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H4.SMR/381-40

COLLEGE ON ATOMIC AND MOLECULAR PHYSICS:
PHOTON ASSISTED COLLISIONS IN ATOMS AND MOLECULES

(30 January - 24 February 1989)

RADIATIVE ELECTRON COLLISION & HALF-COLLISION IN A STRONG LASER FIELD & FLOQUET-GREEN'S FUNCTION (RESOLVENT) METHOD OF ANALYSIS

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Lecture Notes for the Winter College on Atomic and Molecular Physics, Photon Assisted Collisions in Atoms and Molecules, 30. Jan - 24. Feb. 1989, International Center for Theoretical Physics, Trieste, Italy.

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ABSTRACT:

The theory of radiative electron collision as well as of above-threshold ionization (ATI) in a strong laser field are presented from a unified point of view. The powerful method of Floquet-Green's function (or resolvent) is developed from the first principles, and illustrated in details by solving a number of hitherto unsolved model problems of radiative processes. Two supplements are also provided; one on the method of radiative close-coupling technique for the solution of radiative electron scattering in a coulomb field, and one on the general separable potential method for obtaining the ATI spectrum. Results of applications of these methods are presented and discussed. A summary of the physical results is given and two general propensity rules for radiative processes in strong laser fields are formulated.

Introduction

In recent years, investigations of radiative electron collision and half-collision in the presence of a laser field have made significant progress both in experiment and theory. In the group designated "radiative electron-collision" belongs such processes as stimulated Bremsstrahlung, inverse Bremsstrahlung, sub-threshold and simultaneous electron-photon excitation, "capture-escape" resonances and related phenomena. In the group designated "radiative half-collision" belong e.g. the above threshold ionization (ATI), detachment and field-induced capture processes. The present lectures treat the theory of such processes and discuss the results of analysis and their physical implications from a unified point of view. The first part develops the stationary Floquet Green's function (or resolvent) method from the first principles and illustrates it in details by solving a number of hitherto unsolved one-electron model problems exactly. They include, among others, the solution of the radiative scattering amplitudes and ATI spectrum for (a) the generalized Fermi-Breit potential and (b) the generalized separable potential; both of these potentials can support many bound states of different l's and the full free-wave continuum.

Two supplimentary notes are provided which deal with

1. the method of radiative-close-coupling equations for the solution of the radiative scattering problem in the long-range Coulomb field ($e + H^+$ + photon-scattering) and
2. the explicit solution of the spectrum of ATI process, using the technique of resolvent for a general separable potential model.

Finally, a summary of the results is given and two general propensity rules for qualitative understanding of radiative processes in strong laser fields are formulated.

It is convenient ot start with the Schrödinger equation of the system in the photon occupation number representation in the Schrödinger picture which gives

$$[E-H]\ |\psi> = 0 \tag{1.1}$$

where E is the total energy of the system and

$$H = [H_a+\omega\ (a^+a+\tfrac{1}{2})\ -\tfrac{1}{c}\ \hat{p}\cdot\hat{A} + \frac{\hat{A}^2}{2c^2}] \tag{1.2}$$

is the total Hamiltonian.

We shall be using the Hartree atomic units ($e = \hbar = m_e = 1$) throughout.

In(1.2), H_a = atomic Hamiltonian

$$\hat{p} = -i\ \vec{\nabla}$$

$$\hat{A} = \begin{cases} \frac{\beta}{2}\ \hat{\varepsilon}_z\ (a^+ + a) & \text{(linear polarization)} \quad (1.3) \\ \text{or} & \\ \frac{\beta}{2}\ [(\hat{\varepsilon}_x + i\hat{\varepsilon}_y)a^+ + (\hat{\varepsilon}_x - i\hat{\varepsilon}_y)a] & \text{(circular polarization)} \quad (1.4) \end{cases}$$

is the vector potential, where

$$\beta = (\frac{8\pi c^2}{L^3\omega})^{\frac{1}{2}}$$

is the normalization constant of the Field in the quantization volume L^3. $\omega(a^+a + \frac{1}{2})$ is the field Hamiltonian. a^+ and a, with

$$a^+|n> = \sqrt{n+1}\ |\ n> \quad \text{and } a|n> = \sqrt{n}\ |\ n-1>,$$

where $|n>$ is the number-state with occupation number n, are the usual creation and annihilation operators for the laser field of frequency ω;

$\hat{\varepsilon}_x$, $\hat{\varepsilon}_y$ and $\hat{\varepsilon}_z$ are unit polarization vectors in the indicated directions. We shall use the excellent "laser approximation" (see e.g. ref. 1 p. 125) in which

$$\frac{n_0}{L^3} >> \frac{n}{L^3} \tag{1.5}$$

where n_0 is the initial occupation number (very large) of the laser and n is the <u>change</u> in that number during the process of interest.

We note also that

$$A_0 = \frac{c}{\omega} F_0 = \left(\frac{8\pi c^2 n_0}{L^3 \omega}\right)^{\frac{1}{2}} \tag{1.6}$$

is the peak strength of the vector potential and F_0 is the corresponding peak field strength of the monochromatic classical field of frequency ω. We now note that the matrix elements of the interactions in (1.2) between the number states $|n' + n_0>$ and $<n + n_0|$ are simply

$$<n+n_0| -\frac{1}{c}\,\hat{p}\cdot\hat{A}\,|n'+n_0>$$

$$= \begin{cases} -\dfrac{\hat{p}}{2c}\cdot\hat{\varepsilon}_z\,[\beta_n\,\delta_{n',n-1}+\beta_{n+1}\,\delta_{n',n+1}] & \text{(linear polarization)} \quad (1.7) \\ -\dfrac{\hat{p}}{2c}\,[(\hat{\varepsilon}_x + i\hat{\varepsilon}_y)\,\beta_n\,\delta_{n',n-1}+(\hat{\varepsilon}_x - i\hat{\varepsilon}_y)\,\beta_{n+1}\,\delta_{n',n+1}] & \text{(circular polarization)} \quad (1.8) \end{cases}$$

where

$$\beta_n \equiv \left[\frac{8\pi c^2(n_0+n)}{L^3\omega}\right]^{\frac{1}{2}} \tag{1.9}$$

Also,

$$<n+n_0|\,\frac{1}{2c^2}\,\hat{A}^2\,|n' + n_0>$$

$$
= \begin{cases} \frac{1}{8c^2} [\beta_n \beta_{n-1} \; \delta_{n', n-2} + (\beta_n^2 + \beta_{n+1}^2) \delta_{n',n} + \beta_{n+1} \beta_{n+2} \, \delta_{n',n+2}] \\ \qquad \text{(linear polarization)} & (1.10) \\ \frac{1}{8c^2} [\beta^2 \; (\beta_n^2 + \beta_{n+1}^2 \;) \; \delta_{n',n}] \\ \qquad \text{(circular polarization)} & (1.11) \end{cases}
$$

In the "laser approximation" (1.5)

$$
\beta_{n\pm 2} \simeq \beta_{n\pm 1} \simeq \beta_n \simeq \beta_0 = \left(\frac{8\pi c^2 n_0}{L^3 \omega}\right)^{\frac{1}{2}} = A_0 ,
$$

and equations (1.7), (1.8) and (1.10),(1.11) reduce to

$$
\langle n_0 + n \; | - \frac{1}{c} \hat{p} \cdot \hat{A} \; | n_0 + n' \rangle
$$

$$
= \begin{cases} - \frac{A_0}{2c} \hat{p} \cdot \varepsilon_z \, [\delta_{n',n-1} + \delta_{n',n+1}] \\ \qquad \text{(linear polarization)} & (1.12) \\ - \frac{A_0}{2c} \hat{p} \cdot [(\hat{\varepsilon}_x + i \hat{\varepsilon}_y) \, \delta_{n',n-1} + (\hat{\varepsilon}_x - i \hat{\varepsilon}_y) \, \delta_{n',n+1}] \\ \qquad \text{(circular polarization)} & (1.13) \end{cases}
$$

and

$$
\langle n_0 + n \; | \; \frac{A^2}{2c^2} \; | n' + n_0 \rangle
$$

$$
= \begin{cases} \frac{A_0{}^2}{8c^2} [\delta_{n',n-2} + \delta_{n',n+2}] + \frac{A_0{}^2}{4c^2} \; \delta_{n,n'} \\ \qquad \text{(linear polarization)} & (1.14) \\ \frac{A_0{}^2}{2c^2} \cdot \delta_{n,n'} \\ \qquad \text{(circular polarization)} & (1.15) \end{cases}
$$

2. The Floquet-Schrödinger Equation

The Schrödinger equation (1.1) can be reduced, in the excellent laser-approximation, to a most convenient set of equations for practical purposes by expanding the total state vector in number states

$$|\psi> = \sum_{n'=-n_0}^{\infty} \psi_{n'} \quad |n_0 + n'> \tag{2.1}$$

where n_0 is the initial (large) occupation number of the field. We substitute (2.1) in (1.1) and project on to $< n + n_0|$. Adopting the convention of measuring all occupation energy of the field from the top of the initial occupation energy, $\omega(n_0 + \frac{1}{2})$, taking formally the large n_0 and L^3 limits such that n_0/L^3 is a constant, and making use of the matrix elements (1.10) and (1.14) for the linear polarization (or (1.13) and (1.15) for the circular polarization) we immediately obtain the Floquet-Schrödinger equation :

$$E\psi_n = H_n \psi_n \tag{2.2}$$

with

$$H_n = \begin{cases} H_a + n\omega + i\omega \frac{\alpha_0}{2} \nabla_z (s_n^+ + s_n^-) + \frac{A_0^2}{8c^2}(s_n^{++} + s_n^{--}) + \delta_\varepsilon & \text{(linear polarization)} \quad (2.3) \\ H_a + n\omega + i\omega \frac{\alpha_0}{2}(\nabla^+ s_n^- + \nabla^- s_n^+) + 2\delta_\varepsilon & \text{(circular polarization)} \quad (2.4) \end{cases}$$

where we have introduced $\alpha_0 = \frac{A_0}{c\omega} = \frac{F_0}{\omega^2}$, the mean "radius of vibration". The "mean energy of vibration" of the electron in the field, δ_ε (some times also called the quiver energy) is

$$\delta_\varepsilon = \frac{A_0^2}{4c^2} = \frac{F_0^2}{4\omega^2}$$

Note that the expression for the quiver energy in the case of circular polarization is $2\delta_\varepsilon = \frac{A_0^2}{2c^2}$ where A_0 is the amplitude of the circularly polarized vector potential related to the photon number density $(\frac{n_0}{L^3})$ as in (1.6). In eq. (2.3) and (2.4) we have also introduced the index shift operators [1] $s_n^\pm$ which merely shift the index n of the quantities following it, namely

$$\left.\begin{array}{l} s_n^\pm \psi_n \equiv \psi_{n\pm 1} \\ \text{and} \\ s_n^{++} \psi_n \equiv \psi_{n+2} \, , \; s_n^{--} \psi_n \equiv \psi_{n-2} \end{array}\right\} \qquad (2.5)$$

We rewrite (2.2) for an electron in a potential $V(\bar{r})$ and in the laser field as

$$(E - H_n^0)\,\psi_n = V\,\psi_n \qquad (2.6)$$

where

$$H_n^0 \equiv H_n - V$$

or

$$H_n^0 = \begin{cases} -\frac{1}{2}\nabla^2 + n\omega + i\omega\frac{\alpha_0}{2}\,\nabla_z(s_n^+ + s_n^-) + \frac{\delta_\varepsilon}{2\omega}\,(s_n^{++} + s_n^{--}) + \delta_\varepsilon \\ \qquad\qquad \text{(linear polarization)} \qquad (2.7) \\ -\frac{1}{2}\nabla^2 + n\omega + i\omega\frac{\alpha_0}{2}\,(\nabla^+ s_n^- + \nabla^- s_n^+) + 2\delta_\varepsilon \\ \qquad\qquad \text{(circular polarization)} \qquad (2.8) \end{cases}$$

We shall first consider solutions $\phi_n^{(o)}(\bar{r})$ of(2.6) without the potential $V(\bar{r})$:

$$(E-H_n^o)\ \phi_n^o\ (\bar{r}) = 0 \tag{2.9}$$

We shall establish below that a complete set of solutions of (2.9) are

$$\phi^o_{n-N}(\bar{K}|\bar{r}) = \begin{cases} e^{i\bar{K}\cdot\bar{r}}\ J_{n-N}(\bar{K}\cdot\bar{\alpha}_0|b),\ b = \frac{\delta_\varepsilon}{2\omega} & \text{(linear polarization)} \quad (2.10) \\ e^{i\bar{K}\cdot\bar{r}}\ J_n(K^{\perp}\alpha_0)\ e^{in\phi_K} & \text{(circular polarization)} \quad (2.11) \end{cases}$$

for all $(n,N) = 0, \pm 1, \pm 2, \pm \ldots$
and $\bar{K} = (K, \theta_K, \phi_K)$ with $K^{\perp} = K \sin\theta_K$

The Bessel-functions of the type $J_p(a|b)$ containing two arguments a and b, appearing in the case of linear polarization, are defined by

$$J_p(a|b) = \sum_{m=-\infty}^{\infty} J_{p+2m}(a)\ J_m(b)\ ,\ p \text{ integral} \tag{2.12}$$

where $J_p(x)$ are ordinary Bessel-functions of (one) argument x and order p; $\bar{\alpha}_0 \equiv \hat{\varepsilon}_z\ \alpha_0$.

