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"MOMENT METHODS FOR UNDERGRADUATES"

J. VAN - BLADEL

Ursi
Union Radio-Scientifique Internationale
c/o Observatoire Royale de Belgique
3 Avenue Circulaire
Brussels
Belgium

MOMENT METHODS FOR UNDERGRADUATES

Leonard L. Tsai
Charles E. Smith
Electrical Engineering Department
University of Mississippi
University, Mississippi 38677

I. Introduction

In recent years a considerable amount of interest has been devoted by the electromagnetic community to the application of the moment method [1] for solution of boundary value problems. The principle reason for the attraction towards this numerical matrix approach is the tremendous versatility it offers in being able to treat structures of arbitrary configurations. Thus, problems which were hitherto untractable by classical approaches, such as the separation of variables method, are now routinely handled, as evidenced by the steady stream of papers using moment methods which now appear regularly in the literature. Besides its flexibility, the moment method has the advantage that it is conceptually simple and from an applications viewpoint is devoid of complicated mathematics; thus, this approach is readily usable by a large group of the electrical engineering community. It is therefore not difficult to envision the moment method becoming even more popular and, in the future, constituting one of the most important tools for analysis of electromagnetic problems.

At its inception, the moment method was developed primarily

for the researcher [1]. As a consequence, emphasis was placed on formalisms based on concepts from the theory of linear vector spaces, with the results that it is beyond the scope of most undergraduate electrical engineering curriculums. Because of its numerous advantages which have induced its rapid adoption by applications engineers in the field, the need to incorporate it in an undergraduate electromagnetics course is gradually becoming evident. In fact, one recent undergraduate electromagnetics textbook [2] devotes two chapters to the treatment of electrostatics problems by matrix methods. While the advanced researcher finds the formal development of moment method through linear vector space theory straightforward, most undergraduate and beginning graduate students at present educational levels find the transition awkward. Specifically, no pedagogical algorithm exists to help the student to relate the procedures of the moment method to either his intuition or former training.

In this paper, a technique for presenting the moment method in elementary terms is developed. The vehicle employed is the electrostatics problem of determining the charge distribution on a thin wire held at a constant potential. The only prerequisites necessary are elementary physics and calculus concepts. The formalisms of this development will be seen to evolve from familiar basic integration and circuit ideas. The experiences of the Electrical Engineering Faculty at the University of Mississippi with this teaching experiment will be reported. Finally, a number of other sample problems are also supplied.

II. The Static Charge Distribution on a Constant Potential Thin Wire

The ideas of the moment method will be introduced at an elementary level in this section through the example statics problem of determining the charge distribution on a constant potential wire. Usually, a beginning undergraduate electromagnetics field course starts by establishing electrostatics concepts. The notions of charge distributions, $\rho(\bar{r}')$, giving rise to potentials $\Phi(\bar{r})$ from which fields may be determined should already be familiar. Thus,

$$\Phi(\bar{r}) = \frac{1}{4\pi\epsilon_0} \int_{\substack{\text{line}\\ \text{source}}} \frac{\rho(\bar{r}')}{R}\, d\ell' \qquad (1)$$

where ϵ_0 is the permitivity of free space, ℓ' is distance measured along the line source, $\bar{r} = (x,y,z)$ denotes the observation coordinates, and $\bar{r}' = (x',y',z')$ denotes the source coordinates with

$$R = |\bar{r}-\bar{r}'| = ((x-x')^2 + (y-y')^2 + (z-z')^2)^{\frac{1}{2}}$$

and the geometry is as depicted in Fig. 1. Typical uses of this relationship are, for example, determining potentials and then fields from an infinitely long line charge or a circular loop, on which the charge distribution is constant. The inquisitive student, however, may question the usefulness of these idealized problems; specifically, how in practice does one establish a constant charge distribution. If a battery is connected to a wire does the resulting charge assume a constant distribution?

The question may thus be posed of how does one actually determine what the charge distribution is in a practical problem. Hence, the stage is set for the introduction of moment methods through which a myriad of problems may be solved.

