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Basic Concepts in Signals and Systems

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SUMMARY

In this series of lectures, basic concepts in signals and systems will be discussed. Starting with the definition of linear systems, some elementary signals are introduced, which is followed by the notion of time-invariant systems. Signals and Systems are then coupled through impulse response, convolution, and response to exponential signals. This naturally leads to Fourier series representation of periodic signals and consequently, signal representation in the frequency domain in terms of amplitude and phase spectra. Linear system response to periodic signals, discussed next, is then easy to understand. To handle non-periodic signals, Fourier transform is introduced by viewing a non-periodic function as the limiting case of a periodic one, and its application to linear system analysis is illustrated. The concepts of energy and power signals and the corresponding spectral densities are then introduced. The discussion ends with the derivation of the transfer function required for distortionless transmission.

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1. Linear Systems: Definition

The concept of linear systems plays an important role in the analysis and synthesis of most practical systems, be it communication, control or instrumentation. Consider a system S which produces an output y when an input x is applied to it (both y and x are usually functions of time). We shall denote this symbolically as

$$x \overset{S}{\rightarrow} y$$

Then S is said to be linear if it obeys two principles, viz. principle of superposition and principle of homogeneity. The former implies that if $x_1 \rightarrow y_1$ (note that we have omitted S above the arrow: this is implied) and $x_2 \rightarrow y_2$, then $x_1+x_2 \rightarrow y_1+y_2$. The principle of homogeneity implies that if $x \rightarrow y$, then $\alpha x \rightarrow \alpha y$ where α is an arbitrary constant. Note, in passing, that α could be zero i.e. zero input should lead to zero output in a linear system. Combining the two principles, we can now formally define a linear system as one in which

$$x_{1,2} \rightarrow y_{1,2} \Rightarrow \alpha x_1 + \beta x_2 \rightarrow \alpha y_1 + \beta y_2 \qquad (1)$$

where the notation $\Rightarrow$ is used to mean "implies".

As an example, consider the system described by the well-known equation of a straight line

$$y = mx + c \qquad (2)$$

It may seem surprising but (2) does not describe a linear system unless c=0, simply because zero input does not lead to zero output. Another way of demonstrating this is to apply αx as input; then the output is

$$y' = m\alpha x + c \neq \alpha y = m\alpha x + c\alpha \qquad (3)$$

By the same token, the dynamic system described by

$$\frac{dy}{dt} + 5y = 5x + 6 \qquad (4)$$

is not linear, because (x=0, y=0) does not satisfy the equation

Another, and a bit more subtle example is shown in Fig.1. Is this system linear? Obviously x=0 leads to z=0 but then this is only a necessary condition for a linear system. Is it sufficient? To test this, apply $x=x_o$; then $z=z_o$. Now apply $x=-x_o$; the output is still z_o instead of $-z_o$. The obvious conclusion is that the system is nonlinear.

Almost all practical systems are nonlinear, which are usually much more difficult to handle than linear systems. Hence we make our life comfortable by approximating (or idealizing?) a nonlinear system by a linear one. Also, in many situations, a nonlinear system is "incrementally" linear, i.e. the system is linear if an increment Δx in x is considered as the input and the corresponding increment Δy in y is considered as the output. Both (2) and (4) are descriptions of such incrementally linear systems. A transistor amplifier is a highly nonlinear system, but it behaves as a linear one if the input is an ac signal superimposed on a much larger dc bias.

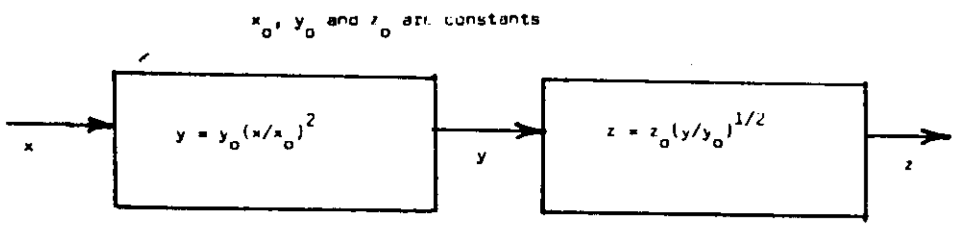

Fig. 1 - Linear system ?

2. Elementary Signals

A signal, in the context of electrical engineering, is a time-varying current or voltage. An arbitrary signal can be decomposed into some elementary or "basic" signals, which, by themselves also occur frequently in nature These are (i) the exponential signal $e^{\alpha t}$ where α may be real or imaginary or complex, (ii) the unit step function u(t) and (iii) the unit impulse function, $\delta(t)$. When α is purely imaginary in $e^{\alpha t}$, we get a particularly important situation, because if $\alpha = j\omega$ and ω is real, then

$$e^{j\omega t} = \cos \omega t + j \sin \omega t \qquad (5)$$

Thus sinusoidal signals, $\cos\omega t$ and $\sin\omega t$, which are so important in the study of communcations, are special cases of the exponential signal. The quantity ω, as is well known, is the frequency in radians/sec, while $f=\omega/(2\pi)$ is the frequency in cycles/sec or Hz.

