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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) · P.O.B. 586 · MIRAMARE · STRADA COSTIERA 11 · TELEPHONE: 2240-1
CABLE: CENTRATOM · TELEX 460392 · I

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COURSE ON BASIC TELECOMMUNICATIONS SCIENCE

9 January - 3 February 1989

Fourier Integral

J. LUCAS

University of Sydney, Dept. of Electrical Eng., Australia

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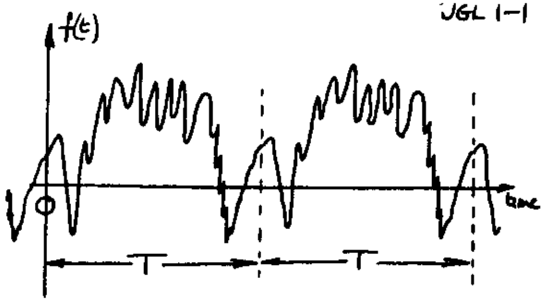
Start with Fourier Series.



Any periodic function of time with period T

ie $f(t) = f(t \pm KT)$
K integer.

can be written as a Fourier Series:

ie. $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n 2\pi f_0 t) + \sum_{n=1}^{\infty} b_n \sin(n 2\pi f_0 t)$$

& the basic (or "fundamental") $f_0 = \frac{1}{T}$ * COMMON INVERSE RELATION.

a_n & b_n are then found from simple integrals:

eg $$a_n = \frac{2}{T}\int_{t'}^{t'+T} f(t) \cos(n 2\pi f_0 t)\, dt$$

Now we can get a more compact form of this Fourier Series by expressing:

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \quad \& \quad \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

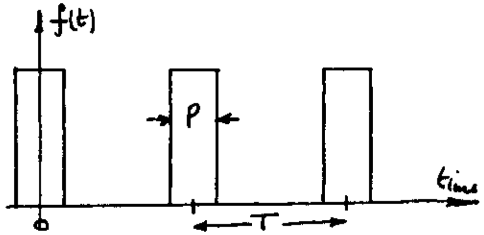
So that: $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left\{ \frac{e^{jn2\pi f_0 t} + e^{-jn2\pi f_0 t}}{2} \right\} + \sum_{n=1}^{\infty} b_n \{ \}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_0 t}$$

where $$C_n = \frac{1}{2}(a_n - j b_n) = \frac{1}{T}\int_{t'}^{t'+T} f(t) e^{-jn2\pi f_0 t}\, dt.$$

Let's now apply this compact Fourier Series to a simple numerical example:

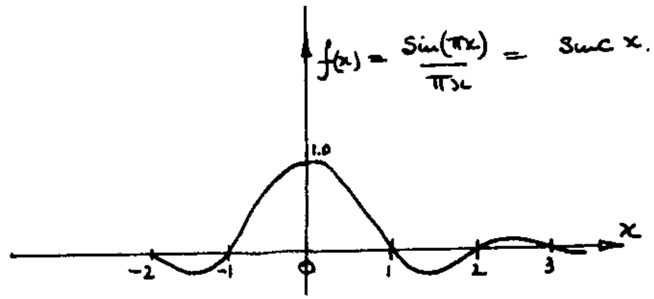
A repetitive pulse wave form.

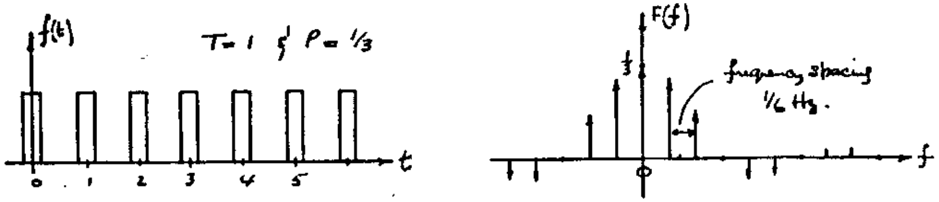


We find that $$C_n = \frac{P}{T} \frac{\sin\left(\frac{n\pi P}{T}\right)}{\left(\frac{n\pi P}{T}\right)}$$