Consider a finite length, straight, conducting thin wire of radius, a, situated in free space to which a constant potential of one volt is applied, [3], as illustrated in Fig. 2. Because the wire is conducting, charges are free to move, eventually redistributing themselves in some final manner. If we know the charge distribution, then Eq. 1 may be used to compute the potential everywhere. However, it is precisely this charge distribution which is the unknown to be solved for in this problem. Let us therefore seek an alternative interpretation to Eq. 1 where the right-hand side for this problems is unknown and the left-hand side, Φ, is known. Since the potential everywhere is governed by Eq. 1, let the observation point now fall on the wire where Eq. 1 remains valid. Here the applied potential, which is known, constrains Φ ($\bar{r}$ on wire) to be exactly one volt, hence Eq. 1 reduces to

$$1 = \frac{1}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{\rho(y')}{|y-y'|}\, dy' \qquad (2)$$

for $-L/2 < y < L/2$ where $\bar{r} \to y$, $\bar{r}' \to y'$, $d\ell' \to dy'$, and $R \to (y-y')$. To reiterate, whatever the form of the unknown charge distribution $\rho(y')$, it must satisfy Eq. 2, or, equivalently, it needs to cause the potential observed anywhere on the wire to be exactly one volt. Equation 2 thus constitutes an integral equation which needs to be solved in order to determine $\rho(y')$ on the wire.

Let us next seek a numerical solution to this problem. Since Eq. 2 applies for observation points anywhere along the wire, it can be specialized to a fixed point y_k as shown in Fig. 3 with the result

$$1 = \frac{1}{4\pi\varepsilon_0} \int_{-L/2}^{L/2} \frac{\rho(y')}{|y_k - y'|} \, dy' \quad . \qquad (3)$$

Because y_k is constant, the integrand becomes a function of y' only. Our task is now the determination of this functional dependence. Before proceeding further let us recall a familiar concept from integral calculus. The integral of a function, $f(y)$, may be regarded as the sum of the areas under rectangular strips, each height of which equals the mean of $f(y)$ over that strip, as illustrated in Fig. 4. (The essence of numerical integration). Specifically,

$$\int_{-L/2}^{L/2} f(y')dy' \cong f(y_1)\Delta y' + f(y_2)\Delta y' + f(y_3)\Delta y' + \ldots + f(y_n)\Delta y' + \ldots f(y_N)\Delta y'$$

$$= f_1\Delta y' + f_2\Delta y' + f_3\Delta y' + \ldots f_N\Delta y' . \qquad (4)$$

Equation 4 applies, of course, when $f(y')$ is a known function but just as importantly it applies even when $f(y')$ is an unknown, if we interpret integrals as merely giving the area under a curve. Hence, from this interpretation of the integral, Eq. 3 may be recast into the form

$$4\pi\varepsilon_0 \cong \frac{\rho_1\Delta}{|y_k-y_1'|} + \frac{\rho_2\Delta}{|y_k-y_2'|} + \frac{\rho_3\Delta}{|y_k-y_3'|} + \ldots + \frac{\rho_n\Delta}{|y_k-y_n'|} + \frac{\rho_N\Delta}{|y_k-y_N'|} \; . \quad (5)$$

Therefore, the wire has been divided up into N segments all of length Δ as illustrated in Fig. 5. Over each small segment, we can now regard the charge as constant, ρ_n. The idea is that once these unknown constants, ρ_n's, are determined then the charge distribution over the wire will be specified. Since the ρ_n's are free to vary, we have not committed the sin of assuming the charge distribution to be a constant over the entire length of the wire. In addition, if we needed a more accurate representation of the unknown $\rho(y')$, then a finer division of segments, or a larger N, can always be used.

Up to this point, we have obtained an equation in terms of N unknown constants. This was accomplished by selecting only one observation point, y_k, (or match point) somewhere on the wire. From backgrounds in circuits, it is realized that if a solution for these N constants is to follow, then N linearly independent equations are required. Even though a specific y_k was used as our match point, any other point on the wire can do just as well. Towards that end let us simply choose, for convenience, additional observation points on the wire as depicted in Fig. 6. The match points y_k (unprimed coordinates) are now simply placed in the center of the original Δ's into which the source has been divided (primed coordinates). Applying then Eq. 5 at these N match point locations successively, one obtains

$$
\begin{aligned}
4\pi\varepsilon_0 &= \frac{\rho_1\Delta}{|y_1-y_1'|} + \frac{\rho_2\Delta}{|y_1-y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_1-y_N'|} \\
4\pi\varepsilon_0 &= \frac{\rho_1\Delta}{|y_2-y_1'|} + \frac{\rho_2\Delta}{|y_2-y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_2-y_N'|} \\
&\vdots \\
4\pi\varepsilon_0 &= \frac{\rho_1\Delta}{|y_k-y_1'|} + \frac{\rho_2\Delta}{|y_k-y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_k-y_N'|} \\
&\vdots \\
4\pi\varepsilon_0 &= \frac{\rho_1\Delta}{|y_N-y_1'|} + \frac{\rho_2\Delta}{|y_N-y_2'|} + \cdots + \frac{\rho_N\Delta}{|y_N-y_N'|}\,.
\end{aligned}
\tag{6}
$$

This result then constitutes the N linear equations which need to be solved for the N unknown constants ρ_n.