The unit step function, shown in Fig.2 is defined by

$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases} \qquad (6)$$

Note that it is discontinuous at t=0. The unit impulse function $\delta(t)$ is related to u(t) through

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \qquad (7)$$

or

$$\delta(t) = \frac{du(t)}{dt} \qquad (8)$$

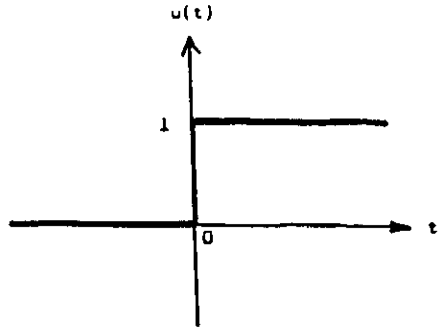

Fig. 2 - The unit step function

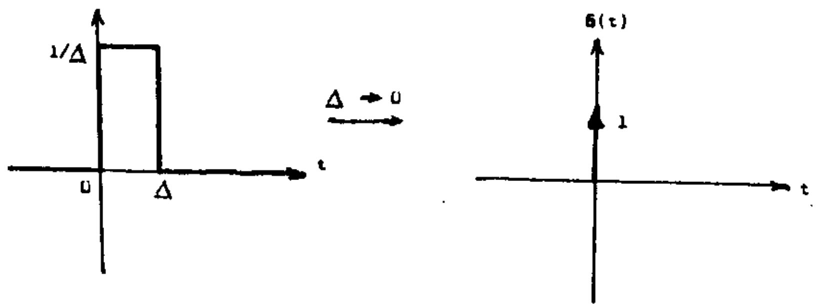

Fig. 3 - A limiting view of δ(t)

Obviously, it exists only at t=0, and the value there is infinitely large, but

$$\int_{-\infty}^{\infty} \delta(\tau)d\tau = \int_{0^-}^{0^+} \delta(\tau)d\tau = 1 \qquad (9)$$

i.e. the area under the plot of $\delta(t)$ versus t is unity. This is called the strength of the impulse; for example, the strength of the impulse $K\delta(t)$ is K. Obviously, there is some formal difficulty with regard to the definition of $\delta(t)$, but we shall not enter into this debate here. $\delta(t)$ can be viewed as the limit of the rectangular pulse shown in Fig 3 as $\Delta\to 0$; Fig.3 also shows the representation of $\delta(t)$. Two important properties of $\delta(t)$ are that

$$x(t)\ \delta(t-t_o) = x(t_o)\ \delta(t-t_o) \qquad (10)$$

and

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = x(t) \qquad (11)$$

Equation (11) easily follows from (9) and (10), and represents the "Sifting" or "Sampling" property of the impulse function.

3. Time Invariance

At this point, we need to introduce another concept viz. that of time-invariance of a system. A system S is time-invariant if a time shift in the input signal causes the same time shift in the output signal i.e. if $x(t)\to y(t)$ implies $x(t-t_o)\to y(t-t_o)$.

Both (2) and (4) are descriptions of time-invariant system. On the other hand, $y(t)=tx(t)$ represents a time-varying system. Most of the practical systems we encounter are time-invariant systems.

Systems which are linear and time-invariant (LTI) are particularly simple to analyze in terms of their impulse response or frequency response function, as will be demonstrated in what follows.

4. Impulse response and convolution

Consider an LTI system whose response to a unit impulse function is h(t), i.e.

$$\delta(t) \to h(t) \qquad (12)$$

By time-invariance, therefore

$$\delta(t-\tau)\to h(t-\tau) \qquad (13)$$

By homogeneity, if we multiply the left hand side of (13) by $x(\tau)d\tau$, the right hand side should also get multiplied by $x(\tau)d\tau$ i.e.

$$x(\tau)\ \delta(t-\tau)d\tau \to x(\tau)\ h(t-\tau)d\tau \qquad (14)$$

By superposition, if we integrate the left hand side of (14) we should do the same for the right hand side i.e.

$$\int_{-\infty}^{\infty} x(\tau)\ \delta(t-\tau)d\tau \to \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \qquad (15)$$

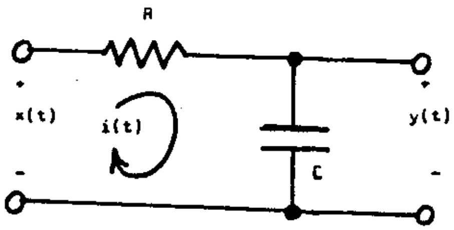

Fig. 4 - RC network

But , by (11), the left hand side of (15) is simply x(t), so the right hand side should be y(t). Thus if the unit impulse response h(t) of an LTI system is known, then one can find the output of the system due to an arbitrary excitation x(t) as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)d\tau \qquad (16)$$

$$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \qquad (17)$$

where the second form follows simply, through a change of variable. The integral (16) or (17) is called the convolution integral and the operation of convolution is symbolically denoted as

$$y(t) = x(t) * h(t)$$

It is a simple matter to prove that convolution operation is commutative (i.e. $x(t) * h(t) = h(t)*x(t)$; infact, this is what equivalence of (16) and (17) implies), associative (i.e. $x(t)* [h_1(t)* h_2(t)] = [x(t)* h_1(t)]* h_2(t)$; this is useful in the analysis of cascade connection of systems) and distributive (i.e. $x(t) * [h_1(t) + h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$; this is useful in the analysis of parallel systems).