Before proving that (2.10) or (2.11) is a complete set of solutions of (2.9) with H_n^o given by (2.7) we derive the following useful properties of $J_p(a|b)$. Besides their usefulness in the context of the present theory, they should be of interest on their own in the general theory of Bessel-function.

3. Properties of the Bessel-function of two-arguments

$$J_n(a|b) = \sum_{m=-\infty}^{\infty} J_{n+2m}(a)\ J_m(b) \tag{3.1}$$

We shall first prove that $J_n(a|b)$ satisfies the recurrence relation

$$\frac{a}{2}\,[J_{n+1}(a|b) + J_{n-1}(a|b)] - b\,[J_{n+2}(a|b) + J_{n-2}(a|b)] = nJ_n(a|b) \qquad (3.2)$$

This is the analog of the well-known recurrence relation [2]

$$\frac{x}{2}\,[J_{n+1}(x) + J_{n-1}(x)] = nJ_n(x) \qquad (3.3)$$

satisfied by the ordinary Bessel function of (one) argument x and order n.

For future use, we also express the well-known summation theorem of ordinary Bessel-function[2] in the more useful symmetrical form

$$\sum_{N=-\infty}^{\infty} J_{n-N}(x)\, J_{n'-N}(x') = J_{n-n'}(x-x') \qquad (3.4)$$

and recall the identities

$$J_{n-n'}(0) = \delta_{n,n'} \qquad (3.5)$$

and

$$\sum_{\bar{K}} e^{i\bar{K}\cdot(\bar{r}-\bar{r}')} = \delta(\bar{r}-\bar{r}') \qquad (3.6)$$

where

$$\sum_{\bar{K}} \equiv \frac{1}{(2\pi)^3} \int d\bar{K} \qquad (3.7)$$

To prove (3.2), consider first

$$\frac{a}{2}\,[J_{n+1}(a|b) + J_{n-1}(a|b)]$$

$$= \sum_{m=-\infty}^{\infty} \{\frac{a}{2}[J_{n+1+2m}(a) + J_{n-1+2m}(a)]J_m(b)\}$$

$$= \sum_{m=-\infty}^{\infty} [n+2m]\, J_{n+2m}(a)\, J_m(b)$$

$$= nJ_n(a|b) + \sum_{m=-\infty}^{\infty} (2m)\, J_{n+2m}(a)\, J_m(b) \qquad (3.8)$$

second,

$$-b\,[J_{n+2}(a|b) + J_{n-2}(a|b)]$$

$$= -b\,\{\sum_{m=-\infty}^{\infty} J_{n+2+2m}(a)\,J_m(b) + \sum_{m=-\infty}^{\infty} J_{n-2+2m}(a)\,J_m(b)\}$$

$$= -b\,\{\sum_{m'=-\infty}^{\infty} J_{n+2m'}(a)\,J_{m'-1}(b) + \sum_{m'=-\infty}^{\infty} J_{n+2m'}(a)\,J_{m'+1}(b)\}$$

$$= -2\sum_{m'=-\infty}^{\infty} J_{n+2m'}(a)\left(\frac{b}{2}\right)[J_{m'-1}(b) + J_{m'+1}(b)]$$

$$= -2\sum_{m'=-\infty}^{\infty} J_{n+2m'}(a)\,[m' J_{m'}(b)] \tag{3.9}$$

Adding (3.8) and (3.9) we get on the left hand side of (30) simply $nJ_n(a|b)$ which equals the right hand side of (3.2). Q.E.D.

We next prove that $J_n(a|b)$'s satisfy the following summation theorem:

$$\sum_{N=-\infty}^{\infty} J_{n-N}(a|b)\,J_{n'-N}(a'|b) = J_{n-n'}(a-a') \tag{3.10}$$

Proof: Using the definition (2.12) the left hand side of (3.10) can be written as

$$\sum_{m,m'=-\infty}^{\infty}\left[\sum_{N=-\infty}^{\infty} J_{n-N+2m}(a)\,J_{n'-N+2m'}(a')\right] J_m(b)\,J_{m'}(b)$$

Using (3.4) to simplify the quantity in the square brackets, we get

$$\sum_{m,m'=-\infty}^{\infty} J_{n-n'+2(m-m')}(a-a')\,J_m(b)\,J_{m'}(b)$$

Changing the summation indices $m' = m-p$ and $m = m$ we get

$$\sum_{p=-\infty}^{\infty} J_{n-n'+2p}(a-a')\sum_{m=-\infty}^{\infty} J_m(b)\,J_{m-p}(b)$$

$$= \sum_{p=-\infty}^{\infty} J_{n-n'+2p}(a-a')\,J_p(0)$$

$$= J_{n-n'}(a-a')$$

where we have used (3.5) and (3.6). Q.E.D.

This theorem immediately allows us to derive the following corollary:

$$\sum_{N=-\infty}^{\infty} J_{n-N}(a|b)\ J_{n'-N}(a|b) = \delta_{n,n'} \tag{3.11}$$

Putting a = a' in(3.10)and noting that

$$J_{n-n'}(0) = \delta_{n,n'}$$

eq.(3.11) follows.
Furthermore, we prove the following two normalization relations for Bessel-functions of two-arguments:

$$\sum_{n=-\infty}^{\infty} J_n(a|b) = 1 \tag{3.12}$$

and

$$\sum_{n=-\infty}^{\infty} J_n^2\ (a|b) = 1 \tag{3.13}$$

To prove(3.12)we write

$$\sum_{n=-\infty}^{\infty} J_n(a|b) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_{n+2m}(a)\ J_m(b);$$

change the summation indices n = p - 2m and m = m, (n,m,p = integer) and interchange the order of summations to get

$$\sum_{m=-\infty}^{\infty} [\sum_{P=-\infty}^{\infty} J_p(a)]\ J_m(b)$$

$$= \sum_{m=-\infty}^{\infty} J_m(b) = 1 \qquad \text{Q.E.D.}$$

In the last two steps we have used the well-known normalization relation [2]

$$\sum_{m=-\infty}^{\infty} J_m(x) = 1 \tag{3.14}$$

for the ordinary Bessel-functions.

To prove(3.12)put n = n' = 0 and change the summation index N→-N, in eq.(3.11) This gives at once the result

$$\sum_{N=-\infty}^{\infty} J_N^{2}(a|b) = 1$$

We summarize the properties of the generalized Bessel-function of two arguments derived above and compare them with the well-known analogous relations satisfied by the usual Bessel-function of one argument.

<u>Table 1</u> Properties of generalized Bessel-function of two arguments and their analogs for the ordinary Bessel-function of one argument

1.	$J_n(a,b) = \sum_{m=-\infty}^{\infty} J_{n-2m}(a)\, J_m(b)$	$J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p (\frac{x}{2})^{n+2p}}{n!(n+p)!}$
2.	$\frac{a}{2}\,[J_{n+1}(a|b) + J_{n-1}(a|b)]$ $-b\,[J_{n+2}(a|b) + J_{n-2}(a|b)] = nJ_n(a|b)$	$\frac{x}{2}\,[J_{n+1}(x) + J_{n-1}(x)] = nJ_n(a|b)$
3.	$\sum_{N=-\infty}^{\infty} J_{n-N}(a|b) J_{n'-N}(a'|b) = J_{n-n'}(a-a')$	$\sum_{N=-\infty}^{\infty} J_{n-N}(x) J_{n'-N}(x') = J_{n-n'}(x-x')$
4.	$\sum_{N=-\infty}^{\infty} J_{n-N}(a|b) J_{n'-N}(a|b) = \delta_{nn'}$	$\sum_{N=-\infty}^{\infty} J_{n-N}(x) J_{n'-N}(x) = \delta_{n,n'}$
5.	$\sum_{n=-\infty}^{\infty} J_n(a|b) = 1$	$\sum_{n=-\infty}^{\infty} J_n(x) = 1$
6.	$\sum_{n=-\infty}^{\infty} J_n^{2}(a|b) = 1$	$\sum_{n=-\infty}^{\infty} J_n^{2}(x) = 1$
7.	$J_n(0|0) = \delta_{n,o}$	$J_n(0) = \delta_{n,o}$
8.	$J_n(0|b) = \begin{cases} J_{\frac{n}{2}}(b); & n \text{ even} \\ 0; & n \text{ odd} \end{cases}$	-
9.	$J_n(a|0) = J_n(a)$	-
10.	$J_n(-a|b) = (-1)^n J_n(a|b)$	$J_n(-x) = (-1)^n J_n(x)$

4. Completeness and Orthogonality of Floquet-Volkov-states

After this digression into the properties of the Bessel-function of two-arguments we return to the problem at hand. Now substitute (2.10) in (2.9) with (2.7), use the recurrence relation (3.2) to simplify, so that

$$[E - H_n^{\,o}]\ \phi^{o}_{n-N}\,(\bar{K}|\bar{r})$$

$$= [E - (K^2/2 + N\omega + \delta_\varepsilon)]\ e^{i\bar{K}\cdot\bar{r}}\ J_{n-N}(\bar{K}\cdot\bar{\alpha}_o|b)$$

$$= 0, \text{for the eigenvalues } E = E' = \frac{K^2}{2} + N\omega + \delta_\varepsilon$$

similarly for the circular polarization case

$$[E - H_n^{o}]\ \phi^{o}_{n-N}(\bar{K}\,|\,\bar{r})$$

$$= [E - (K^2/2 + N\omega + 2\delta_\varepsilon)]e^{i\bar{K}\cdot\bar{r}}\ J_{n-N}(K^{\perp}\alpha_o)\ e^{in\phi_K}$$

$$= 0, \text{for eigenvalues } E = E' = K^2/2 + N\omega + 2\delta_\varepsilon$$

To prove the completness of the solutions $\phi^{o}_{n-N}(\bar{r})$ which will be referred to as the Floquet-Volkov-states, we construct

$$\sum_{\bar{K}}\sum_{N}\ |\phi^{o}_{n-N}\rangle\langle\phi_{n'-N}|$$

$$= \sum_{\bar{K}} e^{i\bar{K}\cdot(\bar{r}-\bar{r}')}\sum_{N=-\infty}^{\infty} J_{n-N}(\bar{K}\cdot\bar{\alpha}_o\,|\,b)\ J_{n'-N}(\bar{K}\cdot\bar{\alpha}_o\,|\,b)$$

$$= \delta\ (\bar{r} - \bar{r}')\cdot\delta_{nn'}\ , \text{ Q.E.D.} \tag{4.1}$$

In the last line we have used (3.6) and the corollary (3.11) of the summation theorem (3.10) proved above.

