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COURSE ON BASIC TELECOMMUNICATIONS SCIENCE

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Electromagnetism: A Refresher Course PT. 2

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6. Faraday effect

Macroscopic parameters of a cloud of charges

Let the cloud consist of particles of charge q and mass m, distributed with density $\underline{n}$. One of the characteristics of the cloud is the plasma frequency, given by

$$f_p = \frac{1}{2\pi}\sqrt{\frac{nq^2}{m\varepsilon_o}} \qquad (6.1)$$

A few values for an electron cloud are

$n\,(cm^{-3})$	10^8	10^{11}	10^{12}	10^{14}	10^{16}
f_p	90MHz	2,85GHz	9GHz	90GHz	900GHz

When the cloud is immersed in a D.C. magnetic field $\bar{b}_o$, the particles are subjected to a circular motion of angular frequency

$$\omega_c = \frac{qb_o}{m} \qquad (6.2)$$

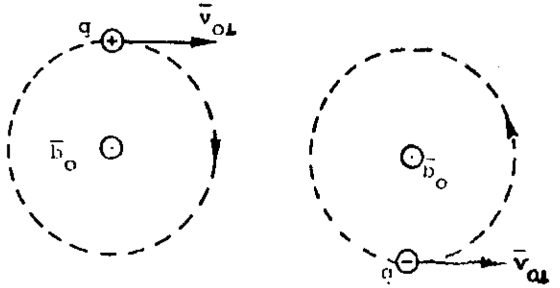


Fig. 6.1

This is the cyclotron frequency. It has a sign, from which the rotation sense may be deduced. For an electron q = -e.

A few values for an electron cloud are (10000 Gauss = 1 T).

b_o (Gauss)	143	357	1070	3570	12500
f_c (GHz)	0.4	1	3	10	35

Given these parameters, it is possible to show that a charged cloud in a $\bar{b}_o$ behaves like an anisotropic medium in which $\bar{D} = \bar{\bar{\varepsilon}}.\bar{E}$, where

$$\bar{\bar{\varepsilon}} = \begin{pmatrix} \varepsilon & j\varepsilon' & 0 \\ -j\varepsilon' & \varepsilon & 0 \\ 0 & 0 & \varepsilon'' \end{pmatrix} \qquad (6.3)$$

If we neglect the collisions, and the associated losses, the parameters are

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = 1 + \frac{\omega_p^2}{\omega_c^2-\omega^2}$$

$$\varepsilon_r' = \frac{\varepsilon'}{\varepsilon_o} = -\frac{\omega_c\omega_p^2}{\omega(\omega_c^2-\omega^2)} \qquad (6.4)$$

$$\varepsilon_r'' = \frac{\varepsilon''}{\varepsilon_o} = 1 - \frac{\omega_p^2}{\omega^2}$$

For the electron $\omega_c = -\frac{eb_{oz}}{m}$. The z-axis is directed along $\bar{b}_o$.

Plane wave propagation

For a wave propagating in the z-direction Maxwell's equations become

$$-\frac{\partial E_y}{\partial z} = -j\omega\mu_o H_x$$

$$\frac{\partial E_x}{\partial z} = -j\omega\mu_o H_y$$

$$-\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x - \omega\varepsilon' E_y = j\omega D_x \qquad (6.5)$$

$$\frac{\partial H_x}{\partial z} = \omega\varepsilon' E_x + j\omega\varepsilon E_y = j\omega D_y$$

To uncouple these 4 equations with 4 unknowns we replace E_x, E_y by the linear combinations

$$A = \frac{1}{2}(E_x - jE_y) \qquad (6.6)$$

$$B = \frac{1}{2}(E_x + jE_y)$$

In terms of these components:

$$\bar{E} = E_x\bar{u}_x + E_y\bar{u}_y = A(\bar{u}_x + j\bar{u}_y) + B(\bar{u}_x - j\bar{u}_y) \qquad (6.7)$$

The electric field has clearly been split into two circularly polarized components. For the magnetic field, analogously,

$$C = \frac{1}{2}\,(H_x - jH_y) \qquad (6.8)$$

$$D = \frac{1}{2}\,(H_x + jH_y)$$

$$\bar{H} = H_x\bar{u}_x + H_y\bar{u}_y = C(\bar{u}_x + j\bar{u}_y) + D(\bar{u}_x - j\bar{u}_y)$$

When these components are inserted in (6.5) two systems of (uncoupled) equations are obtained. For the (A,C) couple :

$$\frac{dA}{dz} = -j\omega\mu_o(jC)$$

$$-\frac{d(jC)}{dz} = j\omega(\varepsilon - \varepsilon')A \qquad (6.9)$$

Elimination of C gives

$$\boxed{\frac{d^2A}{dz^2} + \omega^2\mu_o(\varepsilon - \varepsilon')A = 0} \qquad (6.10)$$

In a similar fashion we obtain

$$\boxed{\frac{d^2B}{dz^2} + \omega^2\mu_o(\varepsilon + \varepsilon')B = 0} \qquad (6.11)$$

Let us assume that $b_{oz} > 0$, i.e. that the wave propagates

in the positive direction of $\bar{b}_o$. For such case the rotation sense of the "A" wave is that of the ions. This "ionic" wave has a propagation constant k_i given by

$$k_i^2 = \omega^2\mu_o(\varepsilon - \varepsilon') = k_o^2\left[1 - \frac{\omega_p^2}{\omega(\omega + |\omega_c|)}\right] \qquad (6.12)$$

The "B" wave is similarly an "electronic" wave, with constant

$$k_e^2 = \omega^2\mu_o(\varepsilon + \varepsilon') = k_o^2\left[1 - \frac{\omega_p^2}{\omega(\omega - |\omega_c|)}\right] \qquad (6.13)$$

It is clear that the wave is propagated when $k^2 > 0$ (passband), but that it is attenuated when $k^2 < 0$ (stop band).

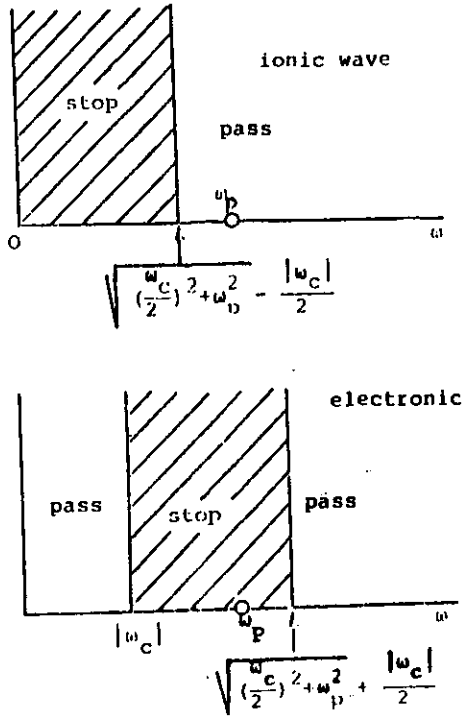


Fig. 6.2

Faraday effect

This effect arises because the two basic waves, A and B, have different propagation constants. Let us assume that both waves propagate, and that $\bar{E}$ is linearly polarized in the x-direction at z = 0. Thus,

$$\bar{E}(0) = \bar{u}_x = \underbrace{\tfrac{1}{2}(\bar{u}_x+j\bar{u}_y)}_{\text{A-wave}} + \underbrace{\tfrac{1}{2}(\bar{u}_x-j\bar{u}_y)}_{\text{B-wave}} \tag{6.14}$$

Farther down the z-axis this field has become

$$\bar{E}(z) = \tfrac{1}{2}(\bar{u}_x+j\bar{u}_y)e^{-jk_i z} + \tfrac{1}{2}(\bar{u}_x-j\bar{u}_y)e^{-jk_e z} \tag{6.15}$$

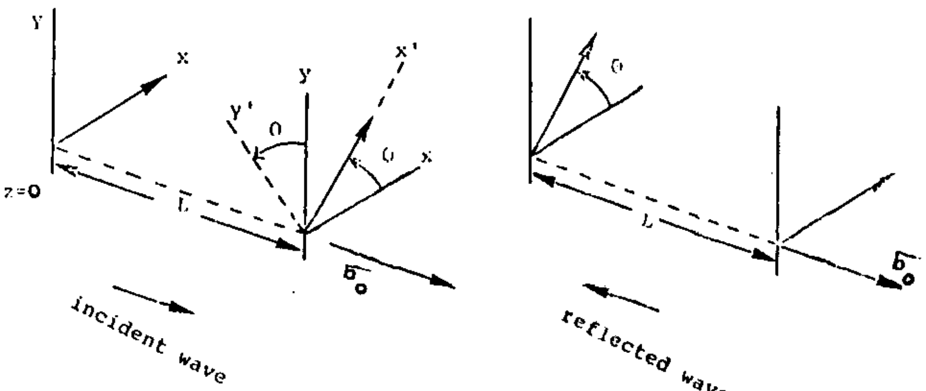


Fig. 6.3

Such a field is again linearly polarized, but in a new direction x' forming an angle θ with x. This is shown by applying the following coordinate transformation

$$\bar{u}_x = \bar{u}_{x'} \cos\theta - \bar{u}_{y'} \sin\theta$$
$$\bar{u}_y = \bar{u}_{x'} \sin\theta + \bar{u}_{y'} \cos\theta \tag{6.16}$$

The electric field is now

$$\bar{E}(z) = \tfrac{1}{2}\bar{u}_{x'}\left[e^{+j\theta}e^{-jk_i z} + e^{-j\theta}e^{-jk_e z}\right] + \tfrac{j}{2}\bar{u}_{y'}\left[e^{+j\theta}e^{-jk_i z} - e^{-j\theta}e^{-jk_e z}\right] \tag{6.17}$$

It can be written as

$$\bar{E}(z) = \bar{u}_{x'}\, e^{-j\frac{k_e+k_i}{2}z} \tag{6.18}$$

provided we set

$$\boxed{\theta = \frac{k_i-k_e}{2}z} \tag{6.19}$$

This is the angle which characterizes the Faraday rotation. If we look at the reflected wave in Fig. (6.3) we see that it propagates against the magnetic field, hence that $b_{oz} < 0$. As a result, the angle θ is the negative of (6.19) with respect to the direction of propagation. It therefore has the same direction in space, a property which has interesting technological applications.