As an example of application of the convolution integral, consider the RC network shown in Fig. 4, where both x(t) and y(t) are voltages, and the capacitor is uncharged before application of x(t) (an alternate way of expressing this is to say that C is initially relaxed). When $x(t)=\delta(t)$, the current in the circuit is $i(t)=\delta(t)/R$. This impulse of current charges the capacitor to a voltage

$$\frac{1}{C}\int_{0-}^{0+}\frac{\delta(\tau)}{R}\,d\tau = \frac{1}{RC} \tag{18}$$

at $t=0^+$. For $t>0^+$, $\delta(t)=0$; hence the capacitor charge decays exponentially; so does the voltage across it, according to

$$y(t) = \frac{1}{RC}\,e^{-t/(RC)} \tag{19}$$

Thus the impulse response of the RC network is

$$h(t) = \frac{1}{T}\,e^{-t/T}u(t) \tag{20}$$

where T=RC is called the time constant of the network.

Now suppose the input is changed to a unit step voltage i e. $x(t)=u(t)$. Then the response is, by (17),

$$y(t) = \int_{-\infty}^{\infty}\frac{1}{T}\,e^{-\tau/T}u(\tau)u(t-\tau)d\tau \tag{21}$$

$$= \frac{1}{T}\int_{0}^{t} e^{-\tau/T}d\tau = (1-e^{-t/T})u(t) \tag{22}$$

where the lower limit arises due to the factor $u(\tau)$ and the upper limit arises as a consequence of the factor $u(t-\tau)$ in the integrand.

5. LTI System Response to Exponential Signals

Let $x(t)=e^{st}$ be applied to a system with impulse response $h(t)$; then by (17), the response is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau \tag{23}$$

$$= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau \tag{24}$$

$$= H(s)\,e^{st} \tag{25}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau \tag{26}$$

is called the system function or transfer function of the system, and is a function of s only. A signal for which the output differs from the input only by a scaling factor (perhaps complex) is called the eigenfunction of the system, and the scaling factor is called the eigenvalue of the system. Obviously, e^{st} is an eigen function of an LTI system, and $H(s)$ is its eigenvalue.

When $s=j\omega$, H represents the frequency response of the system, i.e. if $x(t)=e^{j\omega t}$ or its real part ($\cos\omega t$) or imaginary part ($\sin\omega t$), then the output will be $H(j\omega)e^{j\omega t}$ or $Re\left[H(j\omega)e^{j\omega t}\right]$ or $Im\left[H(j\omega)e^{j\omega t}\right]$ respectively. For example, if $H(j\omega)=|H(j\omega)|\,e^{j\angle H(j\omega)}$ and the input is $\cos\omega t$, then the output shall be $|H(j\omega)|\cos(\omega t+\angle H(j\omega))$. $H(j\omega)$ varies with frequency, and the plots of $|H(j\omega)|$ and $\angle H(j\omega)$ versus ω are known as magnitude and phase responses respectively.

Since the principle of superposition holds in a linear system, the response to a linear combination of exponential signals, $\sum_i a_i e^{s_i t}$, will be of the form $\sum_i a_i H(s_i)e^{s_i t}$. It is precisely this fact which motivated Fourier to explore if an arbitrary signal could be represented as a superposition of exponential signals. As is now well known, this can indeed be done - by a Fourier series for a periodic signal and by the Fourier transform for a general, not necessarily periodic, signal.

6. The Fourier Series

Consider a linear combination of the exponential signal $e^{j\omega_o t}$ with its harmonically related exponential signals $e^{jk\omega_o t}$ $k=0, \pm 1, \pm 2, \ldots$:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \qquad (27)$$

In this, k=0 gives a constant term or d.c., in electrical engineering language; $e^{j\omega_o t}$ is the smallest frequency term, with a frequency ω_o and period $T=2\pi/\omega_o$, and is called the fundamental. The term $e^{j2\omega_o t}$ has a frequency $2\omega_o$, while $e^{-j2\omega_o t}$ has a frequency $-2\omega_o$; the period of either term is T/2, and both the terms represent what is known as the second harmonic. A similar interpretation holds for the general term $e^{jk\omega_o t}$, which has a period $T/|k|$. Note that we take the frequency as positive or negative, but the period is taken as positive. Obviously, the summation (27) is periodic with a period equal

to T, in which there are $|k|$ periods of the general term $e^{jk\omega_o t}$ but only one period of the fundamental.