The analogy of this system of equations to circuit concepts is obvious and we may write it more succinctly in matrix notation as

$$
\begin{bmatrix}
\frac{\Delta}{|y_1-y_1'|} & \frac{\Delta}{|y_1-y_2'|} & \cdots & \frac{\Delta}{|y_1-y_N'|} \\
\frac{\Delta}{|y_2-y_1'|} & \frac{\Delta}{|y_2-y_2'|} & \cdots & \frac{\Delta}{|y_2-y_N'|} \\
\vdots & \vdots & & \vdots \\
\frac{\Delta}{|y_N-y_1'|} & \frac{\Delta}{|y_N-y_2'|} & \cdots & \frac{\Delta}{|y_N-y_N'|}
\end{bmatrix}
\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_N \end{bmatrix}
=
\begin{bmatrix} 4\pi\varepsilon_0 \\ 4\pi\varepsilon_0 \\ \vdots \\ 4\pi\varepsilon_0 \end{bmatrix}
\tag{7}
$$

or in "moment method" notation [1] as

$$[\ell_{mn}][f_n] = [g_n] \tag{8}$$

It can now be concluded that once the matrix equation is solved by any of the several standard inversion or equation solution schemes on a digital computer, the desired charge distribution $\rho(y')$ will be known in descrete form, ρ_n's (i.e., f_n's of Eq. 8).

To recapitulate, the solution of the integral equation in Eq. 2 for the charge distribution on a wire at a constant potential has been accomplished first by dividing the wire into constant charge segments and then by successively enforcing Eq. 2 at the centers of these segments. However, the fact that we chose for convenience match points at the centers of the source segments does present a problem. The astute student will observe that when the match point index coincides with the source summation index in any one equation of Eq. 5; i.e., match point equals source point or $y_k = y'_n$, the denominator $|y_k - y'_n| \to 0$ renders a singular matrix element. (This happens for every diagonal element in Eq. 7). That this anomoly necessarily occurs due to the approximations used will be evident from a more detailed examination. In order for Eq. 5 to be an exact equality, N must approach ∞. Furthermore, even for N finite and large the form of Eq. 5 strictly speaking yields the potential from a collection of N weighted point charges, as illustrated in Fig. 7. It is therefore not suprising that we encounter a singularity when the diagonal term is sought because we have really approximated the continuous wire as a collection of point charges which is related to simply the potential due to a point charge observed at the charge itself.

Evidently, a more elaborate treatment is needed for the diagonal terms or the potential contribution due to a segment of charge itself (the previous treatment has been found to be sufficiently accurate for mutual or non-diagonal terms in most problems) [3].

The wire geometry originally depicted in Fig. 2 shows a finite radius a. The fact that the wire is highly conducting will cause the potential to be uniformly unity throughout the wire, including on its axis, and, furthermore, cause the resulting charge distribution to be a uniform surface charge distribution ρ_s over the wire surface. This observation can now be used to compute the self or diagonal terms of the coefficient matrix.

With the aid of Fig. 8, the self term may be interpreted as the potential due to a uniform tube of surface charge ρ_s, at the center of the tube. Hence,

$$\Phi \text{ (Tube center)} = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} \int_{-\Delta/2}^{\Delta/2} \frac{\rho_s a \, d\phi dy'}{\sqrt{a^2 + y'^2}} \qquad (9)$$

$$= \frac{2\rho_s (2\pi a)}{4\pi\varepsilon_0} \ln(\Delta/a).$$

If the surface charge on the tube is normalized to a line charge, i.e., $2\pi a \rho_s = \rho_\ell$, the desired diagonal term (when m=n) is

$$\ell_{nn} = 2 \ln(\Delta/a) \quad . \qquad (10)$$

If one recalls from Eq. 7 that for $m \neq n$, $\ell_{mn} = \frac{\Delta}{|y_m - y'_n|}$, the final matrix equation[1] representing this problem becomes