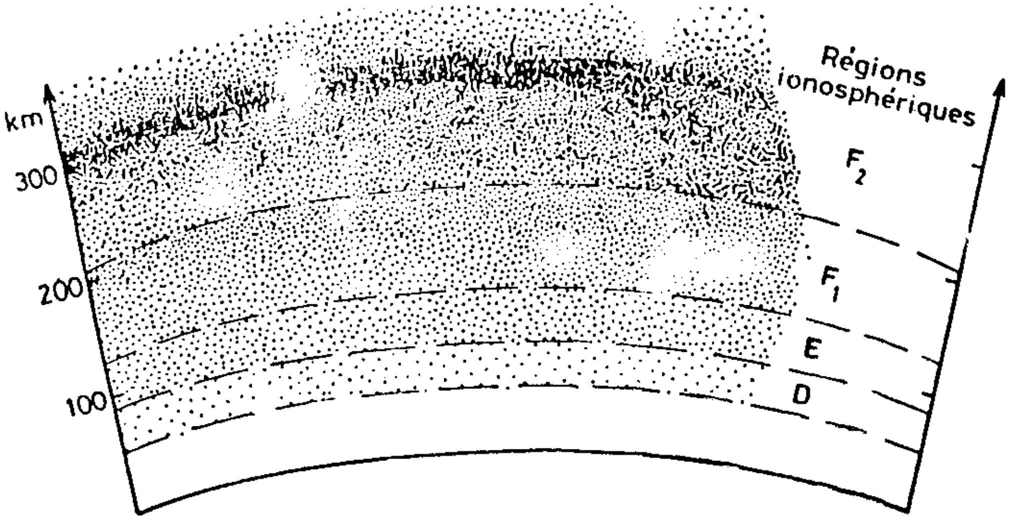
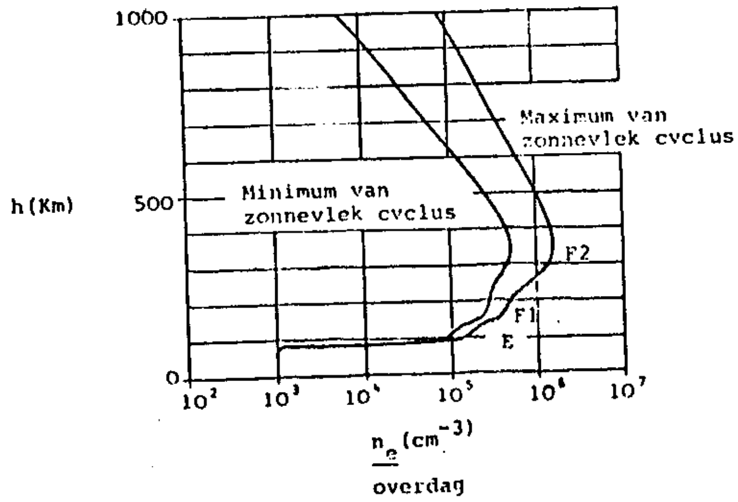
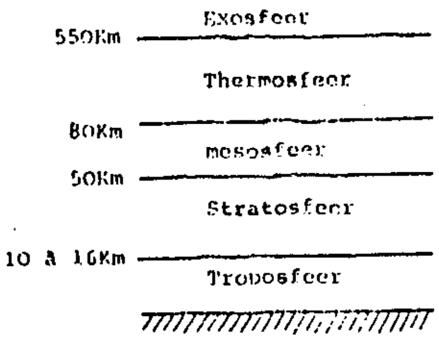
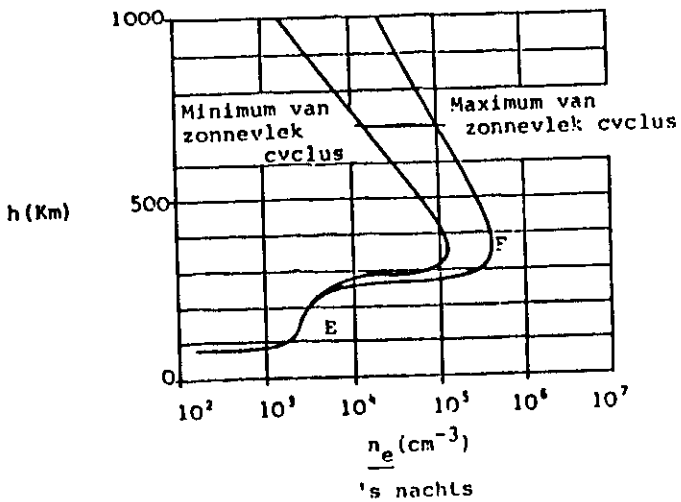


Fig. 5 : *L'ionosphère terrestre, qui comprend les régions E, D, F et la région supérieure, est constituée d'un plasma faiblement ionisé par le rayonnement UV solaire et par le rayonnement cosmique. Les ondes radio de basses fréquences y sont réfléchies et absorbées dans certains cas.*



(a)





(b)

7. Far field

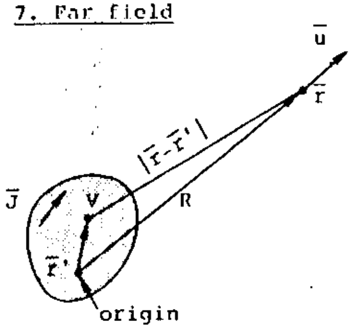


Fig. 7.1

Fig. 7.1 shows time harmonic currents radiating in free space. The complex vector potential $\bar{A}(\bar{r})$, derived from (2.8), is of the form

$$\bar{A}(\bar{r}) = \frac{\mu_o}{4\pi}\iiint \frac{\bar{J}(\bar{r}')e^{-jk_o|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\,dV' \qquad (7.1)$$

At large distances, in a direction of unit vector $\bar{u}$,

$$|\bar{r}-\bar{r}'| \approx R - \bar{u}.\bar{r}' \qquad (7.2)$$

Inserting this value in (7.1) gives the far field expression

$$\lim_{R\to\infty} \bar{A} = \frac{e^{-jk_oR}}{R}\underbrace{\frac{\mu_o}{4\pi}\iiint \bar{J}(\bar{r}')\,e^{jk_o\,\bar{u}.\bar{r}'}\,dV'}_{\text{direction dependent vector } \bar{N}(\bar{u})} \qquad (7.3)$$

The $\bar{E}$ and $\bar{H}$ fields, obtained from (1.17) and (2.1), are

$$\bar{E}(\bar{r}) = \bar{F}(\bar{u})\,\frac{e^{-jk_oR}}{R}$$

$$\bar{H}(\bar{r}) = \frac{1}{R_{co}}\,(\bar{u}\times\bar{F})\,\frac{e^{-jk_oR}}{R} \qquad (7.4)$$

where $\bar{F}$ is the transverse vector

$$\bar{F} = \bar{u}\times(\bar{u}\times\bar{N}) \qquad (7.5)$$

These very important formulas will be discussed further in the lectures on Antenna Theory.

When the dimensions of the source are small with respect to the wavelength λ, the exponential in $\bar{N}(\bar{u})$ may be usefully expanded as

$$e^{jk_o\bar{u}.\bar{r}'} = 1 + jk_o\bar{u}.\bar{r}' + \frac{1}{2}(jk_o)^2(\bar{u}.\bar{r}')^2+\ldots \qquad (7.6)$$

Inserting this value in (7.3) gives

$$\lim_{R\to\infty} \bar{A} = \frac{e^{-jk_oR}}{R}\,\frac{\mu_o}{4\pi}\left[\iiint \bar{J}(\bar{r}')dV' + jk_o\iiint(\bar{u}.\bar{r}')\bar{J}(\bar{r}')dV' + \text{terms in } k_o^2\right] \qquad (7.7)$$

Suitable manipulation of this equation leads to the far fields

$$\bar{E} = \frac{R_{co}}{4\pi}\,\frac{e^{-jk_oR}}{R}\left[-k_o^2c\,\bar{u}\times(\bar{u}\times\bar{P}_e) - k_o^2\bar{u}\times\bar{P}_m + \text{terms in } k_o^3\right]$$

$$\bar{H} = \frac{1}{4\pi}\,\frac{e^{-jk_oR}}{R}\left[k_o^2c\,\bar{u}\times\bar{P}_e - k_o^2\,\bar{u}\times(\bar{u}\times\bar{P}_m) + \text{terms in } k_o^3\right] \qquad (7.8)$$

We recognize in these formulas the contributions of, first, an <u>electric dipole moment</u>

$$\bar{P}_e = \frac{1}{j\omega}\iiint\bar{J}\,dV = \iiint_V \rho\,\bar{r}\,dV + \iint_S \rho_s\,\bar{r}\,dS \qquad (7.9)$$

and, second, a <u>magnetic dipole moment</u>

$$\bar{P}_m = \frac{1}{2}\iiint_V \bar{r}\times\bar{J}\,dV \qquad (7.10)$$

The discussion can be carried further, and the terms in k_o^3 shown to consist of contributions from e.g. quadrupole moments. The fields of the dipoles will be mentioned again in the lectures on Antenna Theory.