What about a given periodic function x(t) with a period T i.e. $x(t+mT)=x(t)$, $m=0, \pm 1, \pm 2, \ldots$? Can it be decomposed into the form (27)? It turns out that under certain conditions which are satisfied by all but a few exceptional cases, one can indeed do so. To determine a_k's, multiply both sides of (27) by $e^{-jn\omega_o t}$ and integrate over the interval 0 to T. Obviously this results in an integral $\int_0^T e^{j(k-n)\omega_o t}\,dt$ on the right hand side, which is zero if $k \neq n$, and T if k=n. Thus $a_n = (1/T)\int_0^T x(t)\, e^{-jn\omega_o t}\,dt$ or

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_o t}\,dt \qquad (28)$$

a_k represents the weight of the k-th harmonic and is called the spectral coefficient of x(t). a_k is, in general, complex. A plot of $|a_k|$ versus k will consist of discrete lines at k=0 $\pm 1, \pm 2, \ldots$; it resembles a spectrum as observed on a spectroscope, and is called the amplitude spectrum. Similarly, one can draw a phase spectrum.

It is obvious from (27) that x(t) could be written as the summation of a sine and a cosine series, and that the corresponding coefficients could be found from $\int_0^T x(t) \cos k\omega t\,dt$ and $\int_0^T x(t) \sin k\omega t\,dt$. It is, however, much more convenient to handle the exponential form of the Fourier series as given in (27).

As an example of application of the Fourier series, consider the pulse stream shown in Fig. 5.

Note , at this point, that in (28), the lower and the upper limit of integration are not important so long as their difference is T This is so because $\int_{t_o}^{t_o+T} e^{j(k-n)\omega_o t}$ is independent of t_o. In the example under consideration, it is obviously convenient to choose the interval $-\frac{T}{2} \leq t \leq T/2$ which virtually becomes $-\tau/2 \leq t \leq +\tau/2$, because $x(t)=0$ at other values of t within the chosen interval. Hence

$$
\begin{aligned}
a_k &= \frac{A}{T} \int_{-\tau/2}^{\tau/2} e^{-jk\omega_o t}\, dt \\
&= \frac{2A}{k\omega_o} \sin \frac{k\omega_o \tau}{2} \\
&= \tau A \, \frac{\sin \frac{k\omega_o \tau}{2}}{\frac{k\omega_o \tau}{2}}
\end{aligned} \qquad (29)
$$

This is of the form $\tau A \sin x/x$, where $x = k\omega_o \tau/2$.

Note that a_k is real, and can be positive, zero or negative. Hence separate amplitude and phase spectrum plots are not necessary; a single diagram suffices and is shown in Fig.6. Note that a_k has a maximum value at d.c. i.e. k=0, the value being A (this checks with direct calculation from Fig.5). The envelope of the spectrum is of the form sin x/x and exhibits damped oscillations with zeros at $x=\pi$ (i.e. $k\omega_o = 2\pi/\tau$), 2π (i.e. $k\omega_o = \cdot\pi/\tau$),...... Further, the sketch is

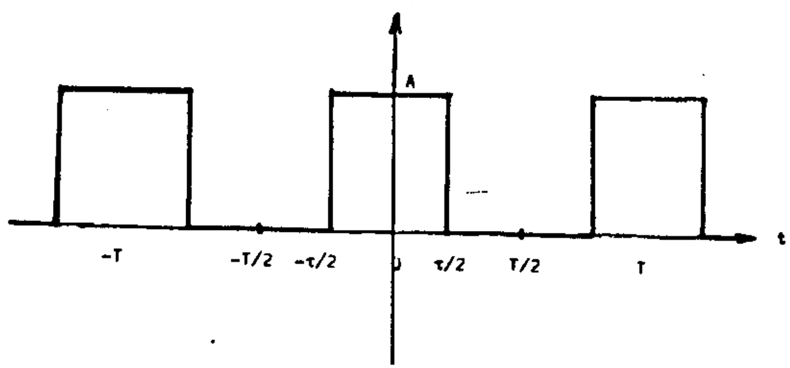

Fig. 5 - A rectangular pulse stream

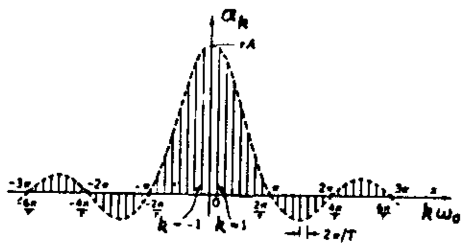

Fig. 6 - Spectrum of x(t) of Fig. 5

symmetrical about x=0, because sinx/x is an even function. The spectrum consists of discrete lines, two adjacent lines being separated by $\frac{2\pi}{T} = \omega_o$ radians/sec.

