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ERROR RATES M-ARY SYSTEMS

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by

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It has been seen that if one wants to transmit one of a set of M known signals { si(t) } over the AWGN channel, it is equivalent to represent the M signals as points { si } in an N-dimensional space and that the relevant components of the noise are also confined to this very same N-dimensional space. Thus, the density function of the noise is given by

$$p_n(u) = (1/\pi\, No)^{N/2} . \exp(-|u|^2/No)$$

By use of the hypothesis testing, specifically the Bayes criterion, the optimum receiver divides the signal space into a set of M disjoint decision regions { Ii }. So $a \in Ik$ iff

$$|a - sk|^2 - No.\ln P(mk) < |a - si|^2 - No.\ln P(mi) \text{ for all } i \neq k \quad (1)$$

and the receiver puts out $\hat{m}$ if $mk \in Ik$. It is noticed that the probability of error is independent of the orthonormal functions and it is unaffected by translation and rotation of the coordinates. In general, a constraint on the signal energy is placed upon. So the average energy of the signal points is

$$Em = \sum_{i=0}^{M-1} P(mi).Ei = \sum_{i=0}^{M-1} P(mi).|si|^2$$

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and the peak energy is

$$Ep = \max \{ Ei \}$$

Although the probability of error is not affected by translation and rotation it is wise to place the coordinate system on the centroid, or center of gravity, of the signal space constellation such that the average or peak energy are minimum.

The geometric characterization of the signal space is very interesting and worthwhile in determining the performance of communication systems. Eventhough this characterization is important only a few constellations do provide ways of finding explicitly its solutions without using numerical methods.

We are going to take into consideration those signal sets which allow us to derive exactly the probability of error (symbol and or bit). These signal sets are essentially those with multi amplitude and/or phase modulation and multi phase modulation.

Let us start by considering the rectangular signal sets. Here, the easiest one is the binary case, that is, there are only two signal points in the constellation. The two possible arrangements are shown below

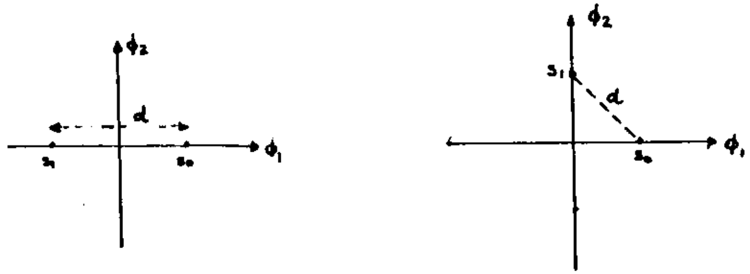

Fig. 1 - a) Antipodal signals; b) Orthogonal signals.

Assuming the signals are equally likely and the channel is the AWGN one, from (1), we have that the optimum decision regions are determined by

$$\min_i \{ |\underline{a} - \underline{s}i|^2 - No.\ln P(mi) \} \qquad (2)$$

Since the signals are equally likely, then (2) becomes

$$\min_i \{ |\underline{a} - \underline{s}i|^2 \}$$

For the case of Fig. 1 a), the locus of all points a equally distant from so and s1 is the axes Ø2. Thus an error occurs if s1 is transmitted and it is decided for so, that is, the noise component n1 exceeds d/2. Thus,

$$P[e/m1] = P[\ \underline{a} \in Io/m1] = P[\ n1 > d/2]$$

where

$$d^2 = \int_{-\infty}^{\infty} (so(t) - s1(t))^2 dt$$

but n1 is a zero mean Gaussian random variable with variance No/2, and so

$$P[e/m1] = \int_{d/2}^{\infty} \sqrt{(1/\pi\, No)}.\exp\{-u^2/No\}du$$

Let $b = a\sqrt{(2/No)}$. Thus

$$P[e/m1] = \int_{d/2/\sqrt{No/2}}^{\infty} (1/2\pi).\exp\{-b^2/2\}db = Q(d/\sqrt{2.No}\ )$$

By symmetry, P[e/mo] = P[e/m1]. So

$$P[e] = \sum_i P[e/mi].P[mi] = P[e/m1] = Q(d/\sqrt{2.No}\ )$$

For the case of Fig. 1 a), we have that the length of the vector so or s1 is $\sqrt{Es}$, and thus, $d = 2.\sqrt{Es}$.