These functions form an orthogonal set

$$\sum_{n=-\infty}^{\infty}\ \langle\phi^{o}_{n-N}(\bar{K}|\bar{r})\ |\ \phi^{o}_{n-N'}(\bar{K}'|\bar{r})\rangle$$

$$= \int e^{i(\bar{K}'-\bar{K})\cdot\bar{r}}\ d\bar{r}\ \cdot\ \sum_{n=-\infty}^{\infty} J_{n-N}\ (\bar{K}\cdot\bar{\alpha}_o|b)\ J_{n'-N'}(\bar{K}'\cdot\bar{\alpha}_o\,|\,b)$$

$$= (2\pi)^3\ \delta(\bar{K}' - \bar{K})\quad \delta_{N',N} \tag{4.2}$$

In the last two steps we have used the summation theorem (3.10) and the property $J_{N-N'}(0) = \delta_{N,N'}$. In a similar way and using the summation theorem of ordinary Bessel-function (3.4) it is easily shown that the set of functions $\phi^{0}_{n-N}(\tilde{K}|\bar{r})$, eq. (3.11) defined for the circularly polarized field also form an orthogonal complete set.

5. The Greens functions of an electron in a linear and a circularly polarized field.

The Greens function (or the resolvent in the coordinate representation) associated with H^{0}_{n} can be defined by [1]

$$(E - H^{0}_{n})\, G^{0}_{nn'}(\bar{r},\bar{r}'|E) = \delta(\bar{r} - \bar{r}')\, \delta_{n,n'} \tag{5.1}$$

In terms of the complete set of solutions (2.10) or (2.11) we may write down the results:

$$G^{0}_{nn'}(\bar{r},\bar{r}'|E) = \begin{cases} \displaystyle\sum_{\bar{K},N=-\infty}^{\infty} e^{i\bar{K}\cdot\bar{r}}\, \frac{J_{n-N}(\bar{K}\cdot\bar{\alpha}_{0}|b)\, J_{n'-N}(\bar{K}\cdot\bar{\alpha}_{0}|b)\, e^{-i\bar{K}\cdot\bar{r}'}}{E - K^{2}/2 - N\omega - \delta_{\varepsilon} + io} & \text{(linear polarization)} \quad (5.2) \\[2ex] \displaystyle\sum_{\bar{K},N=-\infty}^{\infty} e^{i\bar{K}\cdot\bar{r}}\, \frac{J_{n-N}(K^{\perp}\alpha_{0})\, e^{i(n-n')\phi_{K}}\, J_{n'-N}(K^{\perp}\alpha_{0})\, e^{-i\bar{K}\cdot\bar{r}'}}{E - K^{2}/2 - N\omega - 2\delta_{\varepsilon} + io} & \text{(circular polarization)} \quad (5.3) \end{cases}$$

The spherical harmonic expansion of the Greens functions are [3]

$$G^{0}_{nn'}(\bar{r},\bar{r}'|E) = \sum_{lm} \sum_{l'm'} Y_{lm}(\hat{r})\, G^{0}_{nlm,n'l'm'}(r,r'|E)\, Y^{*}_{l'm'}(\hat{r}) \tag{5.4}$$

with

$$G^{0}_{nlm,n'l'm'}(r,r'|E)$$

$$= \sum_{N=-\infty}^{\infty} (i)^{l-l'} A_{nlm}^{n'l'm'}(K_N\alpha_o)\, g_{ll'}^{(N)}(r,r'|E) \tag{5.5}$$

(linear polarization)

where

$$g_{ll'}^{(N)}(r,r'|E) = \begin{cases} -2iK_N\, h_l^{(1)}(K_N r)\, j_{l'}(K_N r'); & r>r' \\ -2iK_N\, j_l(K_N r)\, h_{l'}^{(1)}(K_N r') & r<r' \end{cases} \tag{5.6}$$

$$A_{nlm}^{n'l'm'}(K_N\alpha_o) = \int d\hat{K}\, Y_{lm}^{*}(\hat{K})\, Y_{l'm'}(\hat{K})\, J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_o|b) J_{n'-N}(\bar{K}_N\cdot\bar{\alpha}_o|b) \tag{5.8}$$

and

$$G^{o}_{nlm,n'l'm'}(r,r'|E)$$

$$= \sum_{N=-\infty}^{\infty} (i)^{l-l'} B_{nlm}^{n'l'm'}(K_N\alpha_o)\, g_{ll'}^{(N)}(r,r'|E) \tag{5.8}$$

(circular polarization)

where

$$B_{nlm}^{n'l'm'}(K_N\alpha_o) = \int d\hat{K}\, Y_{lm}^{*}(\hat{K})\, Y_{l'm'}(\hat{K})\, {}^{i(n-n')\phi_K}$$

$$J_{n-N}(K_N^{\perp}\alpha_o)\cdot J_{n'-N}(K_n^{\perp}\alpha_o) \tag{5.9}$$

We are now prepared to derive the fundamental radiative amplitudes of interest.

6. The Radiative Scattering Amplitude

The solution of the Floquet-equation (2.6), $[E-H_n^o]\psi_n = V\psi_n$, satisfying the initial scattering state boundary condition can be written down with the help of $G^o_{nn'}(\bar{r},\bar{r}'|E)$ as

$$\Psi_n(\bar{r}) = \phi_n^o(\bar{K}_o|\bar{r}) + \sum_{n'=-\infty}^{\infty} \int G^o_{nn'}(\bar{r},\bar{r}'|E)\, V(\bar{r}')\, \psi_{n'}(\bar{r}')\, d\bar{r}' \qquad (6.1)$$

with

$$\phi_n^o\,(\bar{K}_o|\bar{r}) = e^{i\bar{K}_o\cdot\bar{r}}\, J_n(\bar{K}_o\cdot\bar{\alpha}_o|b) \quad ; \; b = \frac{\delta_\varepsilon}{2\omega} \qquad (6.2)$$

where $E = K_o^2/2 + o\omega = E_o$ is the initial total energy. We take the asymptotic limit $r \to \infty$ and note that

$$\underset{r\to\infty}{G^o_{nn'}(\bar{r},\bar{r}'|E)} = \sum_{N=-\infty}^{\infty} \frac{e^{iK_N r}}{r}\, J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_o|b)$$

$$\cdot\,(-\frac{1}{2\pi})\, J_{n'-N}\,(\bar{K}_N\cdot\bar{\alpha}_o|b)\, e^{-i\bar{K}_N\cdot\bar{r}'} \qquad (6.3)$$

where

$$K_N = \sqrt{2(E'-N\,\omega)} \qquad \text{with } E' = E - \delta_\varepsilon$$

$$\bar{K}_N = (K_N, \Theta_{K_N}, \Phi_{K_N}) \equiv K_N\hat{\Omega} \quad \text{where } \hat{\Omega} \text{ is an unit vector}$$

in the asymptotic direction of $\vec{r}$.

Hence the radiative scattering amplitude associated with the N th outgoing Volkov spherical wave is readily obtained from the asymptotic behaviour of the second part on the right hand side of (6.1) describing the dressed scattered waves only:

$$f^{(N)}(\bar{K}_o\to\bar{K}_N) = -\frac{1}{2\pi}\sum_{n'=-\infty}^{\infty} J_{n'-N}(\bar{K}_N\cdot\bar{\alpha}_o|b) < e^{i\bar{K}_N\cdot\bar{r}'}|V(\bar{r}')|\psi_{n'}(\bar{r}')> \qquad (6.4)$$

(here and subsequently the brackets with the coordinate representation of wave functions are used to imply the integration over the coordinates). In the case of circular polarization

$$\underset{r\to\infty}{G^o_{nn'}(\bar{r},\bar{r}'|E)} = \sum_{N=-\infty}^{\infty} \frac{e^{iK_N r}}{r}\, J_{n-N}\,(K_N^\perp\alpha_o)\, e^{in\phi_{KN}}$$

$$\cdot\,(-\frac{1}{2\pi})\cdot J_{n'-N}(K_N^\perp\alpha_o)\, e^{-in'\phi_{KN}}\, e^{-i\bar{K}\cdot\bar{r}'} \qquad (6.5)$$

where

$$K_N = \sqrt{2(E'-N\omega)} \quad \text{with } E' = E - 2\delta_\varepsilon$$

Using (6.5) in (6.1) and identifying the amplitude of the N th Volkov-spherical wave we get

$$f^{(N)}\ (\bar{K}_o \to \bar{K}_N) = \frac{-1}{2\pi} \sum_{n'=-\infty}^{\infty} J_{n'-N}(K_N^{\perp}\alpha_o) e^{-in'\phi_{KN}} \langle e^{i\bar{K}_N\cdot\bar{r}'} | V(\bar{r}') | \psi_{n'}(\bar{r}') \rangle \tag{6.6}$$

We have thus reduced the problem of determining the amplitude for elastic scattering (N=0) and of all free-free transitions (N≠0) in a laser field to the determination of the Floquet wave function $\psi_n(\bar{r})$ satisfying the initial condition $\psi_n(\bar{r}) \to \phi_n^o\ (\bar{K}_o|\bar{r})$ given by (2.10) or (2.11).

6.1. The Radiative Born -Approximations:

In fact one systematic method of determing ψ_n is to use the iteration - perturbation method in which one solves eq. (6.1) by iteration starting with the initial solution $\phi_n^o(\bar{K}_o|\bar{r})$. Thus in the first iteration we put

$$\psi_n^{(1)}(\bar{r}) = \phi_n^{o}\ (\bar{K}_o|\bar{r}) \tag{6.7}$$

in (6.4) or (6.6) to get

$$f_{BI}^{(N)}(\bar{K}_o \to \bar{K}_N) = \begin{cases} -\frac{1}{2\pi} \sum_n J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_o|b) J_n(\bar{K}_o\cdot\bar{\alpha}_o|b) \\ \quad \cdot \langle e^{i\bar{K}_N\cdot\bar{r}} | V(\bar{r}) | e^{i\bar{K}_o\cdot\bar{r}} \rangle \\ \quad \text{(linear polarization)} & (6.8) \\ -\frac{1}{2\pi} \sum_n J_{n-N}(K_N^{\perp}\alpha_o)\, e^{-in(\phi_{KN} - \phi_o)} J_n(K_o^{\perp}\alpha_o) \\ \quad \cdot \langle e^{i\bar{K}_N\cdot\bar{r}} | V(\bar{r}) | e^{i\bar{K}_o\cdot\bar{r}} \rangle \\ \quad \text{(circular polarization)} & (6.9) \end{cases}$$

These explicit expressions constitute the generalization of the well-known first Born-approximation to the radiative scattering.