8. Scattering cross-sections

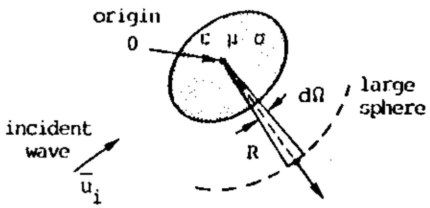


Fig . 8.1

A typical scattering configuration is shown in Fig. 8.1, which displays a target of arbitrary shape and constitutive parameters (ε,μ,σ), immersed in an incident electromagnetic wave of arbitrary time dependence. The determination of the scattered fields is a most difficult task, often made somewhat easier by assuming that the incident wave is plane and time-harmonic. Such a restriction is reasonable because, at large distances, the fields of an arbitrary source behave locally as those of a plane wave (see eq. 7.4). Further, Fourier expansions with respect to time and space coordinates show that an arbitrary incident wave may be considered as the superposition of an infinite number of time-harmonic plane waves.

The power density in a progressive plane wave is obtained from (4.12) and (5.5) as

$$W_i = \frac{1}{2}\,\mathrm{Re} \iint_{\substack{\text{unit}\\\text{area}}} (\bar{E} \times \bar{H}^*)\cdot\bar{u}_i dS = \frac{1}{2R_{co}}\,|\bar{E}_i|^2 \quad \mathrm{W\ m^{-2}} \qquad (8.1)$$

Illuminated by this wave the "scatterer" or "target" becomes a source of induced currents (volume or surface conduction currents, polarization currents ...), and acts as a secundary "antenna" producing a far field,

$$\bar{E}_{sc} = \bar{F}_{sc}(\bar{u})\,\frac{e^{-jkR}}{R} \qquad \mathrm{V\ m^{-1}}$$
$$\bar{H}_{sc} = \frac{1}{R_{co}}\,(\bar{u} \times \bar{F}_{sc})\,\frac{e^{-jkR}}{R} \qquad \mathrm{A\ m^{-1}} \qquad (8.2)$$

Such a formula holds for every observation direction $\bar{u}$. From (4.12) the time averaged power radiated in an elementary solid angle $d\Omega$ centered on $\bar{u}$ is

$$d\mathcal{P}_{sc} = \frac{1}{2}\,\mathrm{Re}(\bar{E}_{sc} \times \bar{H}^*_{sc})\cdot\bar{u}\underbrace{(R^2 d\Omega)}_{dS} = \frac{1}{2R_{co}}\,|\bar{F}_{sc}|^2\,d\Omega \quad \mathrm{W} \qquad (8.3)$$

The total scattered power follows by summing over all solid angles (i.e. over 4π steradians). Thus,

$$\mathcal{P}_{sc} = \frac{1}{2R_{co}} \iint_{4} |\bar{F}_{sc}|^2\,d\Omega \qquad \mathrm{W} \qquad (8.4)$$

A quantity independent of the power level, the total scattering cross-section, is obtained by dividing the power by the incident power density. Thus,

$$\sigma_{sc}(\bar{u}_i) = \frac{\mathcal{P}_{sc}}{W_i} = \frac{\iint_{4\pi} |\bar{F}_{sc}|^2}{|\bar{E}_i|^2}\ \mathrm{m}^2 \qquad (8.5)$$

This cross-section is a function of frequency, of the direction of incidence $\bar{u}_i$, and of the state of polarization of the incident wave. To illustrate the concept : a σ_{sc} of 3 m^2 means that the target, illuminated by 1 kW per m^2, will scatter 3 kW.

The total scattering cross-section does not express how much power is scattered in any given direction $\bar{u}$. This directional sensitivity is expressed by the bistatic cross-section $\sigma_{bis}(\bar{u}|\bar{u}_i)$, which can be most conveniently defined by way of a numerical example. Let $W_i = 1\ \mathrm{kW\ m^{-2}}$, then $\sigma_{bis} = 2\ m^2$ means that a power

$$d\mathcal{P}_{sc} = 2\left(\frac{d\Omega}{4\pi}\right)\ \mathrm{kW} \qquad (8.6)$$

is scattered in an elementary solid angle $d\Omega$ centered on the direction $\bar{u}$. A very important particular case is that of the monostatic or radar cross-section. It is the bistatic cross-section relative to the backscattering direction $(-\bar{u}_i)$. Thus,

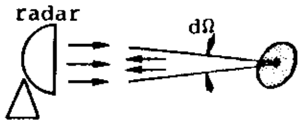


Fig. 8.2

$$\sigma_{rad}(\bar{u}_i) = \sigma_{bis}(-\bar{u}_i|\bar{u}_i) \qquad (8.7)$$

With $W_i = 0.1\mathrm{W\ m^{-2}}$, a $\sigma_{rad} = 3\ m^2$ means that

$$d\mathcal{P}_{sc} = 0.3(\frac{d\Omega}{4\pi}) \quad W \tag{8.8}$$

is scattered back in a solid angle $d\Omega$ (towards the radar set, see Fig. 8.2).

The previous considerations are operational definitions. They

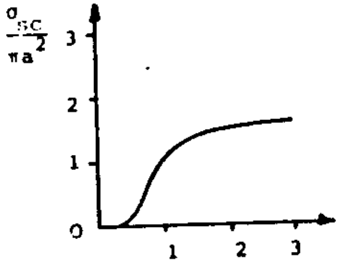


Fig. 8.3

do not solve the real problem, which is to find $\bar{F}(\bar{u})$. We ignore this difficult assignment, and limit ourselves to a display of a few typical results. Fig. 8.3 shows the total cross section of an iron sphere as a function of the radius $\underline{a}$. The frequency is 7.1 10^{14} Hz (green light), and the curve is drawn as a function of the dimensionless parameter $ka = (2\pi a/\lambda)$. Fig. 8.4 shows σ_{rad} for a perfectly conducting sphere. Notice the successive

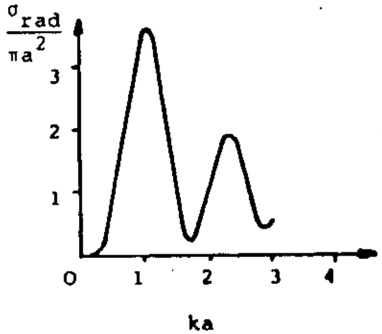


Fig. 8.4

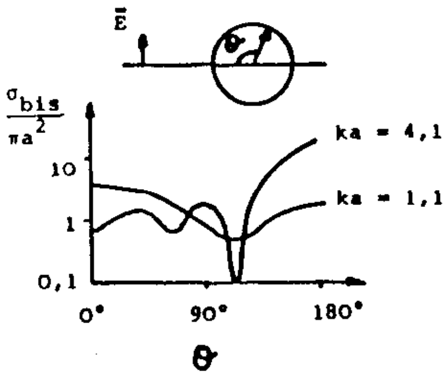


Fig. 8.5

"resonance" peaks. Fig. 8.5 shows, for the same sphere, the bistatic cross-section as a function of the angle of observation Θ.

9. Ray tracing

Ray tracing is a method used at very high frequencies, i.e. at short wavelengths. Wavelengths λ are said to be short when the characteristics of the medium supporting the wave vary little over a distance λ. For such case the wave behaves locally as a plane wave, and its direction of propagation is that of the ray. A typical time-harmonic component is written as

$$E_x = \mathcal{E}_x(\bar{r},k)e^{-jkS(\bar{r})} \tag{9.1}$$

The surfaces $S(\bar{r})$ = const. are the phase fronts. At high frequencies the amplitude $\mathcal{E}_x$ is expanded in a series in the small parameter $(1/k)$. Thus,

$$\mathcal{E}_x(\bar{r},k) = \mathcal{E}_{xo}(\bar{r}) + \frac{1}{k}\mathcal{E}_{x1}(\bar{r}) + \ldots \tag{9.2}$$

Expansions of this kind are inserted in Maxwell's equations. Taking into account that

$$\text{curl}\,\bar{E} = e^{-jkS}(\text{curl}\,\bar{\mathcal{E}} - jk\,\text{grad}\,S \times \bar{\mathcal{E}}) \tag{9.3}$$

yields

$$\begin{aligned} \text{curl}\,\bar{\mathcal{E}} - jk\,\text{grad}\,S \times \bar{\mathcal{E}} &= -jk\,R_{co}\,\mu_r\bar{\mathcal{H}} \\ \text{curl}\,\bar{\mathcal{H}} - jk\,\text{grad}\,S \times \bar{\mathcal{H}} &= \frac{jk}{R_{co}}\,\epsilon_r\bar{\mathcal{E}} + \sigma\bar{\mathcal{E}} \end{aligned} \tag{9.4}$$

Equating the dominant terms (the terms in k) leads to

$$\begin{aligned} \text{grad}\,S \times \bar{\mathcal{E}}_o &= \mu_r\,R_{co}\,\bar{\mathcal{H}}_o \\ \text{grad}\,S \times \bar{\mathcal{H}}_o &= -\frac{\epsilon_r}{R_{co}}\,\bar{\mathcal{E}}_o \end{aligned} \tag{9.5}$$

Equations (9.5) show that grad S is perpendicular to both $\bar{\mathcal{E}}_o$ and $\bar{\mathcal{H}}_o$, and that $\bar{\mathcal{E}}_o$ is perpendicular to $\bar{\mathcal{H}}_o$. They also imply the "eikonal equation"

$$\boxed{|\text{grad}\,S|^2 = \epsilon_r\mu_r = n^2} \tag{9.6}$$

The index of refraction $\underline{n}$ is a function of x,y,z. The $\underline{\text{rays}}$ are the orthogonal trajectories of S(x,y,z); they are therefore tangent to grad S. If ℓ is the length of arc measured along the ray, the equations of the latter take the form

$$\frac{\frac{dx}{d\ell}}{\frac{\partial S}{\partial x}} = \frac{\frac{dy}{d\ell}}{\frac{\partial S}{\partial y}} = \frac{\frac{dz}{d\ell}}{\frac{\partial S}{\partial z}} = \frac{1}{n} \tag{9.7}$$

These equations are obtained by making use of the relationship

$$\left(\frac{dx}{d\ell}\right)^2 + \left(\frac{dy}{d\ell}\right)^2 + \left(\frac{dz}{d\ell}\right)^2 = 1 \tag{9.8}$$