Some important points emerge from the sketch of Fig.6. As T increases, the lines get closer and ultimately when $T \to \infty$, corresponding to a single pulse, the spectrum becomes continuous and will be characterized by the function $\tau A \frac{\sin \omega\tau/2}{\omega\tau/2}$, where $k\,\omega_o$ has become the continous variable ω. This, as we shall see, is the Fourier transform of the single pulse.

Secondly, since the lines concentrated in the lower frequency range are of higher amplitude, most of the energy of the periodic wave of Fig.5 must be confined to lower frequencies. Thirdly, as τ decreases, the spectrum spreads out i.e. there is an inverse relationship between pulse width and frequency spread.

Since the energy of the periodic wave is mostly confined to the lower frequency range, a convenient measure of bandwidth of the signal is from zero frequency to the frequency of the first zero crossing i.e. the bandwidth in Hz, B, can be taken as $1/\tau$.

If x(t) of (27) is the voltage across or the current through a one ohm resistor, then the average power dissipated is $\frac{1}{T}\int_0^T |x(t)|^2\,dt$. If one writes $|x(t)|^2 = x(t)x^*(t)$ and substitutes for x(t) and $x^*(t)$ from (27), there results the following :

$$|x(t)|^2 = \sum_{k=-\infty}^{\infty}\ \sum_{n=-\infty}^{\infty} a_k\, a_n^*\, e^{j(k-n)\omega_o t} \tag{30}$$

As we have already seen, $\int_0^T e^{j(k-n)\omega_o t}\,dt$ is zero if $k \neq n$ and equals T when k=n. Thus average power becomes

$$\frac{1}{T}\int_0^T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2 \tag{31}$$

This is known as Parseval's theorem.

A periodic signal that is of great importance in digital communication is the periodic impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) \tag{32}$$

as shown in Fig.7. If this is expanded in Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_o t} \tag{33}$$

where $\omega_o = 2\pi/T$, then

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_o t}\,dt = \frac{1}{T} \tag{34}$$

The spectrum is sketched in Fig.8.

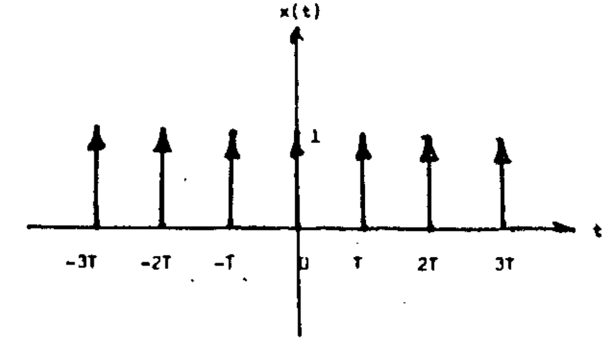

Fig. 7 - A periodic impulse train

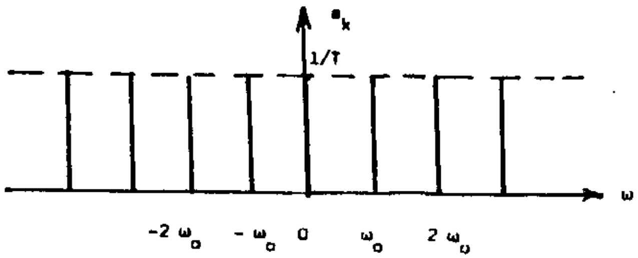

Fig. 8 - Spectrum of the periodic impulse train

What is the bandwidth of this signal? The amplitude is a constant at all frequencies, unlike the spectrum of Fig 6. Hence the bandwidth is infinite. This agrees with our observation about bandwidth and pulse duration, because Fig.7 is the degenerate form of Fig.5 with $\tau \to 0$ and $A \to \infty$.

7. Linear System Response to Periodic Excitation

From the discussion of Section 5, it follows that a linear system, excited by the periodic signal of (27), will produce an output signal, which is also periodic with the same period, and is given by

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \, H(jk\omega_o) e^{jk\omega_o t} \tag{35}$$

where

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) \, e^{-j\omega t} \, dt \tag{36}$$

and h(t) is the unit impulse response.

As a simple example, consider the RC network of Fig.4; we have already derived its impulse response as

$$h(t) = \frac{1}{RC} \; e^{-t/(RC)} \; u(t) \tag{37}$$

so that

$$H(j\omega) = \int_{0}^{\infty} \frac{1}{RC} \, e^{-t(j\omega + \frac{1}{RC})} \, dt \tag{38}$$

$$= \frac{1}{j\omega \, RC+1} \tag{39}$$

When excited by the periodic impulse train of Fig.7, the response y(t) can be found in two ways First, since

$$\delta(t) \rightarrow h(t),$$

it follows that

$$\sum_{k=-\infty}^{\infty} \overline{\delta(t-kT)} \rightarrow \sum_{k=-\infty}^{\infty} h(t-kT)$$

so that

$$y(t) = \frac{1}{RC} \sum_{k=-\infty}^{\infty} e^{-(t-kT)/(RC)} u(t-kT) \qquad (40)$$

Should you try to sketch this waveform, you would realize how messy it looks; also not much information about the effect of the RC network will be obvious from this sketch. On the other hand, the Fourier series method gives, from (35), (34) and (39)

$$y(t) = \sum_{k=-\infty}^{\infty} \frac{1/T}{jk\omega_o RC+1} e^{jk\omega_o t} \qquad (41)$$

Let this be written as $\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_o t}$; then the sketch of $|b_k|$ versus $\omega=k\omega_o$ looks like that shown in Fig.9. Comparing this with Fig.8, we note that the RC network attenuates higher frequencies as compared to lower ones and hence acts as a low-pass filter. The bandwidth of the filter B_f is defined as the frequency at which the $|H(j\omega)|$ falls down by 3 dB as compared to its d.c. value i.e.

