is to have the least upper bound. So, let D be

$$D = \min_{c \geq 0}\ \{\ D(c)\ \}$$

Thus,

$$P(x \rightarrow \hat{x}) \leq D^{\omega(\underline{x},\hat{\underline{x}})}$$

The parameter D depends only on the coding channel and the choice of the metric. When the metric is the maximum likelihood one, D is given by

$$D = \sum_{y} (P(y/x).P(y/x))^{1/2}$$

Therefore, the Bhatacharyya bound is given by

$$P(x \rightarrow \hat{x}) \leq \left\{ \sum_{y} (P(y/x).P(y/x))^{1/2} \right\}^{\omega(\underline{x},\hat{\underline{x}})}$$

The pairwise error probability is the basis of general bit error bounds for coded communication systems. This is based on the union bound where the bit error probability is upper bounded by the sum of the probabilities of all ways a bit error can occur.

For any two coded sequences x and $\hat{x}$ let $a(x,\hat{x})$ be the number of bit errors occuring when x is transmitted and $\hat{x}$ is chosen by the decoder. If $p(x)$ is the probability of transmitting sequence x then the coded bit error bound is given by

$$Pb \leq \sum_{\underline{x},\hat{\underline{x}} \in C}\sum a(x,\hat{x}).p(x).P(x \rightarrow \hat{x})$$

$$\leq \sum_{\underline{x},\hat{\underline{x}} \in C}\sum a(x,\hat{x}).p(x).D^{\omega(\underline{x},\hat{\underline{x}})}$$

where C is the set of all coded sequences.

Example: suppose BPSK is the modulation to be used in an AWGN channel. Assume the additive component of the noise has zero-mean and variance No/2. Also, assume that there is no quantizer, then the conventional ML metric is $m(y,x) = y.x$, which is referred as a soft decision channel. If x is the transmitted symbol then

$$y = x + n$$

and

$$D(c) = E\{\ \exp(c[y.(\hat{x} - x)])\ /x\ \}$$

$$= \exp\{-2cEs + c^2.Es.No\}$$

or

$$D = \min_{c \geq 0}\ \{\ \exp\{-2cEs + c^2.Es.No\}\ \} = \exp\{-Es/No\}$$

Now suppose that we include a two level quantizer forcing a hard decision to be made. This results in a BSC where the coded symbol error probability is

$$p = \Pr\{ n \geq \sqrt{Es} \} = Q(\sqrt{2.Es/No})$$

For the hard decision channel, the ML metric is

$$m(y,x) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}$$

Hence,

$$D(c) = E\{ \exp(c[m(y,\hat{x}) - m(y,x)]) / x \}$$

$$= p.\exp(c) + (1-p).\exp(-c)$$

and

$$D = \min_{c \geq 0} \{ p.\exp(c) + (1-p).\exp(-c) \}$$

$$= \sqrt{4.p(1-p)}$$

Now, suppose we use a convolutional encoder with constraint length K = 7 and rate r = 1/2 with BPSK or QPSK modulation. Using the transfer function technique the bit error probability of this convolutional code is upper bounded by

$$Pb \leq 18.D^{10} + 105.D^{12} + 702.D^{14} + \ldots$$

and so, we have the coded bit error probabilities for soft decision given by

$$Pb \leq 18.\exp(-5.Eb/No) + 105.\exp(-6.Eb/No) + 702.\exp(-7.Eb/No) + .$$

and for hard decision given by

$$Pb \leq 18.(4.p(1-p))^{10} + 105.(4.p(1-p))^{12} + 702.(4.p(1-p))^{14} + \ldots$$

where $p = Q(\sqrt{2.Es/No})$ with $Es = Eb/r$.

The uncoded bit error probability for the BPSK or QPSK is

$$Pb \leq (1/2).\exp[-(1/2).Eb/No]$$

Comparison with the coded bit error probability, we concluded that by use of coding there is a reduction on the average bit energy-to-noise ratio to achieve the same error rate. However, from the asymptotic limit established by Shannon, $C = W.\log\{ 1 + Eb/No \}$, we see that by fixing C, a reduction in Eb/No implies in an increase in bandwidth. This is precisely the trade-off by using coding. In general, the bandwidth expansion factor is 1/r.

From what we just saw, it seems that coding has its applications in communication systems were no constraint is imposed on the bandwidth. However, this is not the case. Recently, a proposed technique using combined coding and modulation allowed the use of coding in band-limited channels without expanding bandwidth and achieving good coding gains. This will be our next topic.

In the combined coding and modulation both the encoder and the modulator are not independent. They have to be designed together in order to achieve good coding gains. It is better to take an example to explain this new procedure. Let us consider a four-state trellis code for 8-PSK modulation. Here, 2 bits of information is required for transmission. Thus, 4 signal points would suffice. However, 8 signal points will be used. This is the expansion of the signal set needed not to produce bandwidth expansion. In Fig. 3, it is shown the signal sets and the trellis diagram for the example.