We now eliminate S by the following manipulation :

$$\frac{d}{d\ell}\left(n\frac{dx}{d\ell}\right) = \frac{d}{d\ell}\left(\frac{\partial S}{\partial x}\right) = \frac{\partial^2 S}{\partial x^2}\frac{dx}{d\ell} + \frac{\partial^2 S}{\partial x \partial y}\frac{dy}{d\ell} + \frac{\partial^2 S}{\partial x\partial z}\frac{dz}{d\ell}$$

$$= \frac{1}{n}\frac{\partial S}{\partial x}\frac{\partial^2 S}{\partial x^2} + \frac{1}{n}\frac{\partial S}{\partial y}\frac{\partial^2 S}{\partial x\partial y} + \frac{1}{n}\frac{\partial S}{\partial z}\frac{\partial^2 S}{\partial x\partial z} \tag{9.9}$$

$$= \frac{1}{2n}\frac{\partial}{\partial x}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right]$$

$$= \frac{1}{2n}\frac{\partial n^2}{\partial x}$$

Combining with similar equations involving y and z yields

$$\boxed{\frac{d}{d\ell}\left(n\frac{d\bar{r}}{d\ell}\right) = \text{grad } n} \tag{9.10}$$

It is clear that $\frac{d\bar{r}}{d\ell}$ is the unit vector $\bar{u}_\ell$ along the ray. Once the rays are found, $S(\bar{r})$ follows from

$$\frac{dS}{d\ell} = n$$

and a straightforward integration. Thus,

$$S(P_1) = S(P_o) + \int_{\ell_o}^{\ell_1} n\, d\ell \tag{9.11}$$

In a homogeneous volume $\underline{n}$ is constant, hence grad n = 0. Eq. (9.10) then implies that $\bar{u}_\ell$ is a constant, which in turn implies that the rays are straight lines. In an inhomogeneous region, however, (9.10) gives

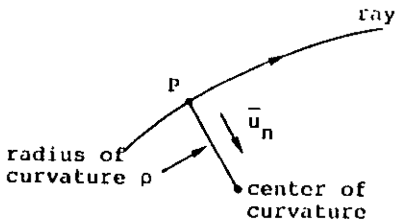


Fig. 9.1

$$n\frac{d\bar{u}_\ell}{d\ell} + \bar{u}_\ell\frac{dn}{d\ell} = \text{grad } n \tag{9.12}$$

The vector $\frac{d\bar{u}_\ell}{d\ell}$ is perpendicular to the ray, as

$$\frac{d\bar{u}_\ell}{d\ell} = \frac{1}{\rho}\bar{u}_n \tag{9.13}$$

It follows that

$$\frac{d\bar{u}_\ell}{d\ell} = \left(\frac{\text{grad } n}{n}\right)_\perp \tag{9.14}$$

where the subscript $\perp$ denotes projection on a plane perpendicular to the ray. A possible graphical construction of the ray follows from (9.14). Assume, indeed, that the direction of the ray is known in P_o (unit vector $\bar{u}_o$). The unit vector in a neighbouring point P_1 may be obtained from (9.14) by the operation

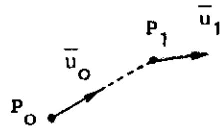


Fig. 9.2

$$\bar{u}_1 = \bar{u}_o + P_oP_1\left(\frac{\text{grad } n}{n}\right)_\perp \tag{9.15}$$

This relationship allows a point by point construction of the ray (Fig. 9.2). Eqs. (9.12) and (9.13) also yield

$$\bar{u}_n \cdot \text{grad}(\log_e n) = \frac{1}{\rho} \tag{9.16}$$

It is clear, from (9.16), that the rays are curved in the direction of high indices $\underline{n}$.

Ray tracing can go further, and generate the laws governing the amplitude and polarization of the fields along a ray. These laws are obtained by equating terms of higher order in (1/k) on both sides of (9.…). The detailed calculations are beyond the scope of the present notes.

10. Elements of Relativity . Doppler effect

The Lorentz transformation

By the turn of the century a series of experiments pointed to the need for an agonizing reappraisal of the laws of Mechanics. In an effort to reconcile theory and experiment, H.A. Lorentz proposed an important set of formulas, aimed at connecting the coordinates of an event measured in two different inertial frames of reference, K and K'. Inertial frames are frames in which a free body moves with uniform velocity. All inertial frames are in uniform translation with respect to each other and K', in particular, moves with velocity $\bar{w}$ with respect to K. Lorentz characterizes a point-event by four numbers, three space coordinates and one time coordinate. Their values in K and K' are related by

$$
\begin{aligned}
x &= x' \\
y &= y' \\
z &= \frac{z' + wt'}{\sqrt{1-\beta^2}} \qquad \beta = \frac{w}{c} \\
t &= \frac{1}{\sqrt{1-\beta^2}}\left(t' + \beta\,\frac{z'}{c}\right)
\end{aligned}
\tag{10.1}
$$

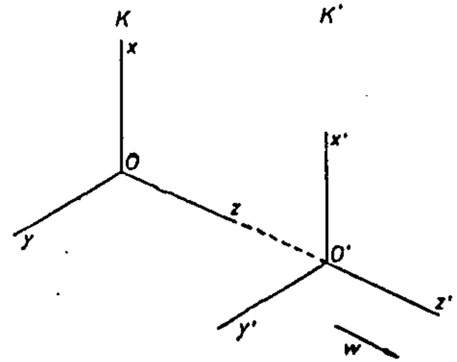


Fig. 10.1

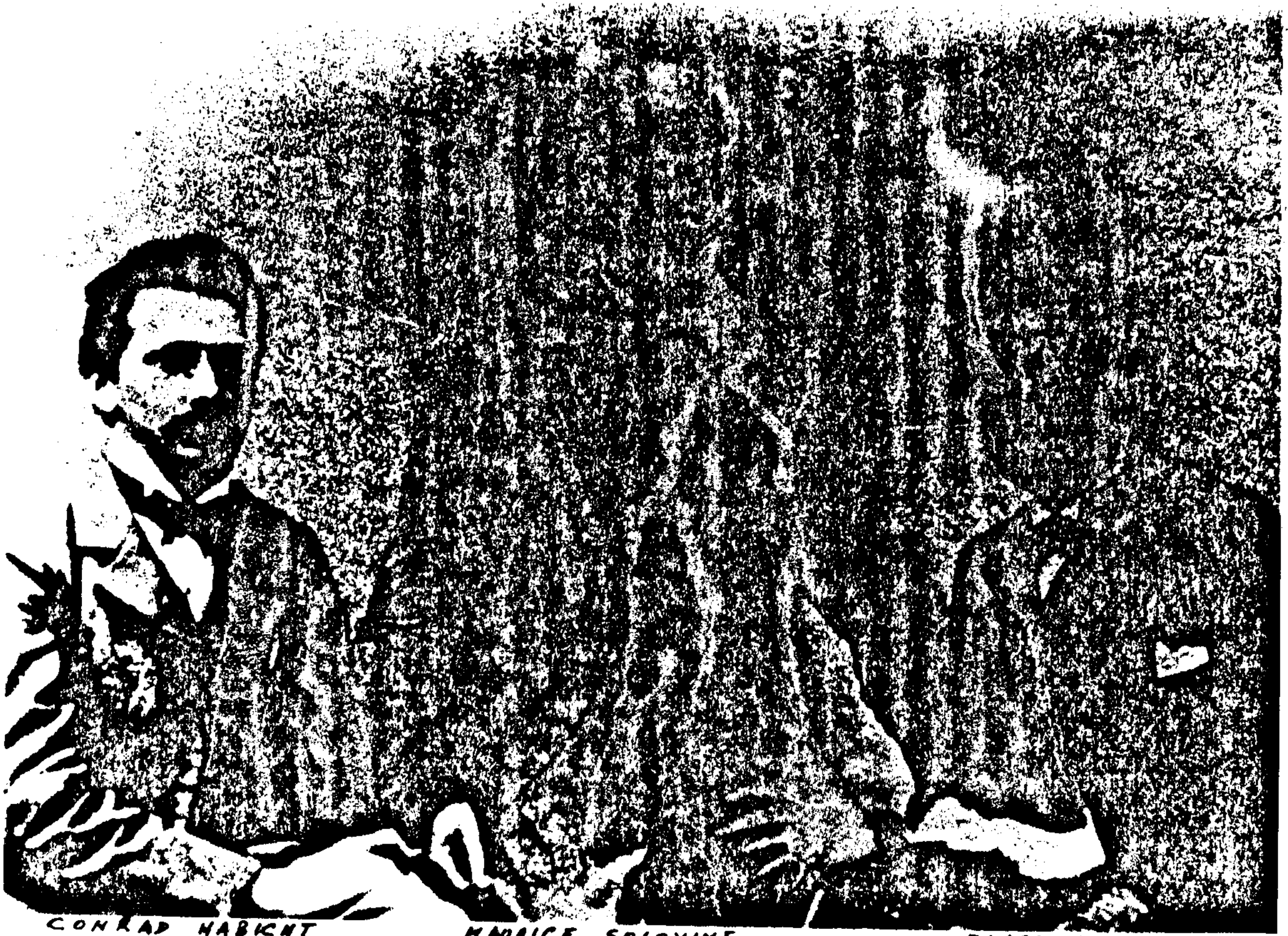
At low velocities (i.e. for $\beta \ll 1$) the third equation yields $z = z' + wt'$, which is the classical Newtonian formula. The fourth equation, however, represents a significant departure from Classical Mechanics. It shows that two events which are simultaneous in K', but occur at different z', are not simultaneous in K. Simultaneity therefore is a relative concept.