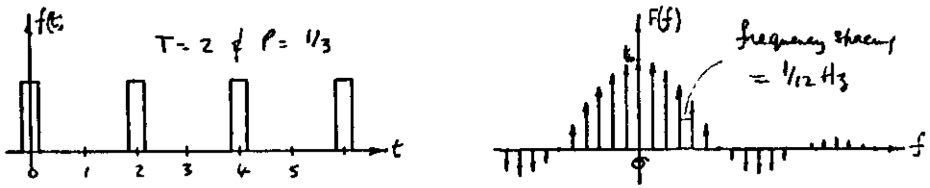
This has the form $\frac{\sin(\pi x)}{\pi x}$ which is called 'Sinc x'

It has the following infinite form:

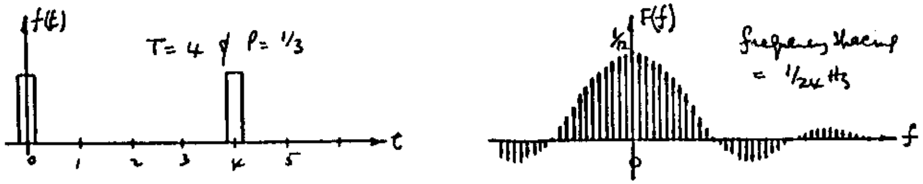




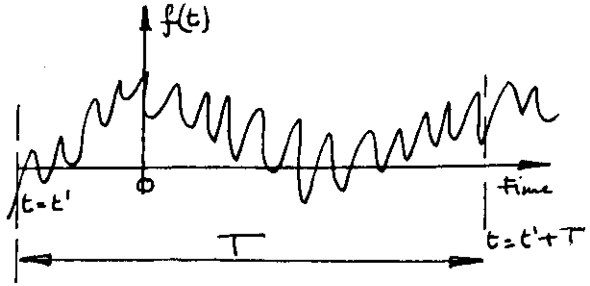
Removing every second pulse:



Removing every second pulse:



Now consider some arbitrary time function which is not periodic.



Choose a segment T seconds long to be "typical" of this signal.

force periodicity.

Apply our Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_0 t} \qquad \text{again } f_0 = 1/T$$

$$\text{where } C_n = \frac{1}{T}\int_{t'}^{t'+T} f(t)\, e^{-jn2\pi f_0 t}\, dt$$

we might just as well make the segment symmetrical about the origin ie $C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn2\pi f_0 t}\, dt.$

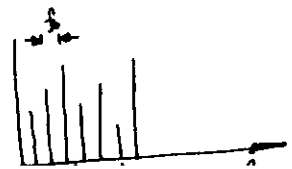
So substituting we have:

$$f(t) = \sum_{-\infty}^{\infty} \left\{ \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn2\pi f_0 t}\, dt \right\} e^{jn2\pi f_0 t}$$

now let $T \to \infty$ $\frac{1}{T}$ has the dimensions of frequency call it Δf

$$\text{So } f(t) = \sum_{-\infty}^{\infty} \left\{ \int_{-T/2}^{T/2} f(t)\, e^{-jn2\pi f_0 t}\, dt \right\} e^{+jn2\pi f_0 t}\, \Delta f$$

as $T \to \infty$ $\Delta f \to df$ & the sum becomes an integral



So finally we have:

$$f(t) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi f t}\, dt \right\} e^{+j2\pi f t}\, df$$

giving the Fourier "pair" definition:

Capitals.

$$F(f) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi f t}\, dt$$

&

$$f(t) = \int_{-\infty}^{\infty} F(f)\, e^{+j2\pi f t}\, df$$

Condition of existence.