[1]Each coefficient can be interpreted as the normalized potential at a match point due to a charge source on the n^{th} segment. Hence, the basic sub-problem in this type numerical approach can be defined in terms of the distance measured in reference to a coordinate system localized on the n^{th} source which greatly facilitates the computation of matrix coefficients.

$$
\begin{bmatrix}
2\ln(\Delta/a) & \frac{\Delta}{|y_1-y_2'|} & \frac{\Delta}{|y_1-y_3'|} & \cdots & \frac{\Delta}{|y_1-y_N'|} \\
\frac{\Delta}{|y_2-y_1'|} & 2\ln(\Delta/a) & \frac{\Delta}{|y_2-y_3'|} & \cdots & \frac{\Delta}{|y_2-y_N'|} \\
\frac{\Delta}{|y_3-y_1'|} & \frac{\Delta}{|y_3-y_2'|} & 2\ln(\Delta/a) & \cdots & \frac{\Delta}{|y_3-y_N'|} \\
\cdots & \cdots & \cdots & \frac{\Delta}{|y_m-y_n'|} & \cdots \\
\frac{\Delta}{|y_N-y_1'|} & \frac{\Delta}{|y_N-y_2'|} & \frac{\Delta}{|y_N-y_3'|} & \cdots & 2\ln(\Delta/a)
\end{bmatrix}
\begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_n \\ \rho_N \end{bmatrix}
=
\begin{bmatrix} 4\pi\varepsilon_0 \\ 4\pi\varepsilon_0 \\ 4\pi\varepsilon_0 \\ 4\pi\varepsilon_0 \\ 4\pi\varepsilon_0 \end{bmatrix} . \quad (11)
$$

The solution of Eq. 11 may be obtained by standard matrix solution methods. With some guidance, the program logic needed is well within the ability of the typical undergraduate having a basic knowledge of computer programming.

In its simplest form, this problem reduces to the computation of the individual elements of the coefficeint matrix of Eq. 8, and, therefore, Eq. 11. The numerical value of these elements can readily be determined as

$$
\ell_{mn} = \begin{cases} 2\ln(\Delta/a) & ; m=n \\ \frac{1}{|m-n|} & ; m\neq n \end{cases} \quad (12)
$$

if one recognizes from the geometry of the problem that

$|y_m-y_n| = \Delta|m-n|$. The generation of a number array for the related ℓ_{mn} terms of the coefficient matrix facilitates the computer solution of Eq. 11 using matrix inversion or other solution routines for systems of linear equations.

The matrix equation for this problem has been solved using a matrix inversion program, and the results for a sample case where the wire length is 1 meter and the radius is 1 millimeter are presented in Fig. 9 for twenty wire segments (i.e., a 20 x 20 matrix). As can be seen, the charge distribution on the constant potential wire is hardly constant, and it exhibits the characteristic singularity at the ends of the wire. This finding together now with some qualitative explanations from the repelling charge viewpoint can hopefully reward the student for his diligence in undertaking the study of this problem. A more accurate presentation of the line charge density for this problem is presented in Fig. 10 for 40 unknowns (40 x 40 matrix) which through comparison with Fig. 9 depicts the convergence of this approximate solution method.

III. Results

Our formal experience with undergraduates solving the static wire moment method problem has been over a period of 5 years: three-times in the beginning undergraduate fields course, and twice in a problems oriented laboratory course. The student's preparation and background is roughly that on static fields in Hayt's electromagnetic fields text [4], before the introduction to time varying fields. For the most part, the students are second semester juniors in a 4-year academic program in electrical engineering. On the average, the response has been quite good, with at least 80% of the class being successful in obtaining the correct solution. This was true regardless of how the problem was assigned, i.e., required, optional, and for extra credit (with all methods having been tried). Perhaps this is another instance of the "Hawthorne Uncertainty Principle" as applied to engineering education [5], where if the students know they are being given some new instruction material on an experimental basis, that fact in itself provides motivation for them to cooperate.

Typical comments offered by students have been favorable. Complaints have mostly been associated with programming errors, and the usual complaints heard relate to the Computer Center's mistreatment of one's program. Several students have also elected to continue on to more sophisticated problems in a later senior design course, involving significantly more complex geometries. Some of these type problems are covered in the following examples.