ISAAC NEWTON 1687

H. A. Lorentz

1890

BERN 1902-1904



CONRAD HABICHT MAURICE SOLOVINE EINSTEIN

Transformation of fields

$$\overline{e}' = \overline{e}_{\parallel} + \frac{1}{\sqrt{1-\beta^2}}(\overline{e}_{\perp} + \overline{w}\times\overline{b})$$

$$\overline{b}' = \overline{b}_{\parallel} + \frac{1}{\sqrt{1-\beta^2}}\left(\overline{b}_{\perp} - \frac{\overline{w}\times\overline{e}}{c^2}\right)$$

$$\overline{d}' = \overline{d}_{\parallel} + \frac{1}{\sqrt{1-\beta^2}}\left(\overline{d}_{\perp} + \frac{\overline{w}\times\overline{h}}{c^2}\right) \qquad (10.2)$$

$$\overline{h}' = \overline{h}_{\parallel} + \frac{1}{\sqrt{1-\beta^2}}(\overline{h}_{\perp} - \overline{w}\times\overline{d})$$

Doppler effect

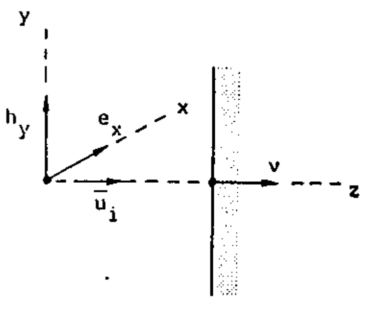


Fig.

Consider an incident plane wave propagating along the z-axis in the "laboratory" frame, i.e. in the axes of a (static) observer. The incident fields are

$$e_{ix} = E\cos(\omega t - \frac{\omega}{c}z)$$
$$h_{iy} = \frac{E}{R_{co}}\cos(\omega t - \frac{\omega}{c}z) \qquad (10.3)$$

This wave impinges at normal incidence on a perfectly conducting plane moving with velocity v. Let K' denote the axes in which the conductor is at rest. According to special relativity the fields in K', at velocities v<<c, are related to those in K by

$$e'_x = e_x - v\mu_o h_y$$
$$h'_y = h_y - v\varepsilon_o e_x \qquad (10.4)$$

Relativity also implies that the coordinates of an event (e.g. the measurement of a field) in K and K' are related by the Lorentz transformation

$$z = z' + vt'$$
$$t = t' + \frac{vz'}{c^2} \qquad (10.5)$$

Combining (10.1), (10.2) and (10.3) gives the transformed incident fields

$$e'_{ix} = E(1 - \frac{v}{c})\cos(\omega' t' - \frac{\omega'}{c}z')$$
$$h'_{iy} = \frac{E}{R_{co}}(1 - \frac{v}{c})\cos(\omega' t' - \frac{\omega'}{c}z') \qquad (10.6)$$

where

$$\omega' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,\omega \approx (1 - \frac{v}{c})\omega = \omega - (\frac{v}{c}\omega) \qquad (10.7)$$

This formula expresses the Doppler shift. More generally, for arbitrary velocities and a wave propagating in a direction making an angle α with the z-axis :

$$\boxed{\omega' = \frac{\omega}{\sqrt{1-\frac{v^2}{c^2}}}(1 - \frac{v}{c}\cos\alpha)} \qquad (10.8)$$

Radar echo

The Doppler shift is the source of numerous technical applications, e.g. in the area of electronic navigation. This shift is also of fundamental importance for the operation of moving target indicator radars (MTI), burglar alarms etc... To clarify this statement consider the simple model shown in Fig. 10.2. In the K' axes the reflected wave is

$$e'_{rx} = -E(1 - \frac{v}{c})\cos(\omega' t' + \frac{\omega'}{c}z')$$
$$h'_{ry} = \frac{E}{R_{co}}(1 - \frac{v}{c})\cos(\omega' t' + \frac{\omega'}{c}z') \qquad (10.9)$$

The reflected fields in K are obtained from those in K' through the "inverse" transformation formulas

$$
\begin{aligned}
e_x &= e'_x + v\mu_o h'_y \\
h_y &= h'_y + v\varepsilon_o e'_x \\
z' &= z - vt \\
t' &= t - \frac{vz}{c^2}
\end{aligned}
\qquad (10.10)
$$

This gives

$$
\begin{aligned}
e_{rx} &= -E(1 - 2\frac{v}{c})\cos(\omega''t + \frac{\omega''}{c} z) \\
h_{ry} &= \frac{E}{R_{co}}(1 - 2\frac{v}{c})\cos(\omega''t + \frac{\omega''}{c} z)
\end{aligned}
\qquad (10.11)
$$

Here ω" is the angular frequency of the radar echo, which is twice Doppler shifted according to the formula

$$
\boxed{\omega'' = \omega \frac{1 - \frac{v}{c}}{1 + \frac{v}{c}} \sim \omega(1 - 2\frac{v}{c})} \qquad (10.12)
$$

11. Transmission lines

Basic equations

The potential difference between points x and x + dx may be written as

$$
v(x+dx,t)-v(x,t) = \frac{\partial v}{\partial x} dx = -r\, dx\, i - \ell \frac{\partial i}{\partial t} dx + v_g dx \qquad (11.1)
$$

In this equation r is the linear resistance in $\Omega\, m^{-1}$, sum of the resistances of conductors 1 and 2, ℓ is the linear inductance (in $H\, m^{-1}$) and v_g the applied voltage (often ze

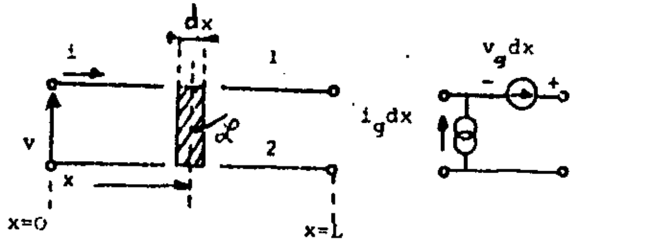


Fig. 11.1

A similar budget may be written for the current in terms of the linear capacitance C (in $F\, m^{-1}$), the linear conductance g (in $S\, m^{-1}$) and a possible current source i_g (in $A\, m^{-1}$). Taking the limit $dx \to 0$ yields the differential equations

$$
\boxed{\frac{\partial v}{\partial x} = -ri - \ell \frac{\partial i}{\partial t} + v_g} \qquad (11.2)
$$

$$
\boxed{\frac{\partial i}{\partial x} = -g\, v - c \frac{\partial v}{\partial t} + i_g} \qquad (11.3)
$$

We shall only consider the sourceless situation ($v_g = 0$, $i_g = 0$), and pay special attention to sinusoidal phenomena. For such case (11.2) and (11.3) become

$$
\frac{dV}{dx} = -r\, I - j\omega\ell\, I = -(r+j\omega\ell)I = -ZI \qquad (11.4)
$$

$$
\frac{dI}{dx} = -g\, V - j\omega c\, V = -(g + j\omega c)V = -YV \qquad (11.5)
$$

Incident and reflected waves

When the line is lossless the basic equations are

$$\frac{\partial v}{\partial x} = -\ell \frac{\partial i}{\partial t} \qquad \frac{\partial i}{\partial x} = -c \frac{\partial v}{\partial t} \tag{11.6}$$

Eliminating i yields

$$\frac{\partial^2 v}{\partial x^2} - \ell c \frac{\partial^2 v}{\partial t^2} = 0 \tag{11.7}$$

The general solution of this equation is obtained by way of the change of variables

$$u = x - \frac{1}{\sqrt{lc}}\, t \qquad v = x + \frac{1}{\sqrt{lc}}\, t \tag{11.8}$$

which converts (11.7) into

$$\frac{\partial^2 v}{\partial u \, \partial w} = 0 \tag{11.9}$$

The general solution of this equation is

$$v = f(u) + g(w) = f(x - \frac{1}{\sqrt{lc}}\, t) + g(x + \frac{1}{\sqrt{lc}}\, t) \tag{11.10}$$

The corresponding current has the form

$$i = \frac{1}{R_c}\left[f(x - v_{ph}t) - g(x + v_{ph}t) \right] \quad (R_c = \sqrt{\frac{\ell}{c}}) \tag{11.11}$$

Here R_c is the characteristic resistance of the line (in Ω), and v_{ph} is the phase velocity $(1/\sqrt{lc})$. The functions f and g are arbitrary, and their actual form is determined by the boundary

conditions at x = 0 and x = L. The function f represents a

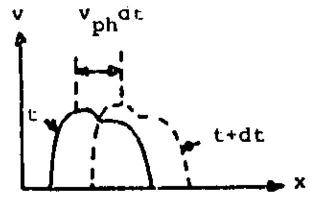


Fig. 11.2

wave to increasing x, as shown clearly in Fig. 11.2. The function g represents a wave to decreasing x. Normally a generator is connected in x = 0 and a load in x = L. The f wave then becomes an incident wave and the g wave a reflected one. It is easy to check that the incident wave is the only one to exist when the line is infinite or, equivalently, when it is loaded by R_c. For such a case there are no reflections, and the line is matched.