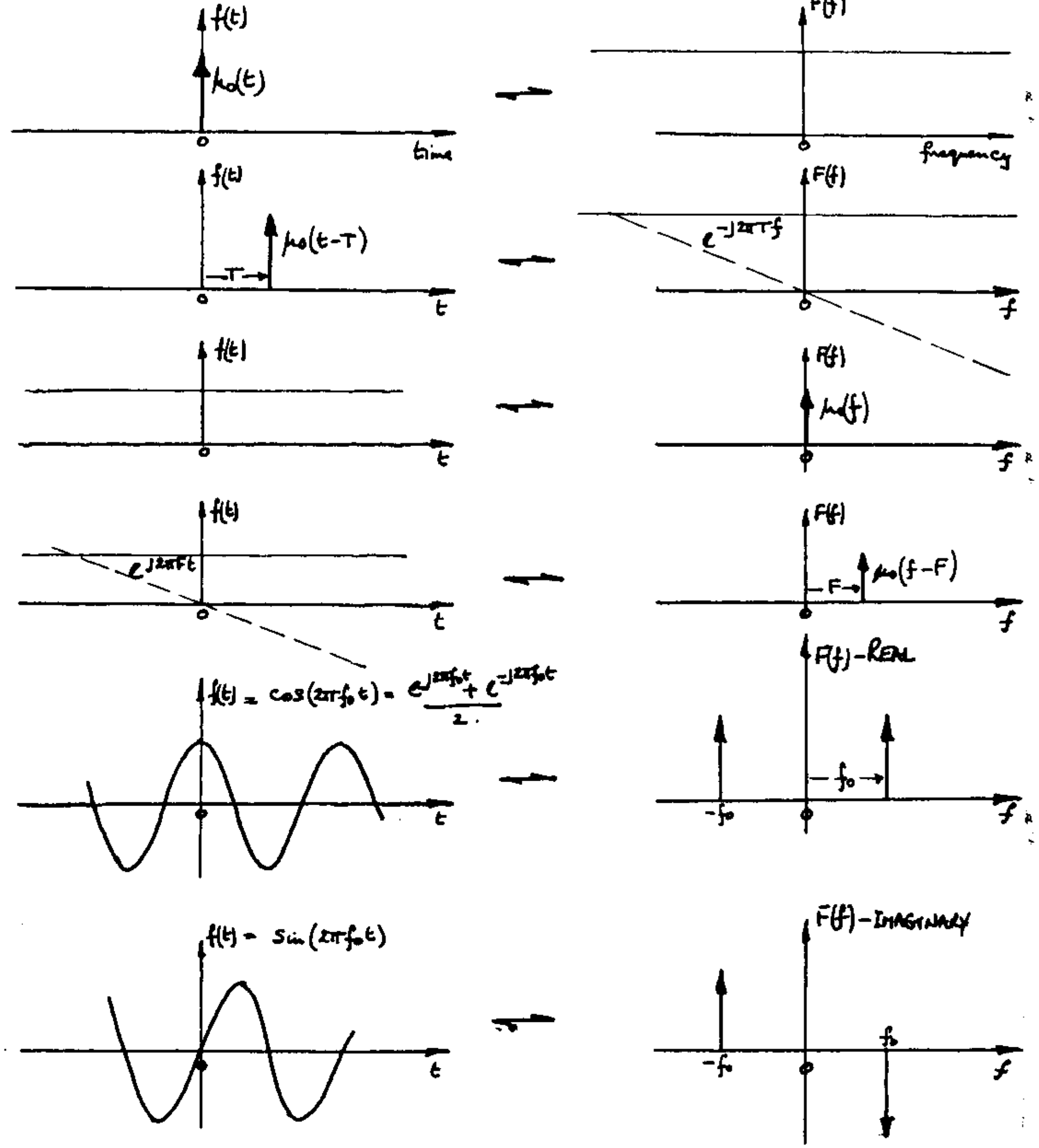
$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