Wire-Type Examples

There are numerous electrostatic problems of the wire-type which can be solved with relative ease once the basic approach has been mastered. Once such extension of the straight wire problem of the previous section is the bent-wire problem shown in Fig. 11. The mathematical formulation of this problem is the same as that of the straight wire problem as stated in Eq. 1; however, the distance between source point and field point, R, does not reduce to a simple length, y-y', in going from wire

References

1. R. F. Harrington, Field Computation by Moment Methods, Macmillan, New York, 1968.

2. A. T. Adams, Electromagnetics for Engineers, Ronald Press, New York, 1971.

3. C. E. Smith, "Applications of Approximate Methods to Electrostatic Fields," University of Mississippi Short Course on Application Moment Methods to Field Problems, May 8-11, 1973.

4. W. H. Hayt, Jr., Engineering Electro-Magnetics, McGraw-Hill Book Company, New York, 1974.

5. G. R. Petersen, "Editorial: The Hawthorne Uncertainty Principle," IEEE Transactions on Education, Vol. E-16, No. 4, p. 181, November 1973.

6. R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw Hill, New York, 1961.

7. ____________, Microwave Engineers Handbook and Buyer's Guide, Horizon House, Dedham, Massachusetts, 1969.

8. J. C. Clements, C. R. Paul, and A. T. Adams, "Computation of the Capacitance Matrix for Systems of Dielectric-Coated Cylindrical Conductors," IEEE Transactions on Electromagnetic Compatibility, Vol. EMC-17, No. 4, November 1975.

9. H. D. Neff, Jr., C. A. Siller, Jr., and J. D. Tillman, Jr., "A Simple Approximation to Current on the Surface of an Isolated Thin Cylindrical Center-fed Dipole of Arbitrary Length," IEEE Transactions on Antennas and Propagation, Vol. AP-18, No. 3, May 1970.

10. R. B. Mack, "A Study of Circular Arrays; Part 2, Self and Mutal Admittances," Craft Laboratory Technical Report 382, Harvard University, 1963.

11. C. M. Butler and M. G. Harrison, "Solution of Hallén's Integral Equation Using the Moment of Methods," University of Mississippi Short Course on the Application of Moment Methods to Field Problems, November 9-11, 1970.

12. C. D. Green, Integral Equation Methods, Barnes & Noble, Inc., New York, 1969.

13. S. A. Schelkunoff, "On Teaching the Undergraduate Electro-Magnetic Theory," IEEE Transactions on Education, Vol. E-15, No.-1, February 1972.

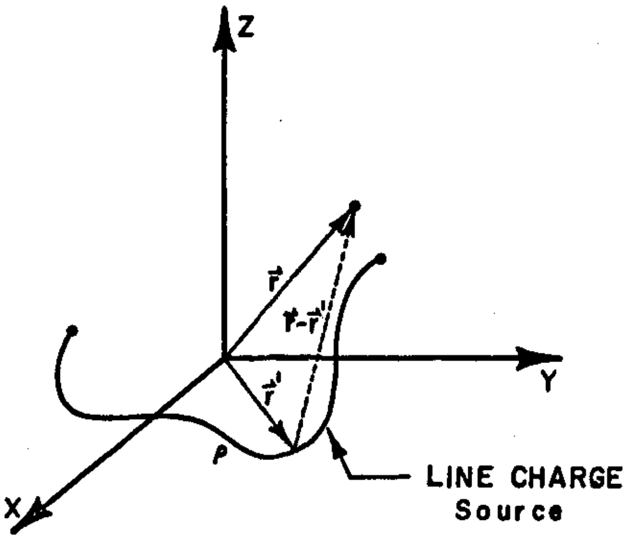

Fig. 1. Geometry for calculation of the potential from a line charge distribution.

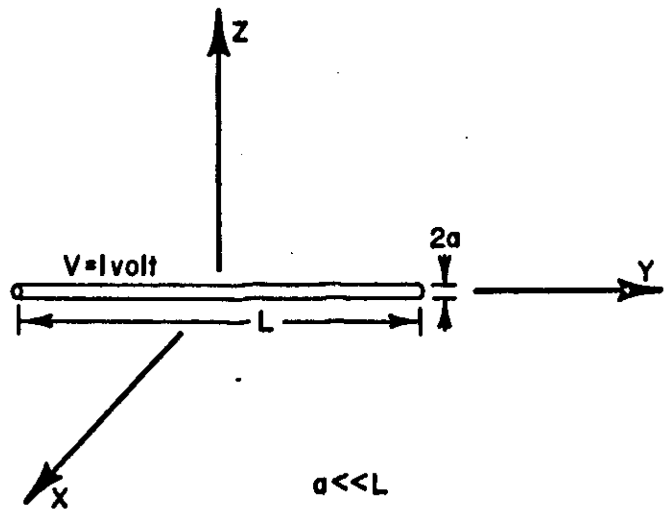

Fig. 2. Finite length wire held at a constant potential.