Finally, the antipodal signals have P[e] given by

$$P[e] = Q(\sqrt{2.Es/No}\ )$$

Following the same steps as before, for the case of Fig. 1 b) (orthogonal signals), the average probability of error is also $P[e] = Q(\ d/\sqrt{(2.No)}\ )$. However, the distance $d = \sqrt{2.Es}$, and so the orthogonal signals have P[e] given by

$$P[e] = Q(\sqrt{Es/No}\ )$$

which is 3 dB worse than the antipodal case.

In this way, we have that the optimum decision boundaries for rectangular signal sets is always rectangular (assuming equally likely signals).

Now, suppose the signal space constellation is the 8-QAM, eight signal points in quadrature amplitude modulation, as shown below

Fig. 2 - Siganl space for 8-QAM

Note that we have two sets of four identical decision regions, that is, set 1 consists of regions So, S3, S4, and S7, and set 2 consists of regions S1, S2, S5, and S6.

For each region of the set 1, we have

$$P[C/mo] = \int_{-\infty}^{d/2} p_n(u)du . \int_{-\infty}^{d/2} p_n(u)du = (1-p)^2$$

where $p = Q(d/\sqrt{2.No})$.

For each region of the set 2, we have

$$P[C/m1] = \int_{-d/2}^{d/2} p_n(u)du . \int_{-\infty}^{d/2} p_n(u)du$$

$$= \{ \int_{-\infty}^{d/2} p\ (u)du - \int_{-\infty}^{-d/2} p(u)du \} \int_{-\infty}^{d/2} p_n(u)du$$

$$= \{(1-p) - p\}.(1-p) = (1-2p).(1-p)$$

Therefore,

$$P[C] = (4/8).(1-p)^2 + (4/8).(1-2p).(1-p)$$

For the case of 16-QAM shown below, we have that there are 3 sets of 4, 8, and 4 equal decision regions. Set 1 consists of So, S3, S12, and S15, set 2 consists of S1, S2, S4, S7, S8, S11, S13, S14, and set 3 of S5, S6, S9, and S10.

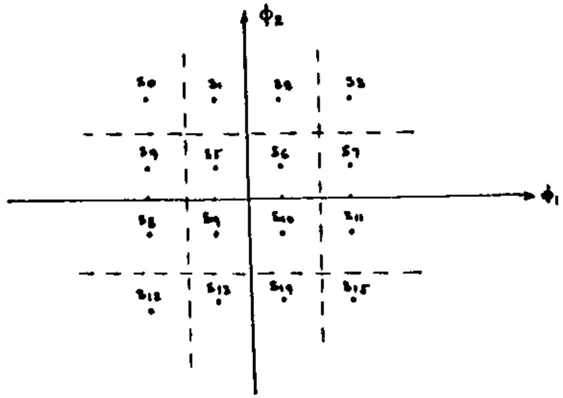

Fig. 3 - Signal points of the 16-QAM

For set 1, we have

$$P[C/m12] = \int_{-\infty}^{d/2} p_n(u)du \int_{-\infty}^{d/2} p_n(u)du = (1-p)^2$$

For set 2, we have

$$P[C/m14] = \int_{-\infty}^{d/2} p_n(u)du \int_{-d/2}^{d/2} p_n(u)du = (1-p).(1-2p)$$

For set 3, we have

$$P[C/m10] = \int_{-d/2}^{d/2} p_n(u)du \int_{-d/2}^{d/2} p_n(u)du = (1-2p)^2$$

Therefore,

$$P[C] = (4/16).(1-p)^2 + (8/16).(1-2p).(1-p) + (4/16).(1-2p)^2$$

The next case is the set of signals lying on the vertices of a hypercube centered on the origin such that the number of signal points is $M = 2^N$.