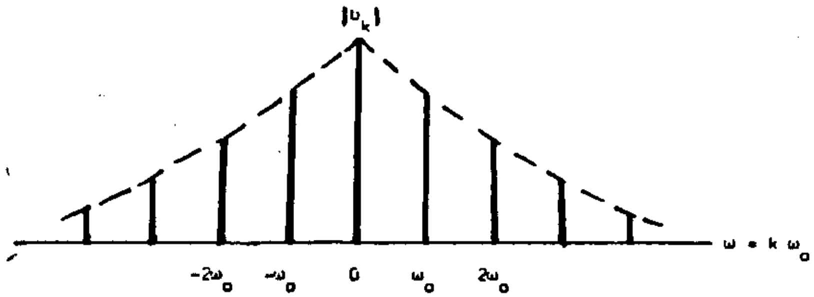

Fig. 5 - Spectrum of output y(t) given by (41)

$$|H(j2\pi B_f)| = H(j0)/\sqrt{2} \tag{42}$$

Combining this with (39) gives $B_f = 1/(2\pi RC)$.

What would be the response of the RC filter to the rectangular pulse stream of Fig.5? This will of course depend on the relative values of T, τ and B_f.

First let us confine ourselves to the time domain. If the product RC is comparable to T, then the output will consist of overlapping pulses, and will retain very little similarity to the input. Let, therefore, RC<<T. then depending on τ, the response during one period will be of the form shown in Fig.10. It is obvious that for fidelity, i.e. if the output is to closely resemble the input, we require $RC \ll \tau < T$.

Now, turn to the frequency domain. If the signal bandwidth is taken as $B = 1/\tau$ Hz, then obviously for fidelity, the RC filter must pass all frequencies upto $1/\tau$ Hz with as little attenuation as possible. Thus B_f must be at least equal to $B = 1/\tau$ Since the attenuation is 3 db instead of zero at B_f and the input spectrum is not limited to B, the pulse shape will be distorted. For reduced distortion, we need to increase B_f and we expect good results if $B_f \gg B$ i.e. $RC \ll \tau$.

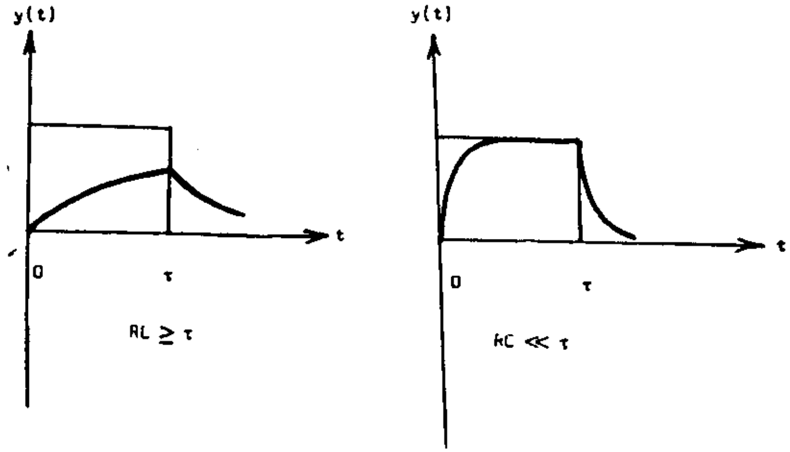

Fig. 10 - RL network response in one period of Fig. 5

8. The Fourier Transform

Now consider a nonperiodic function x(t) which exists in the range $-T/2 \le t \le T/2$, and is zero outside this range.