In the second iteration we obtain the correction to (6.7) in the form

$$\psi_n^{(2)} = \sum_{n'} \int G^{0}_{nn'}(\bar{r},\bar{r}'|E)\, V(\bar{r}')\, \phi^{0}_{n'}(\bar{K}_0|\bar{r}')\, d\bar{r}' \tag{6.10}$$

and substitution on the right hand side of (6.4) or (6.6) gives the correction to the amplitudes

$$f_{B2}^{(N)}(\bar{K}_0\to\bar{K}_N) = \begin{cases} -\frac{1}{2\pi}\sum_{n,n'} J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)\langle e^{i\bar{K}_N\cdot\bar{r}}|V(\bar{r})G^{0}_{nn'}(\bar{r},\bar{r}'|E_0)V(r') \\ \quad |e^{i\bar{K}_0\cdot\bar{r}'}\rangle J_{n'}(\bar{K}_0\cdot\bar{\alpha}_0|b) \\ \qquad \text{(linear polarization)} & (6.11) \\ -\frac{1}{2\pi}\sum_{nn'} J_{n-N}(K_N^{\perp}\alpha_0)e^{-in\phi_{K_N}}\langle e^{i\bar{K}_N\cdot\bar{r}}|V(\bar{r})G^{0}_{nn'}(\bar{r},\bar{r}'|E_0) \\ \quad V(\bar{r}')|e^{i\bar{K}_0\cdot\bar{r}'}\rangle e^{in'\phi_{K_0}} J_{n'}(K_0^{\perp}\alpha_0) \\ \qquad \text{(circular polarization)} & (6.12) \end{cases}$$

They correspond to the usual second Born amplitude generalized [1] to the radiative scattering. Clearly the iteration of the solution $\psi_n(\bar{r})$ can be carried out formally to arbitrary orders and the entire usual Born-series can be generalized to the radiative case. However, beyond the first few terms the actual evaluation of the matrix-elements become extremely laborious in practice.

Eq. (6.4) (or (6.6)) is a typical form of results of the present theory; which allows one to compute the cross sections for stimulated inverse-Bremsstrahlung or Bremsstrahlung for multiple apsorption ($N<0$) or emission ($N>0$) of N photons by the electron which is incident with momentum $\bar{K}_0$ and scatters with the momentum $\bar{K}_N$ due to its interaction with the potential $V(\bar{r})$ and the laser field. The corresponding N-photon cross section is simply given by

$$\frac{d\sigma^{(N)}}{d\hat{K}_N} = \frac{K_N}{K_0}\,|f^{(N)}(\bar{K}_0\to\bar{K}_N)|^2 \tag{6.13}$$

7. Radiative Electron Ejection Amplitudes

In the case of detachment of an electron from a negative ion or ionization of a neutral atom initially the solution of (2.6) must go over to the initial bound-state $\phi_i(\bar{r})$ and no photon should be emitted or absorbed, $|n_i> = |0>$. Thus, $\psi_n^i(\bar{r}) \to \phi_i(\bar{r})\,\delta_{n,o}$ as the field strength goes to zero adiabatically and $E = \epsilon_i + o\,\omega \equiv E_i$, the initial total energy. The Floquet-solution of (2.6) is of the form

$$\psi_n^i(\bar{r}) = \sum_{n'}\int G^o_{nn'}(\bar{r},\bar{r}'|E)\; V(\bar{r}')\,\psi^i_{n'}(\bar{r}')\; d\bar{r}' \qquad (7.1)$$

For large r using (6.3) we get

$$\psi_n^i(\bar{r}) = \sum_{N=-\infty}^{\infty} \frac{e^{iK_N r}}{r}\; J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)\sum_{n'}\left(-\frac{1}{2\pi}\right)J_{n'-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)\cdot$$
$$\cdot < e^{i\bar{K}_N\cdot\bar{r}'}|V(\bar{r}')|\,\psi^i_{n'}(\bar{r}')> \qquad (7.2)$$

where $\psi_n^i(\bar{r}')$ satisfies the boundary condition

$$\psi_n^i(\bar{r}) \underset{\alpha_0\to 0}{\to} \phi_i(\bar{r})\;\delta_{n,o}$$

The corresponding electron-ejection amplitude by obsorption of the N-photon is thus given by

$$f^{(N)}_{i\to f}(\bar{K}_N) \equiv -\frac{1}{2\pi}\; T^{(N)}_{i\to f}(\bar{K}_N) \qquad (7.3)$$

where we have introduced the associated N-photon T-matrix element

$$T^{(N)}_{i\to f} = \sum_{n=-\infty}^{\infty} J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0|b) < e^{i\bar{K}_N\cdot\bar{r}}|V(\bar{r})\;\psi_n^i(\bar{r})> \qquad (7.4)$$

with,

$$K_N = [2(E_i - N\omega - \delta_\epsilon)]^{\frac{1}{2}}$$

As in the case of the radiative scattering amplitude before, we may develop $T^{(N)}_{i\to f}$ as a perturbation series beginning with the first term in which $\psi^i_{n'}(\bar{r}')$ on the right hand side of (6.6) is replaced by the initial condition (6.7) which immediately gives

$$T_{i\to f}^{(N)} = J_{-N}(\bar{K}_N\cdot\bar{\alpha}_0|b) <e^{i\bar{K}_N\cdot\bar{r}}|V(\bar{r})|\phi_i(\bar{r})> \tag{7.5}$$

We note that the unperturbed bound state solution $\phi_i(\bar{r})$ satisfies $[-\frac{1}{2}\nabla^2 + V(\bar{r})]\phi_i(\bar{r})=\varepsilon_i\,\phi_i(\bar{r})$ where ε_i is the binding energy of the state $\phi_i(\bar{r})$. Thus the integral in(7.4) can be rewritten as

$$< e^{i\bar{K}_N\cdot\bar{r}}|(\frac{1}{2}\ \nabla^2 + \varepsilon_i)|\ \phi_i(\bar{r}) >$$

$$= (-\frac{1}{2}\ K_N^{\ 2} + \varepsilon_i)<e^{i\bar{K}_N\cdot\bar{r}}\ |\phi_i(\bar{r})>$$

$$= (\varepsilon_i-E_N)\ \tilde{\phi}_i(\bar{K}_N),\ \text{with}\ E_N = \frac{K_N^{\ 2}}{2} = \varepsilon_i-N\,\omega-\delta_\varepsilon$$

Hence(7.4),where $\tilde{\phi}_i(\bar{K}_N)$ is the Fourier transform of $\phi_i(\bar{r})$,reduces to

$$T_{i\to f}^{(N)} = (\varepsilon_i-E_N)\ \tilde{\phi}_i(\bar{K}_N)J_{-N}(\bar{K}_N\cdot\bar{\alpha}_0|b) \tag{7.6}$$

where we recall that $b \equiv \frac{\delta_\varepsilon}{2\omega}$ and

$$J_{-N}(\bar{K}_N\cdot\ \bar{\alpha}_0|b) = \sum_{m=-\infty}^{\infty} J_{-N+2m}(\bar{K}_N\cdot\bar{\alpha}_0)\ J_m(\frac{\delta_\varepsilon}{2\omega})$$

is the Bessel function of two arguments. For the case of circular polarization we get completely analogously

$$T_{i\to f}^{(N)} = (\varepsilon_i-E_N)\ \tilde{\phi}_i\ (\bar{K}_N)\ J_{-N}(K_N^{\perp}\alpha_0) \tag{7.7}$$

$$\text{with}\ E_N = K_N^2/2 = \varepsilon_i-N\,\omega-2\delta_\varepsilon$$

where $J_N(K_N^{\perp}\alpha_0)$ is an ordinary Bessel function and $K_N^{\perp} = K_N\sin\theta_{K_N}$.

These electron ejection amplitudes,eqs.((7.6)and(7.7)),which correspond to the propagation of the ejected electron(after absorbing N-photons (N<0) from the initial state of ϕ_i) in the continum with the momentum $\bar{K}_N$ have been derived for the first time in [4] using the first order S-matrix theory and once again later on [5].

The individual N-photon differential rates of electron ejection are obtained as usual from

$$\frac{dW^{(N)}}{d\Omega} = 2\pi \, |T_{i\to f}^{(N)} (\bar{K}_N)|^2 \rho = v_N \, |f_{i\to f}^{(N)} (\bar{K}_N)|^2 \qquad (7.8)$$

where $\rho = \frac{K_N}{(2\pi)^3}$ is the density of continum states per unit energy and v_N = is the final velocity ($v_N = K_N$ in a.u.).

Using (7.6) and (7.7) in (7.8) we get

$$\frac{dW^{(N)}}{d\Omega} = \begin{cases} \frac{K_N}{(2\pi)^2} \, |(\varepsilon_i - E_N) \tilde{\phi}_i(\bar{K}_N) \, J_{-N}(\bar{K}_N \cdot \bar{\alpha}_0 ; \frac{\delta_\varepsilon}{2\omega})|^2 & \\ \text{(linear polarization)} & (7.9) \\ \frac{K_N}{(2\pi)^2} \, |(\varepsilon_i - E_N) \tilde{\phi}_i(\bar{K}_N) J_{-N}(K_N^{\perp} \alpha_0)|^2 & \\ \text{(circular polarization)} & (7.10) \end{cases}$$

where $\delta_\varepsilon = \frac{Fo^2}{4\omega^2}$ and all N consistent with a minimum integer N_0 determined by the threshold of electron-ejection in the presence of the field, namely

$$K_0^{\,2} = 2 \, (- \, |\varepsilon_i| \, - N_0 \omega \, - \, \delta_\varepsilon \,) > 0$$

(Note that for absorption processes, by convension , N and N_0 are negative integers).

The total ejection rates integrated over the ejection angles $\hat{K}_N = \Omega$ can also be given analytically [6].

$$W^{(N)} = 4 \, K_N(\varepsilon_i - E_N)^2 |F_{1}(K_N)|^2 \sum_{p=0}^{\infty} [\sum_{m=-\infty}^{\infty} S_{N+2m,p}(\alpha_0 K_N) J_m(\frac{\delta_\varepsilon}{2\omega})]^2 \qquad (7.11)$$

(linear polarization)

where

$$S_{n,p}(\alpha_0 K_N) = \sqrt{\pi} \sum_{j=(p-n)/2} (-1)^j \, c_j^{(n,p)} \, (\alpha_0 K_N)^{n+2j}$$

$$c_j^{(n,p)} = \begin{cases} \dfrac{(\frac{1}{2})^{2n+4j+1}(n+2j)!}{j!(n+j)![\frac{1}{2}(n+2j-p)]!\,\Gamma(3/2+\frac{1}{2}(n+2j+p))} & ;n+p = \text{even} \\ 0 \;;\; n+p = \text{odd} \end{cases}$$

and ,

$$W^{(N)} = 4K_N(\varepsilon_i - E_N)^2 |F_{l_i}(K_N)|^2 (\alpha_0 K_N)^{2N} \cdot$$

$$\cdot \sum_{n=0}^{\infty} (-1)^n \frac{(\alpha_0 K_N)^{2N}}{(2N+n)!n!(2N+2n+1)} \qquad (7.12)$$

(circular polarization)

where $F_{l_i}(K_N)$ is the Fourier transform of the radial part $R_{l_i}(r)$ of the initial bound state $\phi_i(\bar{r}) \equiv R_{l_i}(r)\, Y_{l_i m_i}(\hat{r})$

8. Correction for electron accelaration towards the nucleus due to coulomb attraction

It will be noticed that the transition amplitudes derived above satisfy the plane-wave Volkov outgoing condition which is appropriate for processes in which the atomic potential experienced by the ejected electron is short ranged in nature; for example in the case of detachment of negative ions.