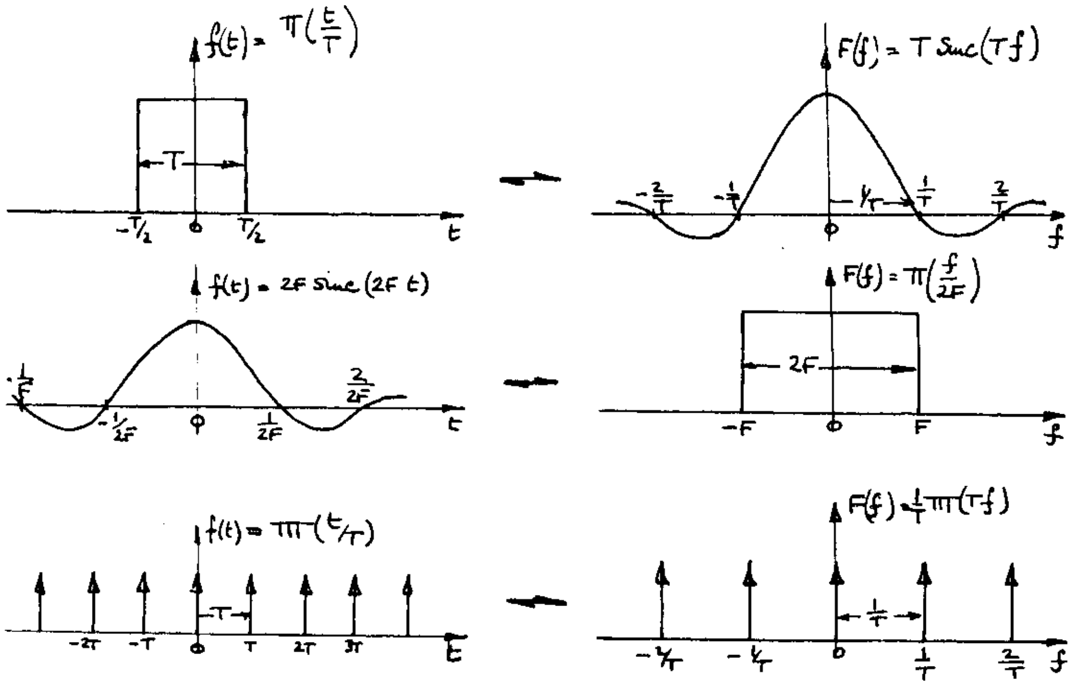
Any real life quantity (signal etc) of engineering interest will have a Fourier transform.

First important result is that any periodic waveform has a transform which is a series of impulses.

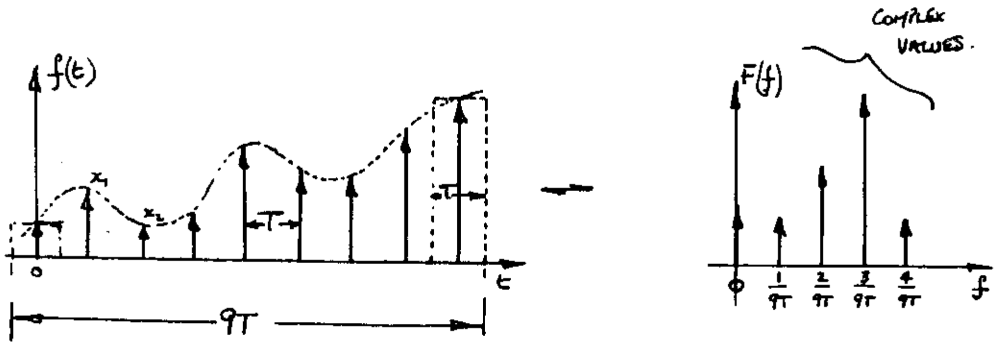
We quickly build up a table of Fourier transform pairs & domain relationships.

Some examples:





D.F.T



$$\hat{f}(t) = x_0\,\mu_0(t) + x_1\,\mu_0(t-T) + x_2\,\mu_0(t-2T) + \dots$$

$$F(f) = x_0 + x_1\,e^{-j2\pi Tf} + x_2\,e^{-j2\pi\cdot 2T\cdot f} + \dots$$

FOURIER THEOREMS.

IN EACH OF THESE THEOREMS IT IS ASSUMED THAT THE TIME FUNCTIONS $f_1(t)$ & $f_2(t)$ ARE TRANSFORMABLE IE $f_1(t) \leftrightarrow F_1(f)$; $f_2(t) \leftrightarrow F_2(f)$

1. LINEARITY.

$$a_1 f_1(t) + a_2 f_2(t) \longleftrightarrow a_1 F_1(f) + a_2 F_2(f)$$

a_1, a_2 constants.