Time harmonic signals

Voltage and current have the general form

$$v(x,t) = v_1 \cos(\omega t - kx + \phi_1) + v_2 \cos(\omega t + kx + \phi_2) \tag{11.12}$$

$$i(x,t) = \frac{1}{R_c}\left[v_1 \cos(\omega t - kx + \phi_1) - v_2 \cos(\omega t + kx + \phi_2)\right]$$

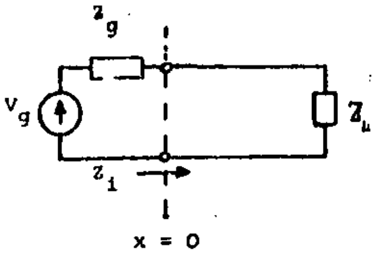
where

$$k = \frac{\omega}{v_{ph}} = \omega\sqrt{\ell c} = \frac{2\pi}{\lambda_g} \tag{11.13}$$

The quantity λ_g is the wavelength along the line. In phasor form :

$$V(x) = V_1\, e^{-jkx} + V_2\, e^{jkx}$$
$$I(x) = \frac{1}{R_c}\left(V_1\, e^{-jkx} - V_2\, e^{jkx}\right) \tag{11.14}$$

where $V_1 = v_1\, e^{j\phi_1}$ and $V_2 = v_2\, e^{j\phi_2}$. Writing (11.14) at x = 0 and x = d leads to the value of the input impedance of the line



$$\frac{Z_i}{R_c} = z_i' = \frac{z_L' + j \ tg \ kd}{1+j \ z_L' \ tg \ kd} \qquad (11.15)$$

Fig. 11.3

When the line is lossy the formulas become

$$V = V_1 \ e^{-\alpha x} e^{-j\beta x} + V_2 \ e^{\alpha x} e^{j\beta x} \qquad (11.16)$$

$$I = \frac{1}{Z_c} \ [V_1 \ e^{-\alpha x} e^{-j\beta x} - V_2 \ e^{\alpha x} e^{j\beta x}]$$

Attenuation and propagation constant are given by

$$\gamma^2 = (\alpha + j\beta)^2 = Y.Z = \alpha^2 - \beta^2 + 2j\alpha\beta = (rg - \omega^2 \ell c) + j(\omega \ell g + \omega rc) \qquad (11.17$$

The characteristic impedance becomes

$$Z_c^2 = \frac{Z}{Y} = (R_c + j \ X_c)^2 = \frac{rg + \omega^2 \ell c + j(\omega \ell g - \omega rc)}{g^2 + \omega^2 c^2} \qquad (11.18)$$

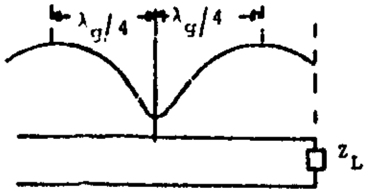
The input impedance is now

$$z_i' = \frac{Z_i}{Z_c} = \frac{z_L' + th\gamma \ d}{1 + z_L' th\gamma \ d} \qquad (11.19)$$

If the load is matched ($z_i' = 1$) the input impedance is Z_c, irrespective of the length of the line.

Reflection coefficient on a lossless line

From (11.14) the ratio between reflected and incident voltages (the reflection coefficient) is



$$K = K_V = \frac{V_2 \ e^{jkx}}{V_1 \ e^{-jkx}} = \frac{V_2}{V_1} \ e^{2jkx} \qquad (11.2$$

Fig. 11.4

At a distance d from the load, therefore,

$$K = K_L \ e^{-2j \ kd} \qquad (11.21$$

Reflection coefficient and impedance are connected by

$$K(x) = \frac{z_i'(x) - 1}{z_i'(x) + 1} \qquad z_i' = \frac{1+K}{1-K}) \qquad (11.22$$

This formula shows that a measurement of Z_i may be obtained from a measurement of K (in amplitude and phase). The value of K may be deduced from an observation of the interference between reflected and incident waves down the line. The amplitude of the voltage is given by

$$|V| = \sqrt{V_1^2 + V_2^2 + 2V_1V_2 \cos(2kx + \phi_2 - \phi_1)} \qquad (11.23$$

$$= V_1 \sqrt{1 + |K|^2 + 2 \ |K| \cos(2kx + \phi_2 - \phi_1)}$$

The general appearance of the curve $|V(x)|$ is as shown in Fig. 11.4. The maxima occur at points where the two waves are in phase, i.e. for

$$- kx + \phi_1 = kx + \phi_2 + n2\pi \qquad (11.24)$$

These points are separated by $(\lambda_g/2)$. The minima (destructive interference) correspond to

$$- kx + \phi_1 = kx + \phi_2 + \pi + 2n\pi \qquad (11.25)$$

They are located $(\lambda_g/4)$ from the maxima. An observation of the ratio of maximum to minimum (the standing wave ratio) gives $|K|$:

$$\mathrm{VSWR} = \frac{v_1+v_2}{v_1-v_2} = \frac{1+\frac{v_2}{v_1}}{1-\frac{v_2}{v_1}} = \frac{1+|K|}{1-|K|} \qquad (11.26)$$

The location of the maxima and minima gives $(\phi_2-\phi_1)$, i.e. the phase angle of K_L. One is then able to construct K_L, and from there to determine the unknown Z_L by use of (11.22). The method is the classical way of determining impedances at high frequencies.

Matching

In most applications it is desirable to match the load to the line. The reasons are :

(1) The power to the load. It is given by

$$\mathcal{P} = \frac{1}{2} \mathrm{Re}\ (VI^*) = \frac{V_1^{\ 2}}{2R_c} - \frac{V_2^{\ 2}}{2R_c} = \frac{V_1^{\ 2}}{2R_c}\ (1 - |K|^2) \qquad (11.27)$$

$\mathcal{P}$ is maximum when $|K| = 0$.

(2) Reflections create peaks of voltage. For a given absorbed power $\mathcal{P}$:

$$|V_{max}| = \sqrt{2\mathcal{P}R_c}\ \sqrt{\frac{1+K}{1-K}} \qquad (11.28)$$

The danger for breakdown increases with $|K|$.

(3) the input impedance is much more sensitive to small frequency excursions when the load is unmatched. This "long line effect" will be explained in the problem session, together with methods to match an (originally unmatched) load, and the use of the Smith chart.

These various considerations may be applied, in slightly modified form, to the lossy line. There,

$$K = K_L\ e^{-2\gamma d} = K_L\ e^{-2\alpha d}\ e^{-2j\beta d} \qquad (11.29)$$

$$K = \frac{Z'-1}{Z'+1}$$

12. Modes and eigenfunctions

We shall discuss the basic ideas of the eigenfunction method on the very simple example of the flexible string, a basic component of several musical instruments (Fig. 12.1). The string is under tension T, and is acted upon by an external force of density p(x) (in N m^{-1}). The force on a small element of string dx is therefore p(x)dx. The equation satisfied by the small displacement y(x,t) of the string is a wave equation, viz.

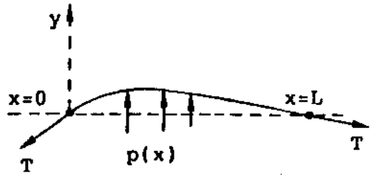


Fig. 12.1

$$\begin{cases} \frac{\partial^2 y}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = -\frac{p(x,t)}{T} \\ y = 0 \text{ in } x = 0 \text{ and } x = L \end{cases} \quad (12.1)$$

The symbol ρ denotes the mass density of the string (in kg m^{-1}). The velocity of propagation is $c = \sqrt{(T/\rho)}$. Under time-harmonic conditions :

$$\begin{cases} \frac{d^2 Y}{dx^2} + \frac{\omega^2 \rho}{T} Y = -\frac{P(x)}{T} \\ Y = 0 \text{ at } x = 0 \text{ and } x = L \end{cases} \quad (12.2)$$

A first question concerns the existence of free vibrations, i.e. of time-harmonic displacements which may be sustained in the absence of external forces. From (12.2) the problem reduces to the determination of functions $y_n(x)$ satisfying

$$\begin{cases} \frac{d^2 y_n}{dx^2} = \lambda_n y_n \\ y_n = 0 \text{ at } x = 0 \text{ and } x = L \end{cases} \quad (12.3)$$

These functions are the eigenfunctions of the operator (d^2/dx^2). Their main property is that, acted upon by the operator, they

reproduce their own form, but with a coefficient λ_n termed the eigenvalue. It is easy to solve (1 .3) explicitly. Thus,

$$y_n = \sin \frac{n\pi x}{L}$$
$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2 \quad (12.4)$$

The frequency of the free vibrations, obtained by equating $(\omega^2\rho/T)$ to $(-\lambda_n)$, is

$$\nu_n = n \frac{1}{2L}\sqrt{\frac{T}{\rho}} \quad (12.5)$$

The eigenfunctions enjoy a crucial property, which is : they are orthogonal. By this we mean that

$$\int_0^L y_n \, y_m \, dx = 0 \qquad (\text{for } m \neq n) \quad (12.6)$$

To solve a system such as (12.2) by the method of eigenfunctions, we expand Y(x) in a series

$$Y(x) = \sum_{n=1}^{\infty} A_n \, y_n(x) \quad (12.7)$$

The problem is to determine the A_n. To do so, we write

$$\frac{d^2 Y}{dx^2} = \Sigma A_n \frac{d^2 y_n}{dx^2} = \Sigma \lambda_n A_n y_n \quad (12.8)$$

$$P(x) = \Sigma B_m \, y_m(x) \quad (12.9)$$

The validity of the term by term differentiation involved in (12.8) is not automatically guaranteed, but in the present case it can be shown to hold. Making use of (12.6) allows determination of B_n. Multiplying, indeed, (12.9) with $y_n(x)$, and integrating from 0 to L, yields

$$\int_0^L y_n(x) P(x) dx = B_n \int_0^L y_n^2 \, dx = \frac{L}{2} B_n \quad (12.10)$$

or

$$B_n = \frac{2}{L}\int_0^L y_n \, P \, dx \qquad (12.11)$$

Inserting (12.7),(12.8) and (12.9) in (12.2) gives, upon equating coefficients of y_n,

$$A_n = -\frac{1}{\rho}\,\frac{B_n}{\omega^2-\omega_n^2} \qquad (12.12)$$

The solution for $Y(x)$ is therefore

$$Y(x) = -\frac{1}{2\pi^2\rho L}\sum \frac{\int_0^L P(x')\sin\frac{n\pi x'}{L}\,dx'}{\nu^2-\nu_n^2}\sin\frac{n\pi x}{L} \qquad (12.13)$$

This solution is obtained in the form of an infinite sum, the "building blocks" of which are the eigenfunctions. Such a sum may easily be programmed on a digital computer. Notice the infinite amplitudes which occur at the resonant frequencies ν_n. These infinities disappear when the frictional and radiative losses of the string are taken into account. Notice that expansion (12.7) can also be used to solve the general wave equation (12.1). Coefficient A_n is now a function of time $a_n(t)$, and steps similar to the previous ones show that a_n satisfies

$$\frac{d^2a_n(t)}{dt^2} + \frac{T}{\rho}\left(\frac{n\pi}{L}\right)^2 a_n(t) = \frac{2}{\rho L}\int_0^L p(x,t)\sin\frac{n\pi x}{L}\,dx \qquad (12.14)$$

This is the equation of an (L,C) circuit.