Let si be represented by

$$\underline{s}i = (si1,\ si2,\ \ldots,\ siN),\ 0 \le i \le 2^N - 1$$

where $sij = +d/2$ or $-d/2$ for all i, j. In order to evaluate the probability of error, assume that

$$so \triangleq (-d/2, -d/2, \ldots, -d/2)$$

is transmitted. Assume that no error is made if

$$nj < d/2 \text{ for all } j = 1, 2, \ldots, N \qquad (3)$$

where $\underline{r} = \underline{a}$ is received, the j-th component of $\underline{a} - \underline{s}i$ is

$$(a_j - s_{ij}) = \begin{cases} n_j \quad , \text{ if } s_{ij} = -d/2 \\ \\ n_j - d, \text{ if } s_{ij} = +d/2 \end{cases}$$

Equation (3) implies that $d - n_j > n_j$ for all j. As a consequence, we have that

$$| \underline{a} - \underline{s}_i |^2 > | \underline{a} - \underline{s}_o |^2 \quad , \text{ for all } s_i \neq s_o \qquad (4)$$

On the other hand, an error is made if for at least one j,

$$n_j > d/2 \qquad (5)$$

From (4) and (5), we conclude that a correct decision is made iff (3) is satisfied. Therefore, given that mo was transmitted, then

$$\begin{aligned} P[C/m_o] &= P(\text{ all } n_j < d/2, \; j = 1,2,\ldots,N \;) \\ &= \prod_{j=1}^{N} P(\; n_j < d/2 \;) = \quad \{ 1 - \int_{d/2}^{\infty} p_n(u)du \} \\ &= (1-p)^N \end{aligned}$$

where $p = Q(\; d/(2.N_o)^{1/2})$. From symmetry,

$$P[C] = P[C/m_o] = (1-p)^N$$

Taking into consideration the signal energy, we have that

$$| s_i |^2 = \sum_{i=1}^{N} d^2/4 = N.d^2/4 = E_s$$

and so $d = \sqrt{(4.E_s/N_o)}$. Thus, $p = Q(\sqrt{2.E_s/N.N_o}\;)$.

When N = 1 and 2, we obtain the same result as before, that is, for the case of antipodal and orthogonal signals, respectively.

The case where the signals in the signal sets have all the same energy and equally spaced in phase (PSK), all values of M but M = 2 and 4 lead to numerical integration. It can be shown that for M - ary PSK the symbol error probability is given by

$$P_s = 1 - \int_{-\pi/M}^{\pi/M} p(u)du$$

where

$$p(u) = (1/2\pi\;)\exp(-v)\{\; 1 + \sqrt{(4.\pi.v)}\cos(u).\exp(v.\cos^2(u)).A(u)\}$$

with

$$A(u) = \sqrt{(1/2\pi)}. \int_{-\infty}^{\sqrt{2v}\cos u} \exp(-y^2/2)dy$$

and $v = a^2.E_s/N_o$.

Below, it is shown some constellations with 2, 4, and 8 signal points.

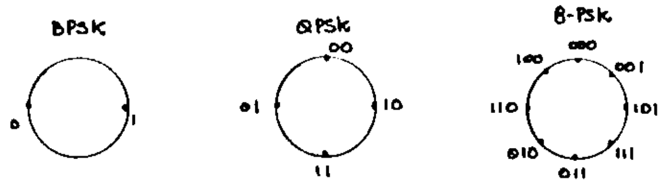

Fig. 4 - 2, 4, and 8-ary PSK

The mapping from the k information bits to the signal points in the constellation follows the Gray encoding. This encoding procedure is interesting since it allows correcting one bit error for high signal to noise ratio. Under this assumption, the noise components will eventually take the transmitted signal to its neighboring decision regions.

Finally, to determine the error rate for the M-ary PAM case it is only a matter of applying the same procedure as was done in the rectangular signal set case.

In the Case Study, we are going to take up this topic again by considering the independent coding and modulation case as well as the combined coding and modulation aspects of power and bandwidth efficient techniques for digital communication systems.