For the important case of ionization of neutral atoms (or ions), the ejected electron, however, experiences a long range coulomb attraction towards the residual nucleus in the final state. This effect is expected to be particularly important in the case of low energy electrons. An exact solution of the coulomb case can be handled only by elaborate numerical methods [7]. Here we shall give an approximate way of accounting for the main effect of coulomb accelaration on the ejected electron by modifying the plane wave results using a property of asymptotic coulomb waves from the theory of coulomb functions. The outgoing coulomb wave function $|\phi_{\bar{K}}>^{+}$ of momentum $\bar{K}$ is related asymptotically to the plane wave state $|\phi_{\bar{K}}>$ by Van Haeringen's asymptotic relation[8]

$$|\phi_K>^{+} = \lim_{p\to K} |\phi_{\bar{K}}> \; [D^{+}(K,p)]^{-1} \tag{8.1}$$

where

$$D^{+}(K,p) = e^{-\pi\eta/2}\,\Gamma(1-i\eta)\,\left(\frac{p^2+K^2-i\epsilon}{4K^2}\right)^{i\eta},$$

$$\eta = -\frac{Z}{K} \quad \text{and} \quad -\pi < \arg(p^2-K^2-i\epsilon) < \pi, \tag{8.2}$$

is a factor which typically arises in the momentum representation of the coulomb wave function. We may rewrite every plane wave asymptotic expressions developed in the previous sections by multiplying each term of them by unity in the form

$$1 = \lim_{p\to K_N} [D^{+}(K_N,p)]^{-1}[D^{+}(K_N,p)]$$

Thus, for example, using it in the asymptotic Green's function we obtain

$$G^{0}_{nn'}(\bar{r},\bar{r}'|E)_{r\to\infty} = \sum_{N=-\infty}^{\infty} \frac{e^{iK_N r}}{r}[D^{+}(K_N,p)]^{-1} J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)$$

$$\cdot \sum_{n'=-\infty}^{\infty} - \frac{1}{2\pi} J_{n'-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)[D^{+}(K_N,p)]e^{-i\bar{K}_N\cdot\bar{r}'} \tag{8.3}$$

Using(8.3) in(7.2) and identifying the coefficient of the Volkov-Van Haeringen outgoing waves

$$\frac{e^{iK_N r}}{r}\,[D^{+}(K_N,p)]^{-1} J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)$$

(in place of the usual Volkov-outgoing waves) we obtain for the corrected N-photon transition matrix

$$T^{(N)}_{i\to f} = \lim_{p\to K_N} \sum_{n'} J_{n'-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)\; D^{+}(\bar{K}_N,p)\,\langle\bar{K}_N|V|\,\psi_{n'}\rangle \tag{8.4}$$

It is seen from $|\tilde{T}^{(N)}_{i\to f}|^2$ that the N-photon Transition rate is modified by the simple factor

$$\lim_{p\to K_N} |D^{+}(K_N,p)|^2 = \frac{(2\pi Z/K_N)}{(1-e^{-2\pi Z/K_N})} \tag{8.5}$$

where Z is the nuclear charge.

The corrected angular distribution for the N-photon ATI is

$$\frac{d\tilde{W}^{(N)}}{d\hat{K}_N} = \frac{2\pi\;Z/K_N}{1-e^{-2\pi Z/K_N}}\;\frac{dW^{(N)}}{d\hat{K}_N} \tag{8.6}$$

where $\frac{dW^{(N)}}{d\hat{K}_N}$ is given by the corresponding plane wave result, Eqs. (7.9)or(7.10).The physical implication of the correction factor found here can be made clear by noting that it exactly corresponds to the relative probability of finding the electron at the nuclear center in the presence and absence of the coulomb acceleration towards it, since[9] it equals

$$\frac{|\phi^{+}_{K_N}(0)|^2}{|\phi_{K_N}(0)|^2} = \frac{2\pi Z/K_N}{1-e^{-2\pi Z/K_N}} \tag{8.7}$$

In the limit $K_{N_0}\to 0$ the plane wave rates (e.g. eq. (7.9)) decrease directly proportionally to the density of states K_{N_α}. However in the same-limit

$$\lim_{K_N \to 0} \frac{2\pi Z/K_N}{|-e^{-2\pi Z}|K_N} \to \frac{2\pi Z}{K_N} \tag{8.8}$$

$\lim_{K_N \to 0}$ cancels the density of state factor K_N. Note, finally that for large K_N, $\eta_N \to 0$ and the Coulomb correction factor

$$\lim_{K_N \to \infty} \frac{2\pi Z/K_N}{1-e^{-2\pi Z/K_N}} \to 1 \tag{8.9}$$

yielding as can be expected the plane-wave rates.

Explicitly the N-photon ATI-rates, including the effect of Coulomb accelaration for the force field of a residual ion of charge Z, are

$$\frac{d\tilde{W}^{(N)}}{d\hat{K}_N} = \frac{1}{(1-e^{-2\pi Z/K_N})} \cdot \frac{Z}{2\pi} \cdot |(\epsilon_i - E_N)\tilde{\phi}_i(\bar{K}_N) J_{-N}(\bar{K}_N \cdot \bar{\alpha}_0 | \frac{\delta_\epsilon}{2\omega})|^2 \tag{8.10}$$

(linear polarization)

and

$$\frac{d\tilde{W}^{(N)}}{d\hat{K}_N} = \frac{1}{1-e^{-2\pi Z/K_N}} \frac{Z}{2\pi} |(\epsilon_i - E_N)\ \tilde{\phi}_i(\acute{K}_N)\ J_{-N}(K_N^{\perp}\ \alpha_0)|^2 \tag{8.11}$$

(circular polarization)

9. Analog of Volkov-states for unperturbed bound states.

The Volkov reference state in the product space corresponds to photon dressing of the plane wave $e^{i\bar{K}\cdot\bar{r}}$. It is given by

$$|\phi_{\bar{K}}^{N}> = \sum_{n=-\infty}^{\infty} e^{i\bar{K}\cdot\bar{r}} J_{n-N}(\bar{K}\cdot\bar{\alpha}_0|b)|n> \qquad (9.1)$$

with $b = \delta_\varepsilon/(2\omega)$, eigenvalue $E = \frac{K^2}{2} + N\omega + \delta_\varepsilon$

(linear polarization)

Or,

$$|\phi_{\bar{K}}^{N}> = \sum_{n=-\infty}^{\infty} e^{i\bar{K}\cdot\bar{r}} J_{n-N}(K^{\perp}\alpha_0)e^{in\phi_K}|n>$$

with eigenvalue $E = K^2/2 + N\omega + 2\delta_\varepsilon$ (9.2)

(circular polarization)

Recall that the coefficients of $|n>$ above are the eigenfuctions of H_n^{0}, eqs.(2.7) or (2.8). We now consider the analog of the plane wave Volkov state for an unperturbed bound state. Let $\phi_i(\bar{r})$ be the atomic bound state. We may express it as a superposition of plane-waves

$$\phi_i(\bar{r}) = \sum_{\bar{K}} \tilde{\phi}_i(\bar{K})e^{i\bar{K}\cdot\bar{r}}$$

where $\tilde{\phi}_i(\bar{K})$ is the Fourier transform of $\phi_i(\bar{r})$

Now, each component plane-wave $e^{i\bar{K}\cdot\bar{r}}$ in this wave packet can evolve freely like the Volkov-state(9.1) or(9.2) in the presence of the laser field only, giving a coherent superposition of Volkov states with weighting amplitudes equal to the Fourier transform of $\phi_i(\vec{r})$. Thus the bound state analog of the Volkov-state is

$$|\phi_i^{N}> = \sum_{\bar{K}} \sum_{n} \tilde{\phi}_i(\bar{K})e^{i\bar{K}\cdot\bar{r}} J_{n-N}(\bar{K}\cdot\bar{\alpha}_0|b)|n> \qquad (9.3)$$

(linear polarization)

or

$$|\phi_i^N> = \sum_{\bar{K}} \sum_n \tilde{\phi}_i(\bar{K}) e^{i\bar{K}\cdot\bar{r}} J_{n-N}(K^{\perp}\alpha_0) e^{in\phi_K} |n> \qquad (9.4)$$

(circular polarization)

The coefficients of $|n>$ above are again solutions of the reference Floquet-Hamiltonian H_n^0 (eqs. (2.7)or (2.8)). These wave packets have average energy $E \approx -\frac{\alpha_i^2}{2} + N\omega + \delta_\varepsilon$ (linear polarization) or $E \approx -\alpha_i^2/2 + N\omega + 2\delta_\varepsilon$ (circular polarization) where $\varepsilon_i = -\alpha_i^2/2$ is the binding energy of $\phi_i(\bar{r})$. Note also that in the weak field limit, $\alpha_0 \to 0$, the wave functions (9.3) and (9.4)go over exactly to the usual product states

$$|\phi_i^N> \to \phi_i(\bar{r})|N> \qquad (9.5)$$

with eigenvalue $E \to -\frac{\alpha_i 2}{2} + N\omega$

The states (9.3)or (9.4) form a convenient set of initial or final reference states inside the field in the strong field condition,in the same sense that the Volkov-states constitute a set of continuum reference states inside the field in the strong field case.

10. Some Exactly Solvable Models for Electron-Atom Interaction in a Laser-Field

The power of the Greens function method within the Floquet theory developed here can be seen from the exact solutions of a number of hitherto unsolved model-problems,which are obtained below. Although they cannot replace the importance of a fully realistic computation in a specific case, the usefullness of having such exact model solutions are many. They often give qualitative but quite general insigths into the nature of the process which otherwise may require too elaborate numerical computations in realistic cases to be practicable. They can sometimes reveal unsuspected aspects which may remain buried in the necessary technicalities of complex computations. They permit considerations of extreme limits in interesting parameter domain which may not be possible in realistic computations.

They can provide test cases against which complex algorithms for realistic computation may be tested. Finally, they are of pedagogical interest in quantum theory generally.

10.1. An Electron in a 3-D δ_l - Potential and a Laser-Field.

The model Hamiltonian of the system is given by

$$H = -\frac{1}{2}\nabla^2 + V(\bar{r}) - \frac{s}{2c}\hat{p}\cdot[(\hat{\varepsilon}_x + i\hat{\varepsilon}_y)a^+ + (\hat{\varepsilon}_x - i\hat{\varepsilon}_y)a] + \frac{s^2}{8c^2}(a^+ + a)^2 \tag{10.1.1}$$

where

$$\hat{V} = |Y_{lm}> U_l(r) < Y_{lm}| \tag{10.1.2}$$

with

$$U_l(r) = \frac{b_l}{2}\frac{\delta(r)}{r^2}\left(\frac{\partial}{\partial r}\right)^{2l+1} r^{l+1} \tag{10.1.3}$$

$$b_l = \frac{[(2l+2)!!]^2}{(2l+1)!}(a_l)^{2l+1} \tag{10.1.4}$$

and $s = \left(\frac{8\pi c^2}{L^3\omega}\right)$ is the normalization constant of the vector potential (chosen here to be circularly polarized for algebraic simplicity)

$$\bar{A} = \frac{s}{2}[(\hat{\varepsilon}_x + i\hat{\varepsilon}_y)a^+ + (\hat{\varepsilon}_x - i\hat{\varepsilon}_y)a] \tag{10.1.5}$$

The potential $\hat{V}$ supports one bound state of angular momentum l and the full set of continuum states. This model is a direct generalization of a single s-state model with

$$\hat{V} = \frac{a}{8\pi}\frac{\delta(r)}{r^2}\frac{\partial}{\partial r} r \tag{10.1.6}$$

investigated for radiative processes by Berson[10] and others[11 - 13], to any angular momentum state l. Following the procedure of section 2 we can write down the Floquet-Schrödinger equation of the system

$$[E - H_n^o]\psi_n = \hat{V}\psi_n \tag{10.1.7}$$

where

$$H_n^0 = [-\frac{1}{2}\nabla^2 + n\omega + i\omega \frac{\alpha_o}{2}(\nabla^+ S_n^- + \nabla^- S_n^+) + 2\delta_\varepsilon] \qquad (10.1.8)$$