13. Closed electromagnetic waveguides

The modal expansion in a waveguide makes use of the eigenfunctions of the Dirichlet problem

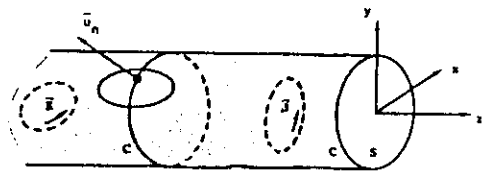


Fig. 13.1

$$\bar{f}_m = \text{grad}\,\phi_m$$
$$\nabla^2\phi_m + \mu_m^2\phi_m = 0 \quad \text{in } S$$
$$\phi_m = 0 \quad \text{on } C \qquad (13.1)$$
$$\iint_S |\bar{f}_m|^2\,dS = 1$$

The index $\underline{m}$ really stands for a double index (m,n). In a rectangle, for example (Fig. 13.3),

$$\phi_{mm} = \frac{2}{\sqrt{\frac{m^2b}{a}+\frac{n^2a}{b}}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (13.2)$$

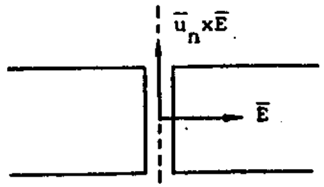
We also need the eigenfunctions of the Neumann problems

$$\bar{g}_m = \text{grad}\,\psi$$
$$\nabla^2\psi_m + \nu_m^2\psi_m = 0 \quad \text{in}$$
$$\frac{\partial\psi_m}{\partial n} = 0 \quad \text{on} \qquad (13.3)$$
$$\iint_S |\bar{g}_m|^2\,dS = 1$$

The field expansions are

$$\bar{E}(r) = \sum_m V_m(z)\,\text{grad}\,\phi_m + \sum_n V_n(z)\,\text{grad}\,\psi_n \times \bar{u}_z + \sum A_m(z)\phi_m\,\bar{u}_z$$
$$\bar{H}(\bar{r}) = \sum_m I_m(z)\,\bar{u}_z \times \text{grad}\,\phi_m + \sum I_n(z)\,\text{grad}\,\psi_n + \sum B_n(z)\psi_n\,\bar{u}_z \qquad (13.4)$$

The problem is to find the expansion coefficients. The method proceeds by requiring (13.4) to satisfy Maxwell's equations. The source terms are volume electric and magnetic currents, $\vec{J}$ and $\vec{K}$, and sidewall



aperture fields. Detailed calculations give, for the E (or TM) modes

$$\frac{dV_m}{dz} + j\omega\mu_o I_m - A_m = -\iint_S \bar{K}.(\bar{u}_z \times \text{grad}\ \phi_m)dS$$

$$-\int_o (\bar{u}_n \times \bar{E}).(\bar{u}_z \times \text{grad}\ \phi_m)d\sigma$$

$$\frac{dI_m}{dz} + j\omega\epsilon_o V_m = -\iint_S \bar{J}.\text{grad}\ \phi_m\ dS \qquad (13.5)$$

$$I_m + \frac{j\omega\epsilon_o}{\mu_m^2} A_m = -\iint_S (\bar{J}.\bar{u}_z)\phi_m\ dS$$

Here, $\bar{u}_n$ is a unit vector perpendicular to the waveguide wall. The contour integrals containing $(\bar{u}_n \times \bar{E})$ represent the excitation through the aperture.

Similar equations hold for the H (or TE) modes : (13.6)

$$\frac{dV_n}{dz} + j\omega\mu_o I_n = -\iint_S \bar{K}.\text{grad}\ \psi_n\ dS - \int_o (\bar{u}_n \times \bar{E}).\text{grad}\ \psi_n\ d\sigma$$

$$\frac{dI_n}{dz} + j\omega\epsilon_o V_n - B_n = -\iint_S \bar{J}.(\text{grad}\ \psi_n \times \bar{u}_z)dS$$

$$V_n + \frac{j\omega\mu_o}{\nu_n^2} B_n = -\iint_S (\bar{K}.\bar{u}_z)\psi_n\ dS - \int_S (\bar{u}_n \times \bar{E}).\bar{u}_z\ \psi_n\ d\sigma$$

For both modes two of the three unknowns may be eliminated, and an equation for the third obtained. For the TE modes, for example, and for perfectly conducting walls,

$$\frac{d^2V_n}{dz^2} + (k^2 - \nu_n^2)V_n = f(z) \qquad (13.7)$$

where f(z) is a source term which vanishes outsides the source region. Far away from the sources, therefore, V_n is a linear combination of exponentials. When $k > \nu_n$, i.e. when the frequency is above cut-off, V_n is of the form

$$A\,e^{-j\sqrt{k^2-\nu_n^2}\,z} + B\,e^{j\sqrt{k^2-\nu_n^2}\,z} \qquad (13.8)$$

outside the sources. The mode is propagated, and transmission line theory may be applied. The wavelength in the guide is

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \frac{\nu_n^2}{k^2}}} \qquad (13.9)$$

and the phase velocity

$$v_{ph} = \frac{c}{\sqrt{1 - \frac{\nu_n^2}{k^2}}} \qquad (13.10)$$

A propagated mode is therefore dispersive. When $k < \nu_n$, i.e. below cut-off and outside the sources, V_n is of the form

$$A\,e^{-\sqrt{\nu_n^2 - k^2}\,z} + B\,e^{+\sqrt{\nu_n^2 - k^2}\,z} \qquad (13.11)$$

The mode is attenuated. The same considerations hold for the TM modes.

It is clear that the number of propagated modes is finite. By suitable choice of the frequency it is possible to launch only one mode (monomode operation). This lowest mode always belongs to the TE family. In a rectangular guide the relevant data are

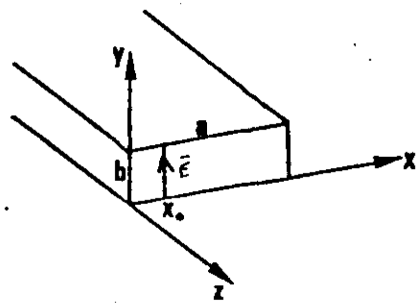


Fig. 13.3

$$\psi_{10} = \cos\frac{\pi x}{a}$$

$$\nu_{10} = \frac{\pi}{a} \qquad (13.12)$$

$$\text{cut off freq} = \frac{c}{2a}$$

14. Optical fibres

The optical fiber is an open dielectric waveguide, of which the circular form is particularly important. A technical realization is shown in Fig. 14.1.a. Rectangular cross-sections may be encountered in other optical guiding structures. We shall concentrate our analysis on a cable consisting of a central core with uniform refraction index N_1, embedded in a cladding medium of smaller index N_2. The field components of the various modes are of the general form

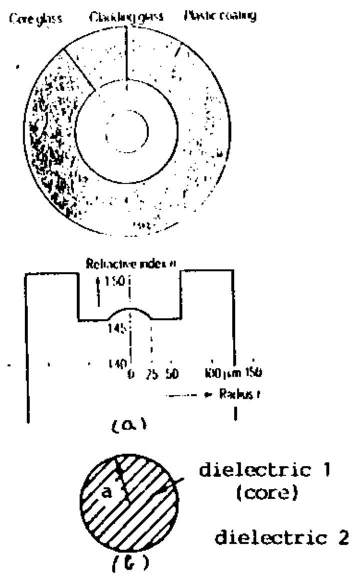


Fig. 14.1

$$E_z = e_z(xy)e^{-\gamma z} = e_z(xy)e^{-\alpha z}\, e^{-j\beta z} \qquad (14.1)$$

If we insert these components in Maxwell's equations we find that the transverse components may be expressed in terms of the longitudinal ones by the formulas

$$\begin{aligned}(k^2N^2+\gamma^2)\bar{e}_t &= -\gamma \text{ grad } e_z + jkR_{co}\mu_r\bar{u}_z \times \text{grad } h_z \\ (k^2N^2+\gamma^2)\bar{h}_t &= -\gamma \text{ grad } h_z - \frac{jk}{R_{co}}\varepsilon_r\bar{u}_z \times \text{grad } e_z\end{aligned} \qquad (14.2)$$

where $k = \omega\sqrt{\varepsilon_o\mu_o}$. The longitudinal components satisfy

$$\begin{aligned}\nabla^2_{xy}e_z + (k^2N^2+\gamma^2)e_z &= 0 \\ \nabla^2_{xy}h_z + (k^2N^2+\gamma^2)h_z &= 0\end{aligned} \qquad (14.3)$$

Eqs. (14.2) and (14.3) must be satisfied in each separate dielectric. To solve these equations for the circular structure, separation of variables may be applied, which leads to the following functional

dependence in the cone $(r < a)$

$$\begin{aligned}e_z &= A\, J_m(u\tfrac{r}{a}) \begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} \\ h_z &= B\, J_m(u\tfrac{r}{a}) \begin{Bmatrix}\sin m\phi \\ -\cos m\phi\end{Bmatrix}\end{aligned} \qquad (14.4)$$

In these expressions we have introduced the dimensionless (and possibl complex) parameter

$$u = a\sqrt{k^2N_1^2+\gamma^2} \qquad (14.5)$$

In the cladding $(r > a)$

$$\begin{aligned}e_z &= C\, K_m(w\tfrac{r}{a}) \begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} \\ h_z &= D\, K_m(w\tfrac{r}{a}) \begin{Bmatrix}\sin m\phi \\ -\cos m\phi\end{Bmatrix}\end{aligned} \qquad (14.6)$$