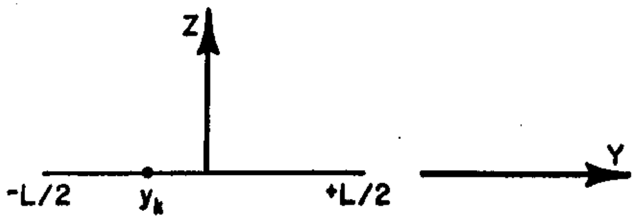

Fig. 3. Geometry for the particular observation point y_k on the wire.

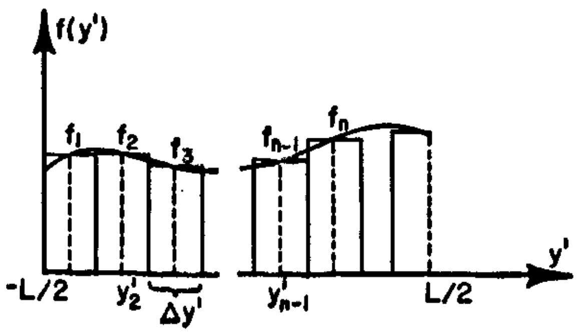

Fig. 4. Representation of $\int_{-L/2}^{L/2} f(y')dy'$ by sum of areas under rectangular strips.

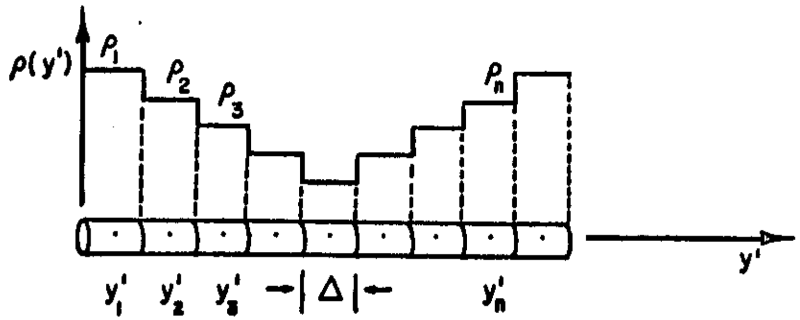

Fig. 5. Division of the wire into segments with individually constant charge distributions.

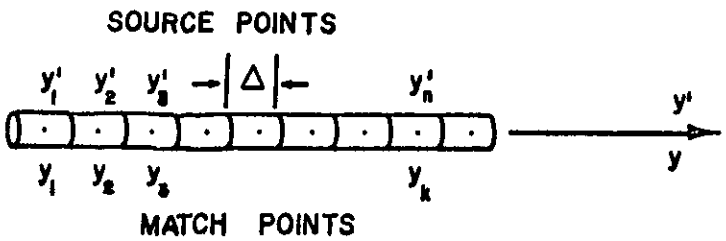

Fig. 6. Location of the N match points to generate N equations for the N unknown ρ_n's.

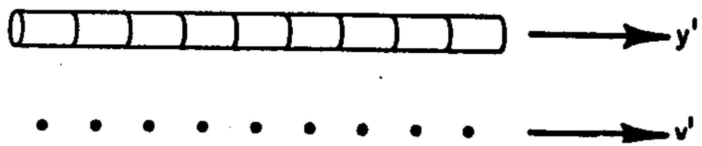

Fig. 7. The occurrance of singularities due to the weighted point charge approximation actually used.

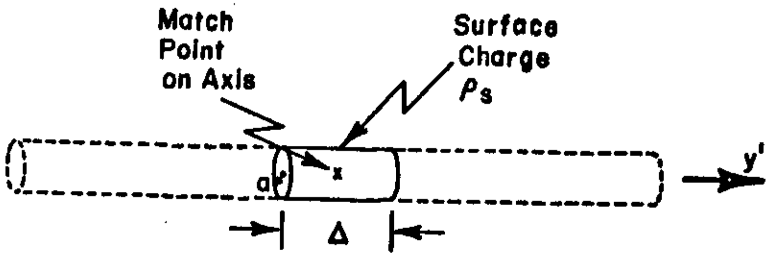

Fig. 8. Geometry for computation of self term.