Consider a periodic extension $x_p(t)$ of $x(t)$, as shown in Fig.11 $x_p(t)$ can be expanded in Fourier series as

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \tag{43}$$

where $\omega_o = 2\pi/T$ and

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_p(t) e^{-jk\omega_o t} dt \tag{44}$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt \tag{45}$$

because for $|t| \leq T/2$, $x_p(t) = x(t)$. Also, since $x(t)=0$ for $|t| > T/2$, we can write

$$a_k = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_o t} dt \tag{46}$$

If we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{47}$$

Then from (46), we get

$$a_k = \frac{1}{T} X(jk\omega_o) = \frac{1}{2\pi} X(jk\omega_o)\omega_o \tag{48}$$

Thus (43) can be written as

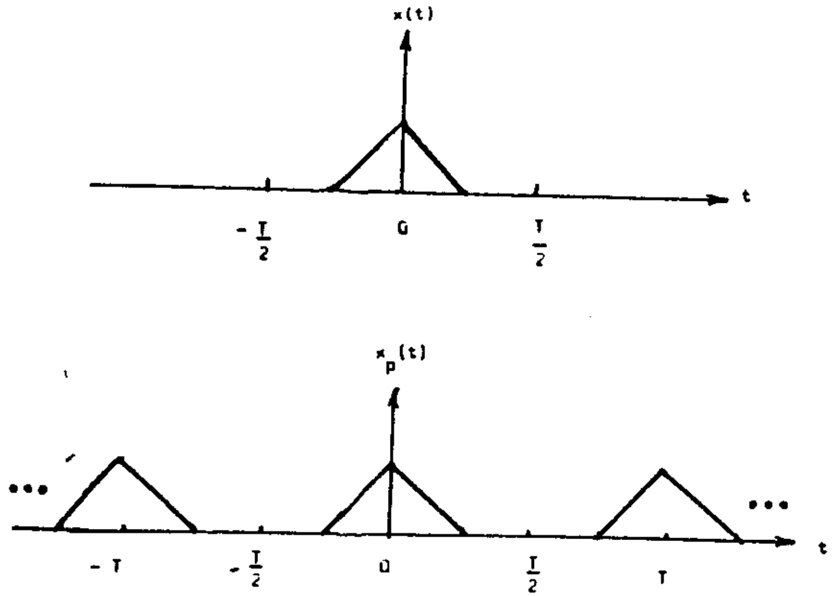

Fig. 11 - A nonperiodic function x(t) and is periodic extension $x_p(t)$

$$x_p(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_o)\,\omega_o\, e^{jk\omega_o t} \qquad (49)$$

Now let $T \to \infty$; then $x_p(t) \to x(t)$, $k\omega_o \to \omega$, a continuous variable, $\omega_o \to d\omega$ and the summation becomes an integral. Thus (49) becomes

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \qquad (50)$$

Combining (47) and (48), we now formally define the Fourier transform of x(t) as

$$X(j\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad (51)$$

and the inverse Fourier transform as

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \qquad (52)$$

Without entering into the question of existence, we simply state below the conditions, named after Dirichlet, under which x(t) is Fourier transformable. These are:

1) $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$
2) finite number of maxima and minima within any finite interval, and
3) finite number of finite discontinuities within any finite interval.

Referring to (26) or (36), it should be obvious that the impulse response h(t) and the frequency response H(jω) are Fourier transform pairs. Explicitly,

$$H(j\omega) = \mathcal{F}[h(t)] \qquad (53)$$

As an example of Fourier transformation, consider the rectangular pulse shown in Fig.12. Notice that this is the limiting form of the periodic function of Fig.5 with $T \to \infty$. Applying (51), we get

$$X(j\omega) = A \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt = \tau A \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} \qquad (54)$$

This, as will be easily recognized, is the limiting form of Fig.6, and is the envelope of the same figure. This verifies the observation made earlier in Section 6.

The Fourier transform has many important properties, the most important in the context of analysis of linear systems being that it converts a convolution in the time domain to a multiplication in the frequency domain i.e. if

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \qquad (55)$$

then, assuming that y(t), x(t) and h(t) are Fourier transformable, and $\mathcal{F}[y(t)] = Y(j\omega)$, we get

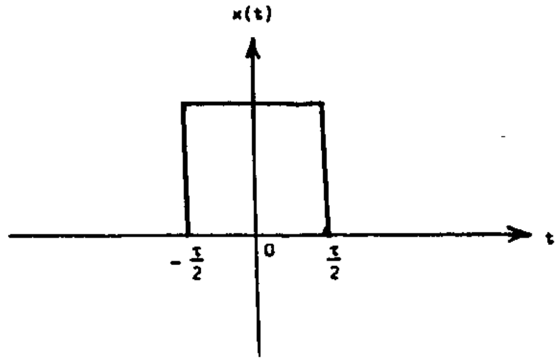

Fig. 12 - A rectangular pulse

$$y(j\omega) = X(j\omega)\ H(j\omega) \tag{56}$$

The proof of (56) is simple and proceeds as follows:

$$Y(jw) = \mathfrak{F}[y(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\ h(t-\tau) d\tau \right] e^{-j\omega t} dt \tag{57}$$

Interchange the order of integration and notice that (t) does not depend on t; the result is

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\ e^{-j\omega t} dt \right] d\tau \tag{58}$$

Let $t-\tau=\xi$; then the integral inside the bracket becomes $e^{-j\omega\tau}H(j\omega)$, so that

$$Y(j\omega) = \int_{-\infty}^{\infty} H(j\omega)\ x(\tau)\ e^{-j\omega\tau} d\tau \tag{59}$$

i.e.

$$Y(j\omega) = H(j\omega)X(j\omega) \tag{60}$$

In words, this amounts to saying that the spectrum of the output of a linear system is simply the product of the spectrum of the input signal and the frequency response of the system. The output in the time domain, y(t) can be simply found by taking the inverse Fourier transform of $Y(j\omega)$.