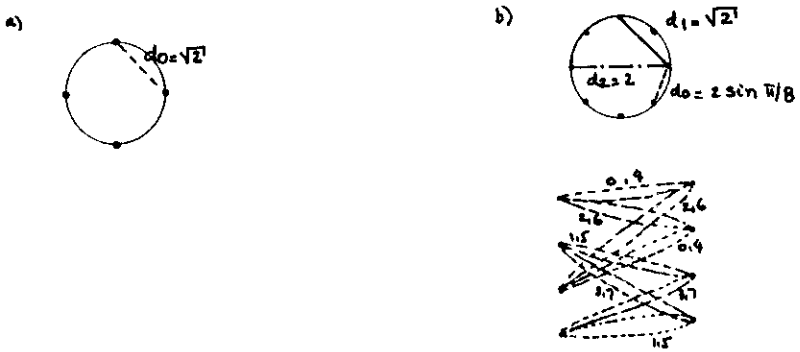

Fig. 3 - a) uncoded 4-PSK; b) 4 state trellis coded 8-PSK

The model of the transmitter of the communication system is as shown in Fig. 4.

Fig. 4 - Transmitter Model.

The mapper is just a mathematical function which maps the encoder output to the signal points. How should one design the matched encoder-modulator to achieve good coding gains? This is still an open question. However, the mapping by set partitioning concept is the key to the solution of the question just posed.

The mapping by set partitioning of an 8-PSK modulation is shown in Fig. 5.

The encoder is determined by choosing any level in this binary tree. For instance, at level 1, we see that in each subset there are 4 signal points. This means that the two bits coming into the encoder will not be encoded and just one bit will be concatenated to decide which subset at level 1 the encoder will use. The resulting encoder will be like the one shown in Fig. 6.

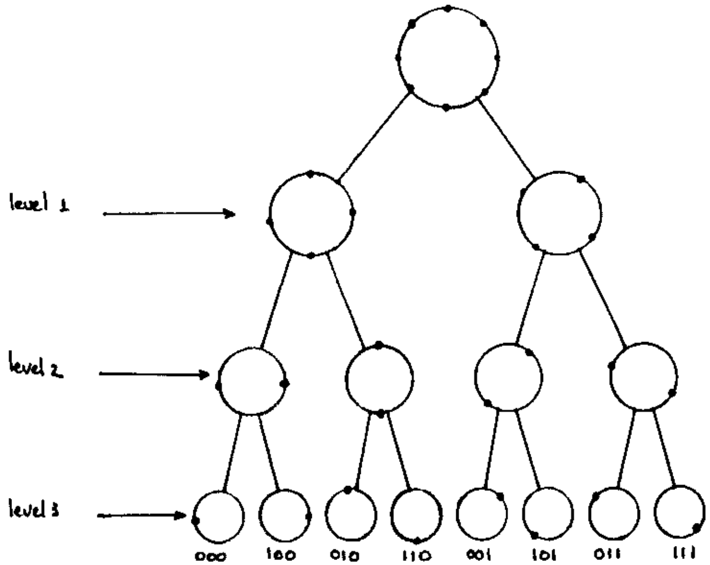

Fig. 5 - Mapping by set partitioning.

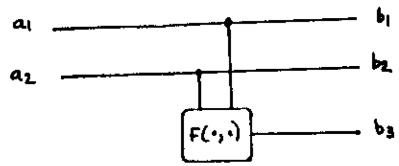

Fig. 6 - Encoder for level 1

At level 2, we see that in each subset there are two signal points. This means that one of the two bits entering the encoder will go directly through it and the other one will be used to feed a convolutional encoder with rate r = 1/2. This is shown in Fig. 7.

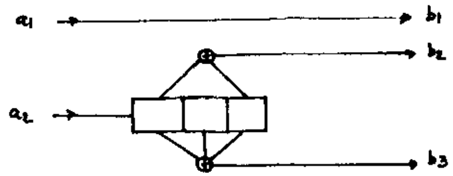

Fig. 7 - Encoder for level 2.

At level 3, we see that in each subset there is just one signal point. This means that the bits coming in will all be encoded. Some of the possible encoders configurations are shown in Fig. 8.

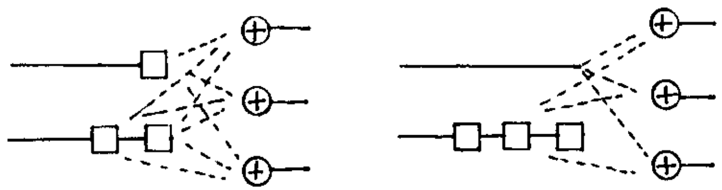

Fig. 8 - Encoder for level 3.

Now a search for good codes must be done for each one of the levels indicated above in order to find the encoder which gives the highest Euclidean free distance. The encoder shown in Fig. 9 is the best for level 2.

The asymptotic coding gain (between the coded 8-PSK and the uncoded 4-PSK) is defined as

$$CG = -20.\log\{ d_{free} / do \}$$

For the case being described, we have a 3 dB coding gain since $d_{free} = 2$ and $do = \sqrt{2}$

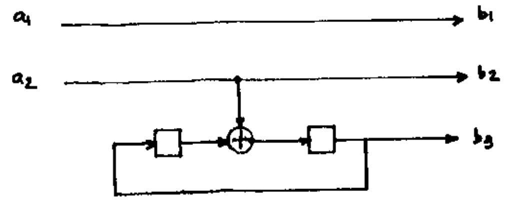

Fig. 9 - The best encoder for level 2.