Using the unperturbed Green's function (5.3) we write

$$\psi_n(\bar{r}) = \phi_n^0(\bar{r}) + \sum_{n'} G_{nn'}^0(\bar{r},\bar{r}'|E)|Y_{lm}(\bar{r}')> U_l(r') <Y_{lm}(\bar{r}')|\psi_{n'}(\bar{r}')> \qquad (10.1.9)$$

with

$$\phi_n^0(\bar{r}) = e^{i\bar{K}_0\cdot\bar{r}} J_n(K_0^\perp\alpha_0)e^{in\phi_0}\ , \qquad (10.1.10)$$

the incident plane-wave Volkov-state in the strong field case. We project (10.1.9) on to $<Y_{lm}(\hat{r})|$, define

$$<Y_{lm}|\psi_n> = F_{nlm}(r)$$

and

$$(\frac{\partial}{\partial r})^{2l+1}(r^{l+1}F_{nlm}(r))|_{r=0} = C_{lm}(n) \qquad (10.1.11)$$

and obtain

$$F_{nlm}(r) = \phi_{nlm}^0(r) + \sum_{n'}\sum_N [-2iK_N h_l^{(1)}(K_N r)\ \delta_{nn'}\cdot$$

$$\cdot B_{nlm}^{n'l'm'}(K_N\alpha_0)\cdot\frac{b_l}{2}\cdot\int_0^\infty dr'\,\delta(r')\frac{j_l(K_N r')}{(r')^l}\ C_{lm}(n') \qquad (10.1.12)$$

where

$$\phi_{nlm}^0(r) = 4\pi i^l j_l(K_0^\perp r)\,Y_{lm}^*(\hat{K}_0)\ J_n(K_0^\perp\alpha_0)\ e^{in\phi_0} \qquad (10.1.13)$$

and we have used the spherical harmonic expansion of the Greens function, eq.(5.8), and of the plane wave

$$e^{i\bar{K}_0\cdot\bar{r}} = \sum_{lm} 4\pi i^l\ j_l(K_0 r)\ Y_{lm}(\hat{r})Y_{lm}^*(\hat{K}_0) \qquad (10.1.14)$$

To obtain the equation satisfied by the unknown constants defined by

$C_{lm}(n)$

we operate on (10.1.12) from the left with $\delta(r)(\frac{\partial}{\partial r})^{2l+1} r^{l+1}$, integrate with respect to dr', make use of the limits

$$j_l(K_N r)\,\frac{1}{r^l}\Big|_{r=0} = \frac{K_N{}^l}{(2l+1)!!}$$

$$\left(\frac{\partial}{\partial r}\right)^{2l+1}(r^{l+1} j_l(Kr))\big|_{r=0} = \left(\frac{\partial}{\partial r}\right)^{2l+1}(r^{l+1} h_l{}^{(1)}(Kr))\big|_{r=0} = \frac{(2l+1)!}{(2l+1)!!} K^l$$

and use the definition (10.1.11) (10.1.15)

to simplify and get

$$C_{lm}(n) = \tilde{\phi}^{o}_{nlm}(\bar{K}_o) - i\, S_{lm}(n|n) C_{lm}(n) \tag{10.1.16}$$

where

$$S_{lm}(n|n) = \sum_N (a_l K_N)^{2l+1}\, B^{nlm}_{nlm}(K_N \alpha_o) \tag{10.1.17}$$

$$\tilde{\phi}_{nlm}(\bar{K}_o) = \frac{4\pi i^l}{(2l+2)!!}(2l+1)!\, K_o^l Y^*_{lm}(\hat{K}_o) J_n(K_o^{\perp}\alpha_o) e^{in\phi_o}$$

$$B^{nlm}_{nlm}(K_N \alpha_o) = \int_0^{\pi} d\theta_K \sin\theta_K |\Theta_{lm}(\theta_K)|^2 J^2_{n-N}(K_N^{\perp}\alpha_o) \tag{10.1.18}$$

and $\Theta_{lm}(\theta_K)$ are defined by [14]

$$Y_{lm}(\theta_K, \phi_K) = \Theta_{lm}(\theta_K)\,\frac{e^{im\phi_K}}{\sqrt{2\pi}} \tag{10.1.19}$$

From (10.1.16) one gets

$$C_{lm}(n) = \frac{\tilde{\phi}^{o}_{nlm}(\bar{K}_o)}{1 + i\, S_{lm}(n|n)} \tag{10.1.20}$$

We take the asymptotic limit $r \to \infty$ in (10.1.9) to obtain

$$\psi_n(\bar{r}) = \phi_n^o(\bar{r}) + \sum_N \frac{e^{iK_N r}}{r} J_{n-N}(K_N^{\perp}\alpha_o) e^{in\phi} \cdot f^{(N)}(\bar{K}_o \to \bar{K}_N), \tag{10.1.21}$$

with

$$f^{(N)}(\bar{K}_0 \to \bar{K}_N) = \{- \frac{1}{2\pi} \sum_{n'} J_{n'-N}(K_N^{\perp} \alpha_0) e^{-in\phi} \frac{b_l}{2} \int_0^\infty dr' 4\pi (-i)^l j_l(K_N r') Y_{lm}(\hat{K}_N) \cdot \frac{\xi(r')}{(r')^l} C_{lm}(n')\} \tag{10.1.22}$$

We have identified the coefficient of the out-going spherical Volkov-state as the amplitude $f^{(N)}$ of the radiative scattering in which N-photons are exchanged ($N < 0$ absorption, $N > 0$ emission and $N = 0$ corresponds to field modified elastic scattering). We simplify it using (10.1.15) to obtain finally

$$f^{(N)}(\bar{K}_0 \to \bar{K}_N) = \sum_{n'} J_{n'-N}(K_N^{\perp} \alpha_0) e^{-in'\phi_{K_N}} \cdot A_l(n';N) \cdot J_{n'}(K_0^{\perp} \alpha_0) e^{in'\phi_0} \tag{10.1.23}$$

where

$$A_l(n';N) = - \frac{4\pi}{K_N} \frac{(K_0 a_l)^{2l+1} (\frac{K_N}{K_0})^l Y_{lm}(\hat{K}_N) Y^*_{lm}(\hat{K}_0)}{1 + i S_{lm}(n'|n')} \tag{10.1.24}$$

with

$$\left.\begin{aligned} K_0^{\perp} &= K_0 \sin \theta_0 \\ K_N^{\perp} &= K_N \sin \theta_{K_N} \end{aligned}\right\} \tag{10.1.25}$$

It is interesting to take the zero field limit, $\alpha_0 = 0$, of (10.1.23). In this limit

$$\left.\begin{aligned} J_{n-N}(K_N^{\perp}\alpha_0 = 0) &= \delta_{n,N} \\ B^{n'lm}_{nlm}(0) &= \delta_{n,N} \\ S_{lm}(n|n) &\Rightarrow (K_0 a_l)^{2l+1} \end{aligned}\right\} \tag{10.1.26}$$

and

$$f^{(N)}(\bar{K}_0 \to \bar{K}_0') = \delta_{N,0}\{- \frac{4\pi}{K_0} \frac{(K_0 a_l)^{2l+1}}{1+i(K_0 a_l)^{2l+1}}\} Y_{lm}(\hat{K}_0') Y^*_{lm}(\hat{K}_0) \tag{10.1.27}$$

We show below that it exactly corresponds to the (lm)th partial scattering amplitude in the central potential for which a_l is the l th scattering-length.

To prove this let us first note that $(K_0 a_l)^{2l+1}$ corresponds[14] to the low energy limit of the tangent of the scattering phase shift $\delta_l(E)$ at an energy $E = \frac{K_0^2}{2}$, or

$$(K_0 a_l)^{2l+1} = - \tan \delta_l(K_0) \quad \text{Lim } K_0 \to 0 \tag{10.1.28}$$

Using the relation (10.1.28) we may re-express the quantity in curly brackets (10.1.27) as the low energy limit of

$$\frac{4\pi}{K_0} \frac{\tan \delta_l(K_0)}{1 - i \tan \delta_l(K_0)}$$
$$= \frac{4\pi}{K_0} \sin \delta_l (K_0) e^{i \delta_l(K_0)} \tag{10.1.29}$$

Eq. (10.1.29) is identified as the well-known expression for the lth partial amplitude for potential scattering expressed in terms of the phase shift $\delta_l(E)$. One sees that this correspondence also justifies our anticipated notations in the definition of the model δ_l-potential, eqs. (10.1.2)-(10.1.4).

Interestingly, this result provides a very simple solution to the well-known inverse scattering problem[15] which requires the construction of a pseudo-potential from a knowledge of the phase-shifts $\delta_l(E)$ (or, at low energies from the knowledge of the scattering lengths a_l.) In the next section we shall introduce and solve the most general δ_l-potential model including the laser field. It will be shown there that a bye product of this solution will in fact provide a solution of the inverse scattering problem just mentioned.
Before concluding this section we note that in the one-term pseudo-potential case l=0, the radiative scattering amplitude (10.1.23) reduces immediately to

$$f^{(N)}(\bar{K}_0 \to \bar{K}_N) = \sum_{n=-\infty}^{\infty} J_{n-N}(K_N^{\perp} \alpha_0)\, e^{-in(\phi_{K_N} - \phi_0)} .$$
$$\cdot \frac{a}{1 + i S_{00}(n|n)} J_n(K_0^{\perp} \alpha_0) \tag{10.1.30}$$

where

$$S_{00}(n|n) = \sum_{N=-\infty}^{\infty} \int_0^{\pi}(aK_N)\ \sin\theta_K\ J_{n-N}^2(K_N^{\perp}\ \alpha_0)\ d\theta_K \tag{10.1.31}$$

Eq. (10.1.30) is identical to the solution originally obtained by Berson[10] by a different method.

10.2. Exact Solution of the generalized Fermi-Breit Potential Model plus Laser Field.

The most general Fermi-Breit Potential model is the extension of the previous case to all partial waves:

$$\hat{V} = \sum_{j=1}^{J} |\,Y_{l_j m_j}\rangle U_{l_j}\ (r)\ \ \langle Y_{l_j m_j}\,| \tag{10.2.1}$$

where J is an arbitrary integer, and

$$U_{l_j}\ (r) = \frac{b_{l_j}}{2}\ \frac{\delta(r)}{r^{l_j+2}}\ (\frac{\partial}{\partial r})^{2l_j+1}\ r^{l_j+1} \tag{10.2.2}$$

and

$$b_{l_j} = \frac{(2l_j+1)!!}{(2l_j+1)!}\ (a_{l_j})^{2l_j+1} \tag{10.2.3}$$

This pseudo-potential supports an arbitrary number (J) of bound states of angular momenta ($l_j m_j$) for all j and all the continuum states.

The Floquet-wave-function $\psi_n\ (\bar{r})$ for this potential can be written as (see section 2)

$$\psi_n(\bar{r}) = \phi_n^0(\bar{r}) + \sum_{j',n'} \int_0^{\infty} G_{nn'}^0\ (\bar{r},\bar{r}'|E)|\ Y_{l_j m_j}(\hat{r}') > U_{l_{j'}}(r')\ r'^2 dr' \cdot < Y_{l_j m_j}\ |\psi_{n'}> \tag{10.2.4}$$

where

$$\phi_n^0\ (\bar{r}) = e^{i\bar{K}_0\cdot\bar{r}}\ J_n(K_0^{\perp}\ \alpha_0)\ e^{in\phi_0} \tag{10.2.5}$$

is the incident wave.