To illustrate the application of (60), consider a linear system having the impulse response

$$h(t) = e^{-\alpha t}\,u(t)\,, \qquad \alpha>0 \tag{61}$$

which is excited by an input signal

$$x(t) = e^{-\beta t}\,u(t), \quad \beta>0 \tag{62}$$

By direct integration, it is easily shown that

$$H(j\omega) = \frac{1}{\alpha+j\omega} \text{ and } X(j\omega) = \frac{1}{\beta+j\omega} \tag{63}$$

Thus

$$Y(j\omega) = \frac{1}{(\alpha+j\omega)(\beta+j\omega)} \tag{64}$$

To determine $y(t)$, one may write

$$Y(j\omega) = \frac{A}{\alpha+j\omega} + \frac{B}{\beta+j\omega} \tag{65}$$

and find A and B as

$$A = -B = \frac{1}{\beta-\alpha} \tag{66}$$

so that

$$y(t) = \mathfrak{F}^{-1}\left[\frac{1}{\beta-\alpha}\left(\frac{1}{\alpha+j\omega} - \frac{1}{\beta+j\omega}\right)\right] \tag{67}$$

$$= \frac{1}{\beta-\alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]u(t) \tag{68}$$

Things are of course, different if $\alpha=\beta$; then one goes back to (64) and uses the property that if $\mathfrak{F}[x(t)] = X(j\omega)$ then $\mathfrak{F}[tx(t)] = j\,dX(j\omega)/d\omega$. Accordingly if $\alpha=\beta$, then

$$y(t) = t\,e^{-\alpha t}\,u(t) \tag{69}$$

9. Spectral Density

In using Fourier transform to calculate the energy or power of a signal, the notion of spectral density is an important one. The total energy and average power of a signal $x(t)$ are defined as

$$E = \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2\,dt = \int_{-\infty}^{\infty}|x(t)|^2\,dt \tag{70}$$

and

$$P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2\,dt \tag{71}$$

respectively. A signal $x(t)$ is called an energy signal if $0<E<\infty$ and a power signal if $0<P<\infty$. A given signal $x(t)$ can be either an energy signal or a power signal but not both. A periodic signal (e.g. the one of Fig.5) is usually a power signal, while a non-periodic signal (e.g. the one of Fig.12) is usually an energy signal. Power and energy signals are mutually exclusive because the former has infinite energy while the latter has zero average power. Depending on the nature of the signal, the spectral density is also to be qualified as power or energy.

Consider an energy signal x(t). Using the facts that $|x(t)|^2 = x(t)x^*(t)$ and $\mathcal{F}[x^*(t)] = X^*(-j\omega)$ and combining with the inversion integral (52), it is not difficult to show that

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega \tag{72}$$

The right most expression in (72) shows that $|X(j\omega)|^2/(2\pi)$ has the dimension of energy per unit radian frequency i.e. $|X(j\omega)|^2$ has the dimension of energy per unit Hz. For this reason, $|X(j\omega)|^2$ is referred to as the energy density spectrum of the signal x(t). Incidentally, (72) is known as the Parseval's relation (cf. (31)).

For a periodic signal, which is a power signal, we have already seen in (31) that the average power P is given by

$\sum_{k=-\infty}^{\infty} |a_k|^2$ where $|a_k|$ is the amplitude of the k-th harmonic. If we define, in similarity with (72),

$$P = \int_{-\infty}^{\infty} S_x(f) \, df \tag{73}$$

then $S_x(f)$ qualifies as the power per unit Hz and is called the power spectral density. In terms of $|a_k|$ it is easily seen that

$$S_x(f) = \sum_{k=-\infty}^{\infty} |a_k|^2 \, \delta(f-kf_o) \tag{74}$$

where $f_o = \omega_o/(2\pi)$ is the fundamental frequency.

10. Distortionless Transmission

A transmission channel is said to be distortionless if the output is a replica of the input. There may be a change of level, that is, amplitude scaling is permissible. Also, any physical channel will require some nonzero amount of time for transmission, so that a delay is inevitable. Hence, distortionless transmission occurs if the output y(t) is of the form

$$y(t) = K\, x(t-\tau) \tag{75}$$

where K and τ are constants. Taking the Fourier transform of both sides, we get

$$Y(j\omega) = K e^{-j\omega\tau} X(j\omega) \tag{76}$$

so that the frequency response of the distortionless channel becomes

$$H(j\omega) = K e^{-j\omega\tau} \tag{77}$$

The important result we have arrived at is that amplitude characteristic should be flat and the phase shift should be linear. Such ideal characteristics cannot of course be realized. Deviation from constancy of $|H(j\omega)|$ results in amplitude distortion and deviation from linear phase causes phase distortion.