2. TIME SCALING.

$$f(at) \longleftrightarrow \frac{1}{a}\cdot F\left(\frac{f}{a}\right)$$

3. TIME SHIFT THEOREM.

$$f(t-a) \longleftrightarrow e^{-j2\pi af}\cdot F(f)$$

4. FREQUENCY SHIFT THEOREM.

$$e^{j2\pi f_0 t}\, f(t) \longleftrightarrow F(f-f_0)$$

5. MODULATION THEOREM.

$$f(t)\cos(2\pi f_0 t) \longleftrightarrow \tfrac{1}{2}F(f-f_0) + \tfrac{1}{2}F(f+f_0)$$

6. TIME DIFFERENTIATION.

$$\frac{d^n f(t)}{dt^n} \longleftrightarrow (j2\pi f)^n\cdot F(f)$$

7. FREQUENCY DIFFERENTIATION.

$$(-j2\pi t)^n f(t) \longleftrightarrow \frac{d^n F(f)}{df^n}$$

8. TIME CONVOLUTION.

$$\int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \longleftrightarrow F_1(f)\cdot F_2(f)$$

9. FREQUENCY CONVOLUTION

$$f_1(t)\, f_2(t) \longleftrightarrow \int_{-\infty}^{\infty} F_1(g)\, F_2(f-g)\, dg.$$

10. Rayleigh's POWER THEOREM

$$\int_{-\infty}^{\infty} \left|f^2(t)\right| dt = \int_{-\infty}^{\infty} \left|F(f)\right|^2 df.$$

The more usual formulation of the DFT is obtained as follows:

The transform is $F(f) = \int_{\text{TOTAL SAMPLE WINDOW}} f(t)\, e^{-j2\pi f t}$

particular samples occur at $t = k\Delta T$ ΔT is the sample spacing, k is the sample counter

Total sample window $= N * \Delta T = T$ where N is the total number of samples.

So fundamental frequency $= \dfrac{1}{N * \Delta T} = \Delta f$

& the output will be harmonics of this

ie output $= m * \Delta f$

So we could approximate the above transform as:

$$F(m\Delta F) = \sum_{k=0}^{N-1} f(k\Delta T)\, e^{-j2\pi \frac{m}{N*\Delta T}\cdot k\Delta T} \cdot \frac{T}{N}$$

$$= \left(\frac{T}{N}\right) \sum_{k=0}^{N-1} f(k\Delta T)\cdot e^{-j2\pi \frac{mk}{N}}$$

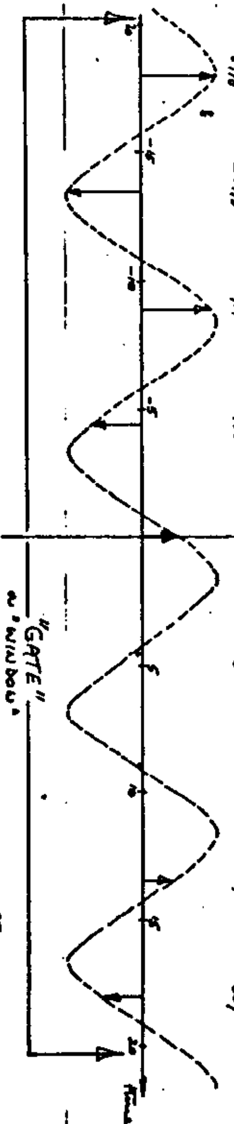
So $F(0 * \Delta F)$ = The simple numeric sum of the samples
'dc' term

$F(1 * \Delta F) = \sum_{k=0}^{N-1} f(k\Delta T)\, e^{-j2\pi \frac{k}{N}}$
"FIRST" HARMONIC

etc

The following example shows a set of samples which are symmetric about the time origin.