Defining

$$\left(\frac{\partial}{\partial r}\right)^{2l_j+1} \left[r^{l_j+1} \langle Y_{l_j m_j} | \psi_n \rangle\right] \Big|_{r=0} \equiv C_j(n) \qquad (10.2.6)$$

projecting (10.2.4) on to $\langle Y_{l_j m_j}|$, multiplying throughout from the left with $\delta(r)\left(\frac{\partial}{\delta r}\right)^{2l_j+1} r^{l_j+1}$, substituting the spherical harmonic expansion of the Green's function, eq.(5.8), and integrating with respect to dr $d\hat{r}$ and dr' we get the equations for the determination of the unknown constants $c_j(n)$:

$$C_j(n) = \tilde{\phi}_j^{o}(n|\bar{K}_0) + \sum_{j'=1}^{J} \sum_{n'=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} B_{nj}^{n'j'}(K_N\alpha_0) \cdot (K_N)^{l_j+l_{j'}+1} (a_{l_{j'}})^{2l_{j'}+1} C_{j'}(n') \cdot \delta_{n',n-m_j+m_{j'}} \qquad (10.2.7)$$

where

$$B_{nj}^{n'j'}(K_N\alpha_0) = \int_0^{\pi} d\theta_K \sin\theta_K \, J_{n-N}(K_N^{\perp}\alpha_0)\, J_{n'-N}(K_N^{\perp}\alpha_0) \cdot \Theta_{l_j m_j}(\theta_K)\, \Theta_{l_{j'} m_{j'}}(\theta_K) \qquad (10.2.8)$$

with

$$K_N = [2(E - N\omega - 2\delta_\varepsilon)]^{\frac{1}{2}} \qquad (10.2.9)$$

and

$$K_N^{\perp} = K_N \sin\theta_{K_N} \qquad (10.2.10)$$

with

$$\tilde{\phi}_j^{o}(n|\bar{K}_0) = 4\pi i^{l_j} (K_0)^{l_j} \frac{(2l_j+1)!}{(2l_j+2)!!} Y^{*}_{l_j m_j}(\hat{K}_0)\, J_n(K_0^{\perp}\alpha_0)\, e^{in\phi_0}$$

and

$$K_0^{\perp} \equiv K_0 \sin\theta_0. \qquad (10.2.11)$$

We transform the index n to p where (n,p)=0, ±1, ±2,...±α are integers in the same domain by

$$n = p + m_j \qquad (10.2.12)$$

rewrite eqs. (10.2.7) as

$$\sum_{j'=1}^{J} [\delta_{jj'} + i\, S_{jj'}(p+m_j|p+m_{j'})]C_{j'}(p+m_{j'}) = \tilde{\xi}_j^{o}(p+m_j|\bar{K}_o) \qquad (10.2.13)$$

where

$$S_{jj'}(p+m_j|p+m_{j'})$$

$$= \sum_{N'=-\infty}^{\infty} B_{p+m_j,j}^{p+m_{j'},j'}(K_N\alpha_o)\,(a_{l_j})^{2l_{j'}+1}\,(K_{N'})^{l_j+l_{j'}+1}$$

$$\cdot \frac{(2l_j+1)!(2l_{j'}+1)!!}{(2l_{j'}+1)!(2l_j+1)!!} \qquad (10.2.14)$$

The set of algebraic eqs. (10.2.13) can be solved to get

$$C_j(p+m_j) = \sum_{j'=1}^{J} [W^{-1}(p)]_{jj'}\; \hat{\xi}_{j'}^{o}(p+m_{j'}|\bar{K}_o) \qquad (10.2.15)$$

where W(p) is the JxJ matrix defined by

$$[W(p)]_{jj'} = [\delta_{jj'} + i\, S_{jj'}(p+m_j|p+m_{j'})] \qquad (10.2.16)$$

for any given $p = 0, \pm 1, \pm 2, \ldots \pm\infty$.

We now take the asymptotic limit $r' \to \infty$ in (10.2.4) to get

$$\psi_n(\bar{r}) = \phi_n^{o}(\bar{r}) + \sum_{N=-\infty}^{\infty} \frac{e^{iK_N r}}{r}\, J_{n-N}(K_N^{\perp}\alpha_o)\, e^{in\phi K_N} \cdot f^{(N)}(\bar{K}_o \to \bar{K}_N) \qquad (10.2.17)$$

where

$$f^{(N)}(\bar{K}_o \to \bar{K}_N) = \sum_{j'=1,n'=-\infty}^{J,\infty} [J_{n'-N}(K_N^{\perp}\alpha_o)\, e^{-in\phi K_N}\, (i)^{-l_{j'}}\, Y_{l_{j'}m_{j'}}(\hat{K}_N)$$

$$\cdot \frac{(2l_{j'}+1)!!}{(2l_{j'}+1)!}\,(a_{l_{j'}})^{2l_{j'}+1}(K_N)^{l_{j'}}]C_{j'}(n') \qquad (10.2.18)$$

Multiplying (10.2.17) throughout by the number state $|n>$ and suming over n, we get the asymptotic behaviour of the total state vector in the product space:

$$|\psi>_{r\to\infty} = |\phi_{\bar{K}_o}^{o}> + \sum_{N=-\infty}^{\infty} [\sum_{n=-\infty}^{\infty} \frac{e^{iK_N r}}{r} J_{n-N}(K_N^{\perp}\alpha_o)\, e^{in\phi K}|n>]$$

$$\cdot f^{(N)}(\bar{K}_o \to \bar{K}_N) \qquad (10.2.19)$$

where $|\phi^{o}_{\bar{K}_o}>$ is the incident Volkov-plane wave state(9.2) in the product space and the quantity in [ ]-brackets is the outgoing Volkov-state associated with the absorption or emission of N-photons. Hence $f^{(N)}(\bar{K}_o-\bar{K}_N)$ given by(10.2.18)is provedto be the amplitude of the N-photon process of interest.

Changing the index j' to j and replacing the infinite summation over n' by that over $p = n'-m_j$ in(10.2.18)and substituting for $c_{j'}(p+m_j)$ from (10.2.15)and (10.2.11) we finally get the explicit result for the amplitude of the N-photon radiative scattering process:

$$f^{(N)}(\bar{K}_o\to\bar{K}_N) = \sum_{j,j'=1}^{J}\sum_{p=-\infty}^{\infty}\{J_{p+m_j-N}(K_N{}^{\perp}\alpha_o)e^{-i(p+m_j)\phi_{K_N}} Y_{l_jm_j}(\hat{K}_N)\, Y^{*}_{l_{j'}m_{j'}}(\hat{K}_o)\, A_{jj'}(p;N)J_{p+m_{j'}}(K_o{}^{\perp}\alpha_o)e^{i(p+m_{j'})\phi_o}\} \quad (10.2.20)$$

where

$$A_{jj'}(p;N) = -\frac{4\pi}{K_o}(i)^{l_{j'}-l_j}[\bar{W}^{1}(p)]_{jj'}\frac{(2l_j+1)!!(2l_{j'}+1)!}{(2l_j+1)!(2l_{j'}+1)!!}$$

$$\cdot\,(K_oa_{l_j})^{2l_j+1}\left(\frac{K_N}{K_o}\right)^{l_j}(K_o)^{l_{j'}-l_j} \quad (10.2.21)$$

when $[W^{-1}(p)]_{jj'}$ is the inverse of the JxJ-matrix [W(p)] defined by (10.2.16).

It is instructive to take the field free limit of(10.2.20)by putting $\alpha_0=0$. In this limit

$$J_{n-N}(0) \Rightarrow \delta_{n,N}$$

We also find from eqs. (10.2.8) and (10.2.14) that

$$B^{n'j'}_{nj}(K_N\alpha_o) \Rightarrow \delta_{n,n'},\ \delta_{n,N},\ \delta_{jj'}$$

$$S_{jj'}(p+m_j|p+m_{j'}) \Rightarrow (K_{p+m_j}a_{l_j})^{2l_j+1}\ \delta_{jj'}$$

and from (10.2.16) and (10.2.14)

$$[\hat{W}^{-1}(p)]_{jj'} \Rightarrow \frac{1}{1+i(K_{p+m_j}a_{l_j})^{2l_j+1}} \delta_{jj'} \tag{10.2.22}$$

Hence from (10.2.20) and (10.2.21) we find that $f^{(N)}(\bar{K}_0 \to K_N)$ goes over to the limit

$$f^{(N)}(\bar{K}_0 \to \bar{K}'_N) = \sum_{j=1}^{J} \sum_{p=-\infty}^{\infty} \delta_{N,p+m_j}\, e^{-i(p+m_j)\phi K_N} \cdot$$

$$\cdot\left(-\frac{4\pi}{K_0}\right) \frac{(K_0 a_{l_j})^{2l_j+1}}{1+i(K_{p+m_j}a_{l_j})^{2l_j+1}} \left(\frac{K_N}{K_0}\right)^{l_j}$$

$$\cdot\, Y_{l_j m_j}(\hat{K}_N)\, Y^{*}_{j m_j}(\hat{K}_0)\, \delta_{p+m_j,0}$$

$$= \delta_{N,0} \sum_{l_j} (2l_j+1) \frac{(K_0 a_{l_j})^{2l_j+1}}{1+i(K_0 a_{l_j})^{2l_j+1}} \cdot P_{l_j}(\cos\theta) \tag{10.2.23}$$

since

$$\sum_{m_j=-l_j}^{l_j} Y_{l_j m_j}(\hat{K}_N)\, Y^{*}_{l_j m_j}(\hat{K}_0) = \frac{2l_j+1}{4\pi} P_{l_j}(\cos\theta) \tag{10.2.24}$$

with

$$\cos\theta = \cos(\hat{K}_0, \hat{K}_N) \tag{10.2.25}$$

Letting (see (10.1.28))

$$(K_0 a_{l_j})^{2l_j+1} \underset{\lim K_0 \to 0}{\Longleftrightarrow} -\tan \delta_{l_j}(K_0) \tag{10.2.26}$$

we may rewrite (10.2.23) as the low energy limit of

$$f^{(0)}(\cos\theta) = \frac{1}{K_0} \sum_{l_j} (2l_j+1) \frac{\tan\delta_{l_j}(K_0)}{1-i\tan\delta_{l_j}(K_0)} P_{l_j}(\cos\theta)$$

$$= \frac{1}{K_0} \sum_{l_j} (2l_j+1) \sin\delta_{l_j}(K_0)\, e^{-i\delta_{l_j}(K_0)} P_{l_j}(\cos\theta) \tag{10.2.27}$$

where θ is the elastic scattering angle. Eq.(10.2.27) is of course, the well-known exact expression[14] for the scattering amplitude in terms of phase shift $\delta_{l_j}(K_0)$.

Thus we have shown that given the scattering lengths a_{l_j}, the general

Fermi-Breit pseudo-potential (10.2.1) - (10.2.3), yield the exact scattering amplitude determined by the given a_{l_j} s. Furthermore, when $(a_{l_j}K_0)^{2l_j+1}$ is substituted by the energy dependent phase shift $\delta_{l_j}(K_0)$ according to the continuation formula

$$(K_0 a_{l_j})^{2l_j+1} \rightarrow -\tan\delta_{l_j}(K_0) \tag{10.2.28}$$

in (10.2.3) then the general Fermi-Breit-Potential reproduces the exact scattering amplitude.

An interesting application of this result is in the case of radiative coulomb scattering with the known phase-shifts[14]

$$\left.\begin{aligned} &\delta_{l_j}(K_0) = \arg\Gamma(i\eta_0 + l_j + 1) \\ &\text{and} \\ &\delta_{l_j}(K_N) = \arg\Gamma(i\eta_N + l_j + 1) \end{aligned}\right\} \tag{10.2.29}$$

where $\eta_N = \frac{Z}{K_N}$ (nuclear charge Z).