The dimensionless parameter is now

$$w = ja\sqrt{k^2N_2^2+\gamma^2} \qquad (14.7)$$

The radial dependence is governed by the modofied Bessel function

$$K_m(x) = \frac{\pi}{2}\, j^{m+1}\left[J_m(jx)+jN_m(jx)\right] = \frac{\pi}{2}\, j^{m+1}H_m^{(1)}(jx) \qquad (14.8)$$

which has the interesting property

$$\lim_{x\to\infty} K_m(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \qquad (14.9)$$

To determine γ we shall require e_z, h_z, e_ϕ, h_ϕ to be continuous at $r = a$. This yields four homogeneous equations with four

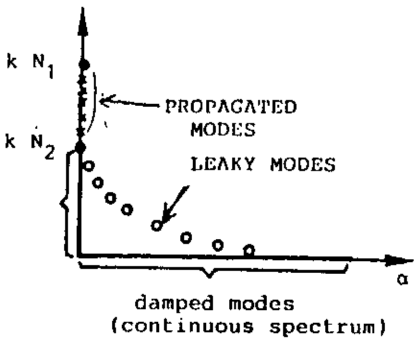
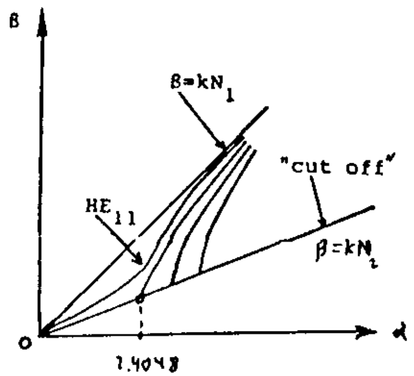


Fig. 14.2



$v = ka\sqrt{N_1^2 - N_2^2}$

Fig. 14.3

unknowns. Setting the determinant equal to zero yields the $\gamma(\omega)$ law, a typical example of which is shown in Fig. 14.2. The crosses on the imaginary axis, finite in number, correspond to the propagated modes. For these modes

$$k N_2 < \beta < kN_1 \qquad (14.10)$$

and (u,w) are real. Cut-off is obtained from the condition $\beta = kN_2$. The $\beta(\omega)$ variation (Fig. 14.3) shows that the HE_{11} mode (m = 1) is always propagated, and that this monomode situation holds up to v = 2.4048. It is therefore desirable to take N_1 and N_2 close to each other to extend the monomode region to high frequencies. Introducing the relative contrast

$$\Delta = \frac{N_1 - N_2}{N_1} \qquad (14.11)$$

allows us to write v as

$$v \simeq ka\, N_1 \sqrt{2\Delta} \qquad (14.12)$$

Example : N_1 = 1.58, N_2 = 1.52, λ = 0.633μm (He-Ne laser), Δ = 3.8%. Monomode transmission for a < 1.12μm. For such low contrasts, the mode turns out to have transverse components which are much larger than the longitudinal ones (quasi TEM transmission).

The first practical fibres were used in the multimode type of operation, with the typical dimensions shown in Fig. 14.4.

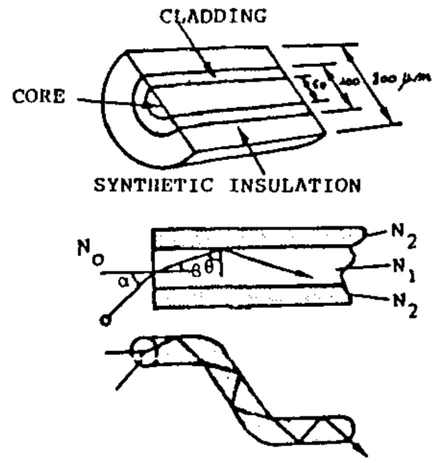


Fig. 14.4

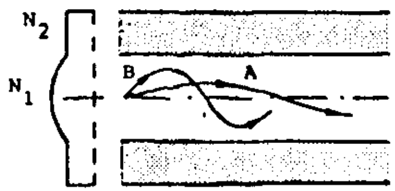


Fig. 14.5

The high number of modes make a ray tracing analysis more efficient than the modal one. Fig. 14.4 shows how the rays bounce on the N_2 medium, where they are totally reflected provided sin θ >(N_2:N_1) (see 5.15). This requires α to be sufficiently small. The "acceptance angle" α must, in fact, be smaller than the "numerical aperture" $\sqrt{N_1^2 - N_2^2}$. The various beams cover different distances in medium 1, i.e. the propagation times differ, which results in a distortion of the signal. A more uniform distribution of times may be obtained by using graded-index fibres based on the index law

$$N = N_1 \left[1 - K\left(\frac{r}{a}\right)^{\alpha}\right] \qquad (14.13)$$

The rays are bent, and their curvature is oriented towards high N's (Fig. 14.5). Paraxial ray A covers a shorter distance than ray B, but at a lower velocity. This compensation results in an equalization of propagation times.

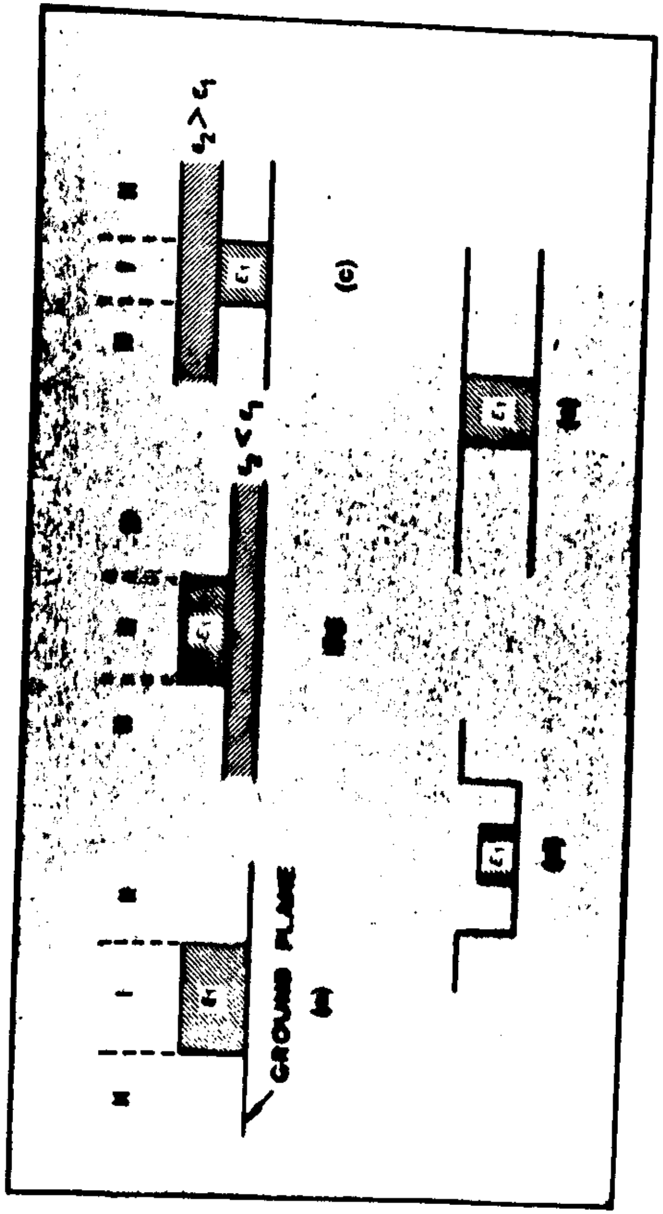


Fig. 1 Cross sections of typical dielectric waveguides. (a) Image guide, (b) Insulated image guide, (c) Inverted strip dielectric guide, (d) Trapped image guide, (e) Non-radiative dielectric guide.

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Typical notations and symbols

$\bar{a}$ = magnetic potential (T m)

$\bar{b}$ = magnetic induction (T)

c = propagation velocity (m)

$\bar{d}$ = electric induction (C m^{-2})

$\bar{e}$ = electric field (V m^{-1})

$\bar{e}_a$ = impressed electric field (V m^{-1})

$\bar{e}_i, \bar{h}_i$ = incident fields

$\bar{h}$ = magnetic field (A m^{-1})

$\bar{j}$ = volume current density (A m^{-2})

$\bar{j}_a$ = applied volume current density (A m^{-2})

$\bar{j}_s$ = surface current density (A m^{-1})

k_o = $\frac{\omega}{c} = \frac{2\pi}{\lambda}$ = wave number in vacuo (m^{-1})

$\bar{m}_e$ = electric polarization density (C m^{-2})

$\bar{m}_m$ = magnetic polarization density (A m^{-1})

n = $(\varepsilon\mu)^{0.5}$ = index of refraction

$\bar{u}_a$ = unit vector in direction a

$\bar{F}$ = radiation vector (V)

I = current (A)

$\bar{P}_e$ = electric dipole moment (C m)

$\bar{P}_m$ = magnetic dipole moment (A m^2)

R = distance to the origin (m)

R_c = $(\mu/\varepsilon)^{0.5}$ = characteristic resistance of a lossless medium (Ω)

W = flux density of electromagnetic power (W m^{-2})

Z_c = characteristic impedance of a medium (Ω)

$\mathcal{E}$ = energy (J)

$\mathcal{P}$ = power (W)

ε_o = $\frac{1}{36\pi} 10^{-9}$ F m^{-1}

λ = wavelength in vacuum (m)

μ_o = 4 10^{-7} H m^{-1}

ν = frequency (Hz)

σ = conductivity (S m^{-1})

σ_{bis} = bistatic cross-section (m^2)

σ_{rad} = radar cross-section (m^2)

σ_{sc} = total scattering cross-section (m^2)

ρ = volume charge density (C m^{-3})

ϕ = electric potential (V)

Φ = magnetic flux (Wb)

Ω = solid angle (sr)