Sampling $\sin(2\pi \cdot 100t + 30°)$ at 4.5 msec intervals.



$$F(m\Delta F)\Big|_{\substack{m=-4\\ +4\ \text{by}\ +1}} = \sum_{k=-4}^{+4} f(k\Delta T)\, W_N^{-mk} \quad : \text{where } W_N = e^{j\frac{2\pi}{9}} \;;\; \Delta F = \frac{1}{40.5\,(\text{msecs})} = 24.69\ \text{Hz}$$

$$F(-4 \times 24.69\,\text{Hz}) = (.978)e^{-j\frac{2\pi}{9}\cdot -4\cdot -4} + (-.995)e^{-j\frac{2\pi}{9}\cdot -4\cdot -3} + (.914)e^{-j\frac{2\pi}{9}\cdot -4\cdot -2} + (-.743)e^{-j\frac{2\pi}{9}\cdot -4\cdot -1} + (.5)e^{-j\frac{2\pi}{9}\cdot -4\cdot 0} + (-.208)e^{-j\frac{2\pi}{9}\cdot -4\cdot 1} + (-.105)e^{-j\frac{2\pi}{9}\cdot -4\cdot 2} + (.407)e^{-j\frac{2\pi}{9}\cdot -4\cdot 3} + (-.669)e^{-j\frac{2\pi}{9}\cdot -4\cdot 4}$$

$= 4.37\angle 57.3° = 2.36 + j3.67$

$F(-3) = 0.19\angle -155° = -.17 - j.08$

$F(-2) = 0.11\angle 13° = +.105 + j.024$

$F(-1) = 0.09\angle -174° = -.08 - j.008$

$F(0) = 0.08$ Simply the sum of the samples

$F(1) = 0.09\angle 174°$

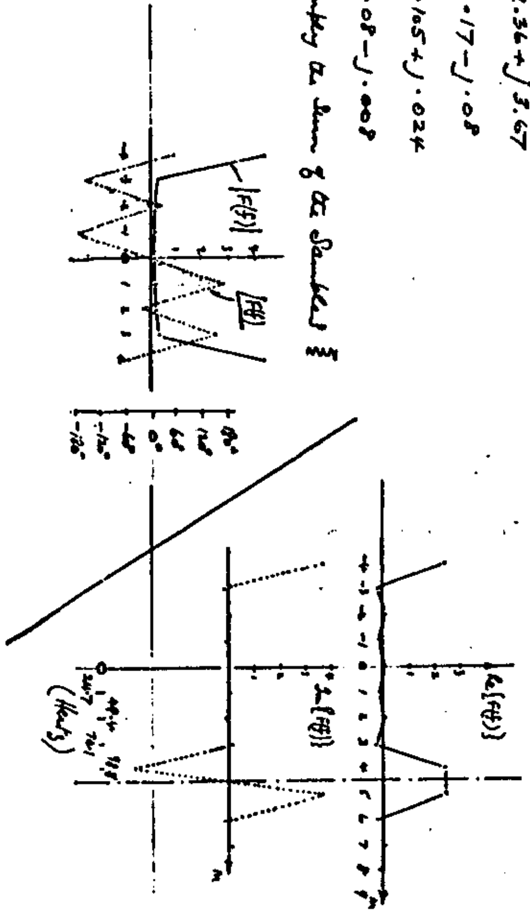
$F(2) = 0.11\angle -13°$

$F(3) = 0.19\angle 155°$

$F(4) = 4.37\angle -57.3°$

$F(5) = 4.37\angle 57.3°$

$F(6) = 0.19\angle -155°$



JGL 1-11

Rectangle function of unit height and base, $\Pi(x)$

The rectangle function of unit height and base, which is illustrated in Fig. 4.1, is defined by

$$\Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ (\frac{1}{2} & |x| = \frac{1}{2}) \\ 1 & |x| < \frac{1}{2}. \end{cases}$$

It provides simple notation for segments of functions which have simple expressions, for example, $f(x) = \Pi(x)\cos\pi x$ is compact notation for

$$f(x) = \begin{cases} 0 & x < -\frac{1}{2} \\ \cos\pi x & -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \frac{1}{2} < x \end{cases}$$

(see Fig. 4.2). We may note that $h\Pi[(x-c)/b]$ is a displaced rectangle function of height h and base b, centered on $x = c$ (see Fig. 4.3). Hence, purely by multiplication by a suitably displaced rectangle function, we

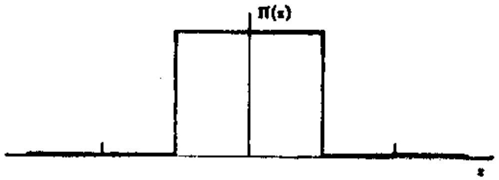


Fig. 4.1 The rectangle function of unit height and base, $\Pi(x)$.