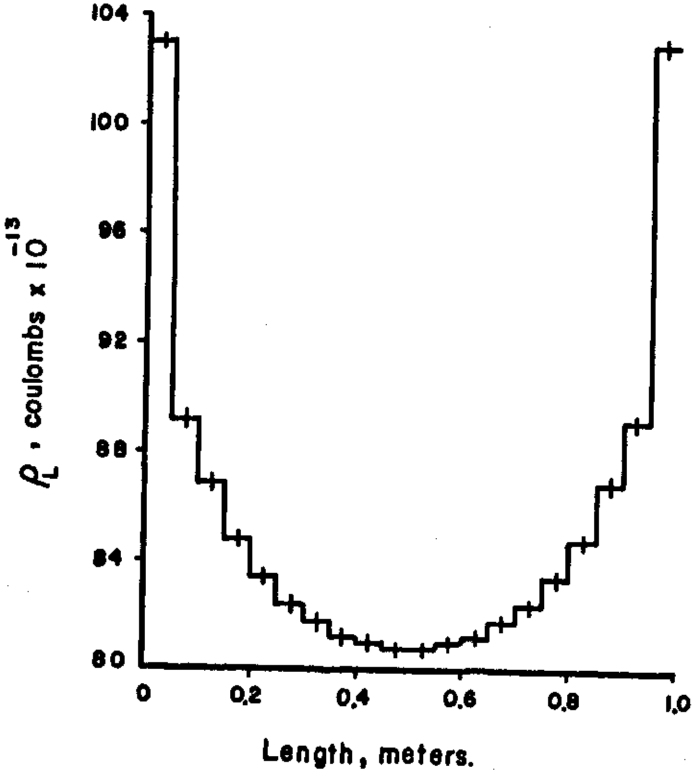

Fig. 9. Line charge density, ρ_L, on a wire of radius 1 mm. and length 1 meter at a 1 volt potential (20 unknowns).

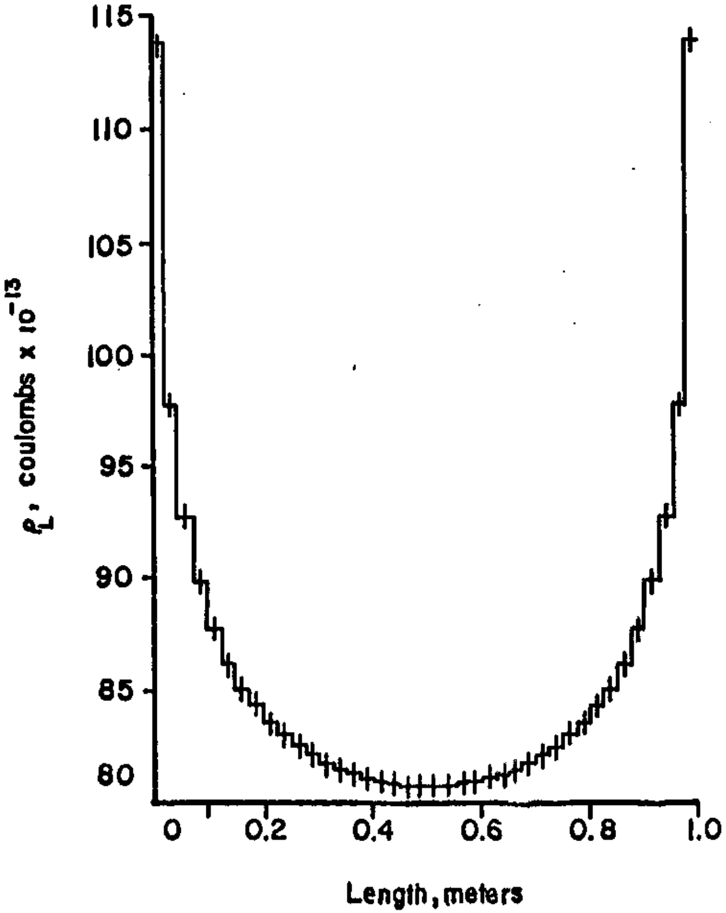

Fig. 10. Line charge density, ρ_L, on a wire of radius 1 mm. and a length of 1 meter at a 1 volt potential (40 unknowns).