INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR.379/18

COURSE ON BASIC TELECOMMUNICATIONS SCIENCE

9 January - 3 February 1989

Probability

S.C. DUTTA ROY

Indian Institute of Technology, Dept. of Electrical Engineering,
New Delhi, India

Probability

Triplet $(S, \mathcal{E}, P)$ where S is the sample space
$\mathcal{E}$ set of outcomes (events)
P measure

Basic axioms: 1) If $A_i \in \mathcal{E}$ and $A_j \in \mathcal{E} \rightarrow A_i \cup A_j \in \mathcal{E}$ ($\bar{A}$ = complement of A)
2) If $A \in \mathcal{E} \rightarrow \bar{A} \in \mathcal{E}$
3) For $A_1, A_2, \ldots, A_N \in \mathcal{E} \rightarrow \bigcup_i A_i \in \mathcal{E}$

Postulates: 1) $P(A) \geq 0$ for $A \in \mathcal{E}$
2) $P(\mathcal{E}) = 1$
3) Mutually Exclusive events, say, $A_i \cap A_j = \phi$ for all $j \neq i$ then $P(\bigcup_i A_i) = \sum_i P(A_i)$

Joint Events: Let A_i, $1 \leq i \leq N$, and B_j, $1 \leq j \leq M$ be events then (A_i, B_j) is a joint event and its measure is $P(A_i, B_j)$ where $0 \leq P(A_i, B_j) \leq 1$.

If B_j are mutually exclusive events then

$$\sum_j P(A_i, B_j) = P(A_i)$$

If A_i are mutually exclusive events then

$$\sum_i P(A_i, B_j) = P(B_j)$$

Therefore, $\sum_i \sum_j P(A_i, B_j) = 1$

Conditional Probabilities: a) $P(A/B) = \dfrac{P(A,B)}{P(B)}$ for $P(B) > 0$

b) $P(B/A) = \dfrac{P(A,B)}{P(A)}$ for $P(A) > 0$

Bayes' Theorem is an extremely useful relationship for conditional probabilities, which states that if A_i, $i=1,2,\dots,n$, are mutually exclusive events such that

$$\bigcup_i A_i = \mathcal{E}$$

then for B an arbitrary event with nonzero probability

$$P(A_i/B) = \frac{P(A_i,B)}{P(B)} = \frac{P(B/A_i)\,P(A_i)}{\sum_j P(B/A_j)\,P(A_j)}$$

Statistical Independence: If A and B are statistically independents then $P(A/B) = P(A)$ so $P(A,B) = P(A/B).P(B) = P(A).P(B)$

~~If ... statistically independents~~

Let A, B and C be events. They are statistically independents iff $P(AB) = P(A)P(B)$, $P(AC) = P(A)P(C)$, $P(BC) = P(B)P(C)$ and

$$P(A,B,C) = P(A).P(B).P(C)$$

Random variable - is a real valued function $X(\cdot)$ with domain the sample space.

Example 1: $X(s) = \begin{cases} 1, & s = \text{head} \\ 0 & s = \text{tail} \end{cases}$

(discrete random variable)

Example 2: $S = \{1,2,3,4,5,6\}$ $\quad X(s) = s^2$

$X(s) \in \{1,4,9,16,25,36\}$

If $P(S = s) = 1/6$ for $1 \le s \le 6$

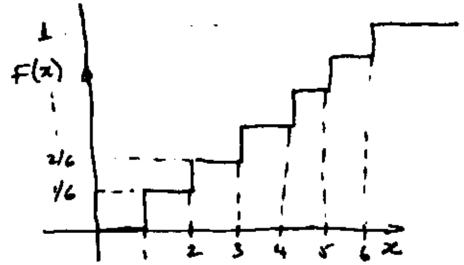
then $P(X(s)) = 1/6$

Let $\{X \le x\}$ be an event then $P(X \le x) \triangleq F(x)$.

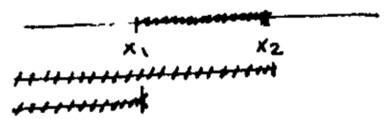
F(x) is called Probability Distribution Function or Cumulative Distri-

Example 3: $X \in \{1,2,3,4,5,6\}$ $\quad P(X = i) = 1/6$ for $1 \le i \le 6$

Find $P(X \le 4) = ?$ $\quad P(X \le 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4) = 4/6$



(Find) $P(x_1 < X \le x_2) \stackrel{?}{=} P(X \le x_2) - P(X \le x_1) = F(x_2) - F(x_1)$



or $P(X \le x_2) = ?$ Assume any $x_1 < x_2$. Find two mutually exclusive events, say, $\{X \le x_1\}$ and $\{x_1 < X \le x_2\}$ then

$\{X \le x_1\} \cup \{x_1 < X \le x_2\} = \{X \le x_2\}$. $\rightarrow$

$P(X \le x_1) + P(x_1 < X \le x_2) = P(X \le x_2) \therefore$

$$P(x_1 < X \le x_2) = P(X \le x_2) - P(X \le x_1)$$

This holds for discrete and continuous random variable.

For continuous, we have that

$$P(x < X \le x + \Delta x) = P(X \le x+\Delta x) - P(X < x)$$

$\div$ by Δx and let $\Delta x \to 0$.

$$p(x) = \lim_{\Delta x \to 0} \frac{P(x < X \le x+\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{P(X \le x+\Delta x) - P(X < x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{dF(x)}{dx}$$

So $p(x) = \frac{dF(x)}{dx}$ is called probability density function

$$dF(x) = p(x)\,dx \xrightarrow{\text{Integrate}} \int_{-\infty}^{x} dF(x) = \int_{-\infty}^{x} p(x')\,dx' \rightarrow$$

$$F(x) - F(-\infty) = \int_{-\infty}^{x} p(x')\,dx'$$

Since $F(-\infty) = 0$ and $F(\infty) = 1$ then

$$F(x) = \int_{-\infty}^{x} p(x')\,dx'$$

Find $P(x_1 < X \le x_2) = ?$

$$P(x_1 < X \le x_2) = F(x_2) - F(x_1) = \int_{-\infty}^{x_2} p(x')\,dx' - \int_{-\infty}^{x_1} F(x')\,dx' =$$

$$= \int_{-\infty}^{x_1} p(x')\,dx' + \int_{x_1}^{x_2} p(x')\,dx' - \int_{-\infty}^{x_1} p(x')\,dx'$$

so $$\boxed{P(x_1 < X \le x_2) = \int_{x_1}^{x_2} p(x')\,dx'}$$