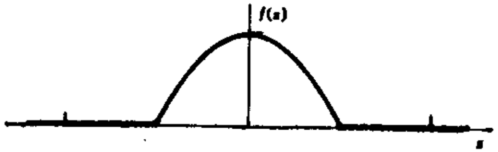


Fig. 4.2 A segmented function expressed by $\Pi(x)\cos\pi x$.

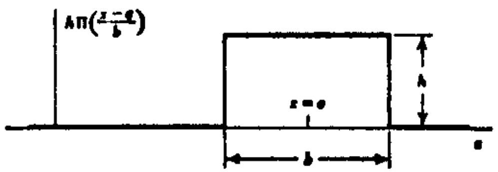


Fig. 4.3 A displaced rectangle function of arbitrary height and base expressed in terms of $\Pi(x)$.

JGL 1-12

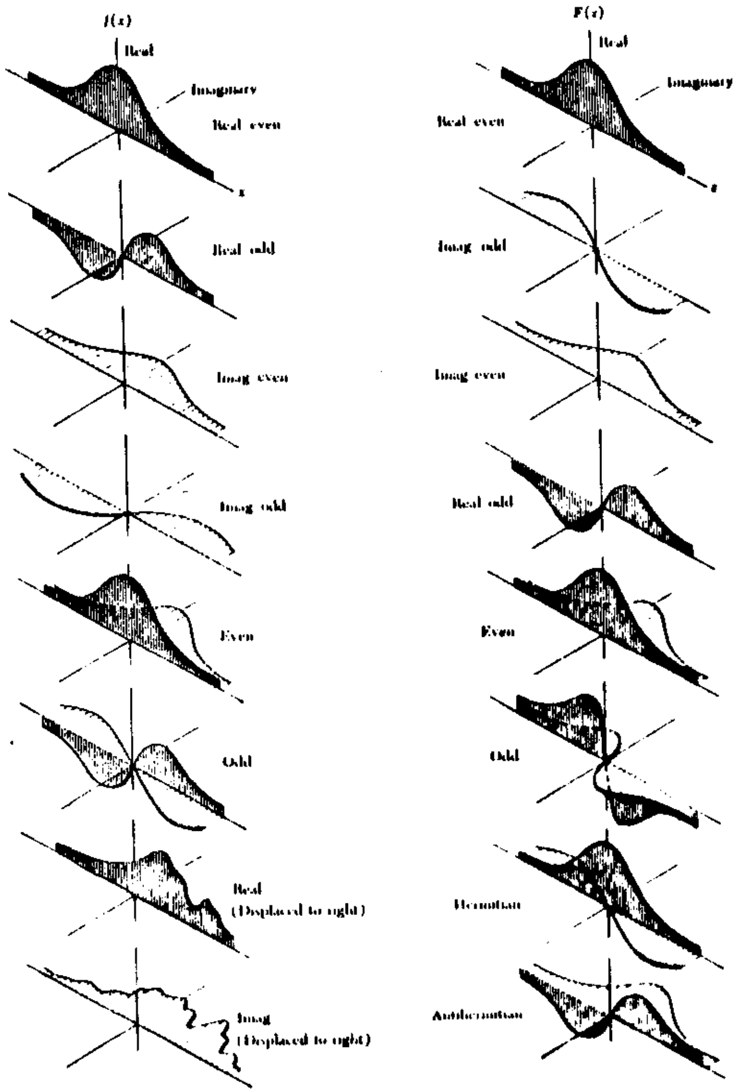


Fig. 2.5 Symmetry properties of a function and its Fourier transform.