The general Fermi-Breit model for the coulomb radiative scattering amplitude for emission or absorption of N-photons becomes:

$$f_c^{(N)}(\bar{K}_0 \rightarrow \bar{K}_N) = \sum_{j,j'=1}^{J} \sum_{p=-\infty}^{\infty} J_{p+m_j-N}(K_N^{\perp}\alpha_0) e^{-i(p+m_j)\phi_{K_N}} \cdot A_{jj'}^{(c)}(p;N)\, J_{p+m_{j'}}(K_0^{\perp}\alpha_0)\, e^{i(p+m_{j'})\phi_0} \tag{10.2.30}$$

where

$$A_{jj'}^{(c)}(p;N) = \frac{4\pi}{K_0}(i)^{l_{j'}-l_j}[W_c^{-1}(p)]_{jj'} \frac{(2l_j+1)!!(2l_{j'}+1)!}{(2l_j+1)!(2l_{j'}+1)!!} \cdot \tan[\arg\Gamma(l_j+1+i\eta_0)] \cdot \left(\frac{K_N}{K_0}\right)^{l_j}(K_0)^{l_{j'}-l_j}\, Y_{l_j m_j}(\hat{K}_N) Y^*_{l_{j'}m_{j'}}(\hat{K}_0) \tag{10.2.31}$$

where $[W_c(p)]$ is the matrix defined by

$$[W_c(p)]_{jj'} = [\delta_{jj'} + i S^{(c)}_{jj'} (p+m_j|pm_{j'})] \tag{10.2.32}$$

$$S^{(c)}_{jj'} (p+m_j|p+m_{j'}) = - \sum_{N=-\infty}^{\infty} B^{p+m_{j'},j'}_{p+m_j,j} (K_N \alpha_0) \tan[\arg\Gamma(l_j+1+i\gamma_N)]$$

$$\cdot (K_N)^{l_j - l_{j'}} \frac{(2l_j+1)!(2l_{j'}+1)!!}{(2l_j+1)!!(2l_{j'}+1)!} \tag{10.2.33}$$

$$B^{p+m_{j'},j'}_{p+m_j,j} (K_N \alpha_0) = \int_0^{\pi} d\theta_K \sin\theta_K \, J_{p+m_j-N}(K_N^{\perp} \alpha_0) \, J_{p-m_{j'}-N}(K_N^{\perp} \alpha_0)$$

$$\cdot \Theta_{l_j m_j} (\theta_K) \, \Theta_{l_{j'} m_{j'}} (\theta_K) \cdot \delta_{m_j, m_{j'}} \tag{10.2.34}$$

with

$$K_N = [2(E_i - N\omega - 2\delta_{\varepsilon})]^{\frac{1}{2}}, \; K_N^{\perp} = K_N \sin\theta_K .$$

Similar results can be easily obtained, by proceeding exactly analogously as above, for the linear polarization case.

10.3. Hard Sphere Potential plus Laser Field

We give an exact solution of the problem of electron scattering from a Hard sphere in a laser field. The hard sphere potential is defined by

$$V(\bar{r}) = \begin{cases} \infty & r < r_0 \\ 0 & r > r_0 \end{cases} \tag{10.3.1}$$

where r_0 is the radius of the hard-sphere.

The total wave function must satisfy the condition that it vanishes for $r \leq r_0$.

Circular Polarization:

We express the Floquet wave function $\psi_n(\bar{r})$ in terms of the reference Green's function (5.2) as

$$\psi_n(\bar{r}) = e^{i\bar{K}_0\cdot\bar{r}}\, J_n(K_0^{\perp}\alpha_0)\, e^{in\phi_0} + \sum_{n'} G^0_{nn'}(\bar{r},\bar{r}_0|E)C_{n'}(\bar{r}_0)\, d\hat{r}_0 \qquad (10.3.2)$$

with

$$E = \frac{K_0^{\,2}}{2} + o.\omega = E_0$$

where the first term corresponds to the incident Volkov-wave, and $C_{n'}(\vec{r}_0)$ are constants to be determined. We first show that $\psi_n(\bar{r})$satisfies the Floquet-Schrödinger equation with the potential (10.3.1) at all points of space except at the point $r=r_0$. Thus

$$(E-H_n^{\,o})\ \psi_n(\bar{r}) = (E-H_n^o)\ [e^{i\bar{K}_0\cdot\bar{r}}J_n(K_0^{\perp}\alpha_0)\ e^{in\phi_{K_0}}]$$

$$+ \sum_{n'} \int \delta_{nn'}\ \delta(\bar{r}-\bar{r}_0)c_{n'}(\bar{r}_0)d\hat{r}_0$$

$$= 0 + \quad C_n\ (r_0\hat{r})\ \frac{\delta(r-r_0)}{r_0^{\,2}} = 0 \qquad (10.3.3)$$

for both $r > r_0$ and $r < r_0$ (for all finite C_n's).

The solution is fully determined from the requirement that at $r = r_0$

$$\psi_n(\bar{r}) = 0 \text{ for all } n = 0,\pm1,\pm2,\ \ldots\pm\infty. \qquad (10.3.4)$$

This determines the unknown C_n's.

Thus projecting (10.3.2) on to $\langle Y_{l_jm_j}|$, using the partial wave expansion of the Green's function in the outer region, eq. (5.3), for $r\geq r'=r_0$, introducing the index j to denote the pair of quantum numbers (l_jm_j), expanding

$$\psi_n(\bar{r}) = \sum_j F_{nj}(r)\ Y_{l_jm_j}(\hat{r}) \qquad (10.3.5)$$

$$C_n(r_0\hat{r}) = \sum_j C_j(n)\ Y_{l_jm_j}(\hat{r})$$

and setting $r = r_0$, we get from (10.3.2)

$$F_{nj}(r_o) = \phi_j^o(n|\bar{K}_o) - \sum_{j=i,n'=-\infty}^{J,\infty} Q_{jj'}(n|n')\, c_j(n')\, \delta_{n',n-m_j+m_{j'}} \qquad (10.3.6)$$

where

$$Q_{jj'}(n|n') = \sum_{N=-\infty}^{\infty} 2iK_N\, h_{l_j}^{(1)}(K_N r_o)\, j_{l_{j'}}(K_N r_o)\, B_{nj}^{n'j'}(K_N \alpha_o) \qquad (10.3.7)$$

$$B_{nj}^{n'j'}(K_N\alpha_o) = \int_0^\pi J_{n-N}(K_N^{\perp}\alpha_o)\, J_{n'-N}(K_N^{\perp}\alpha_o)\, \Theta_{l_j m_j}(\theta_K)\Theta_{l_{j'} m_{j'}}(\theta_K)\sin\theta_K d\theta_K \qquad (10.3.8)$$

with

$$\phi_j^o(n|\bar{K}_o) = 4\pi i^{l_j}\, j_{l_j}(K_o r_o)\, Y^{*}_{l_j m_j}(\hat{K}_o) J_n(K_o^{\perp}\alpha_o)\, e^{in\phi_o} \qquad (10.3.9)$$

We change the index n by $p + m_j$ in the range$(-\infty,+\infty)$ where p is an integer and impose the boundary condition:

$$F_{p+m_j,j}(r_o) = 0 \text{ for all p and j}$$

which gives the required set of equations for uniquely determining the unknown constants $c_j(p+m_j)$:

$$\sum_{j'} Q_{jj'}(p+m_j|p+m_{j'})\, C_{j'}(p+m_{j'}) = \phi_j^o(p+m_j|\bar{K}_o) \qquad (10.3.10)$$

From the asymptotic behaviour (6.5) of the Green's function in eq. (10.3.2) we get

$$\psi_n(\bar{r}) \underset{r\to\infty}{=} \phi_n^o(\bar{r}) + \sum_N \frac{e^{iK_N r}}{r} J_{n-N}(K_N^{\perp}\alpha_o)\, e^{in\phi_{K_N}} \cdot f^{(N)}(\bar{K}_o \to \bar{K}_N) \qquad (10.3.11)$$

where the N-photon radiative scattering amplitude is exactly given by

$$f^{(N)}(\bar{K}_o \to \bar{K}_N) = \sum_{j,j',p} J_{p+m_j-N}(K_N^{\perp}\alpha_o)\, e^{i(p+m_j)\phi_{K_N}} \cdot A_{jj'}(p;N)\, J_{p+m_{j'}}(K_o^{\perp}\alpha_o)\, e^{-i(p+m_{j'})\phi_o} \qquad (10.3.12)$$

with,

$$A_{jj'}(p;N) = -8\pi(i)^{l_{j'}-l_j} Y_{l_j m_j}(\hat{K}_N)\, j_{l_j}(K_N r_o)\, [\bar{Q}^{1}(p)]_{jj'} \cdot$$

$$\cdot j_{l_{j'}}(K_o r_o) Y^{*}_{l_{j'} m_{j'}}(\hat{K}_o) \qquad (10.3.13)$$

To check the self-consistency of this result, take the zero field limit $\alpha_0=0$. Remembering that $J_p(0) = \delta_{p,o}$ we at once find from (10.3.12):

$$\lim_{\alpha_0 \to 0} f^{(N)}(\bar{K}_o \to \bar{K}_N) = \delta_{N,0} \left[\frac{i}{K_o} \sum_{l_j} (2l_j+1) \frac{j_{l_j}(K_o r_o)}{h_{l_j}^{(1)}(K_o r_o)} P_{l_j}(\cos\theta)\right] \qquad (10.3.14)$$

where we have restored the index $j = (l_j, m_j)$ and used the identily

$$\sum_{m_j=-l_j}^{l_j} Y_{l_j m_j}(\hat{K}_N)\, Y^{*}_{l_j m_j}(\hat{K}_o) = \frac{2l_j+1}{4\pi} P_{l_j}(\cos\theta) \qquad (10.3.15)$$

where $\cos\theta = \hat{K}_N \cdot \hat{K}_o$ and θ is the scattering angle.Eq.(10.3.14)exactly reproduces the well-known[14] expression for the scattering amplitude by the hard sphere potential in the absence of the laser.

Linear Polarization:

In this case the Floquet-Schrödinger wave function may be expressed in the form

$$\psi_n(\bar{r}) = e^{i\bar{K}_o \cdot \bar{r}} J_n(\bar{K}_o \cdot \bar{\alpha}_o) + \sum_{n'} G^{o}_{nn'}(\bar{r}, \bar{r}_o|E)\, C_n(\bar{r}_o) d\hat{r}_o \qquad (10.3.16)$$

where $G^{o}_{nn'}(\bar{r}, \bar{r}_o|E)$ is given by the Green's function (5.2) appropriate for the linear polarization.

Expanding

$$\psi_n(\bar{r}) = \sum_j F_{jn}(r) Y_{l_j m_j}(\hat{r}) \qquad (10.3.17)$$

and

$$C_n(\bar{r}_o) = \sum_j C_{jn}(r_o)\, Y_{l_j m_j}(\hat{r}_o) \qquad (10.3.18)$$

and proceeding similarly as in the case of circular polarization above and setting $r = r_o$ we get

$$F_{jn}(r_o) = 4\pi i^{l_j} j_{l_j}(K_o r_o) Y^{*}_{l_j m_j}(\hat{K}_o)\, J_n(\bar{K}_o \bar{\alpha}_o)$$

$$- \sum_{j'n'} Q_{jn,j'n'} C_{j'n'}(r_o) \tag{10.3.19}$$

where

$$Q_{jn,j'n'} = \sum_{N=-\infty}^{\infty} 2iK_N h_{l_j}^{(1)} (K_N r_o) j_{l_{j'}} (K_N r_o) L_{nj}^{n'j'} (K_N \alpha_o) \tag{10.3.20}$$

with

$$L_{nj}^{n'j'} = \int_0^{\pi} d\theta_K \sin\theta_K \, J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_o | b) J_{n'-N}(\bar{K}_N \cdot \bar{\alpha}_o | b)$$

$$\Theta_{l_j m_j}(\theta_K) \; \Theta_{l_{j'} m_{j'}} (\theta_K) \; \delta_{m_j, m_{j'}} \tag{10.3.21}$$

We impose the boundary condition

$$F_{jn}(r_o) = 0 \text{ for all } j \text{ and } n \tag{10.3.22}$$

which gives the equations for c_{jn}:

$$\sum_{j'=1}^{J} Q_{jn,j'n'} C_{j'n'}(r_o) = \phi_j^o (r \bar{K}_o) \tag{10.3.23}$$

with

$$\phi_j^o (r|\bar{K}_o) = 4\pi i^{l_j} j_{l_j} (K_o r_o) Y^*_{l_j m_j}(\hat{K}_o) J_n(\bar{K}_o \cdot \bar{\alpha}_o | b) \tag{10.3.24}$$

From the asymptotic behaviour for $r \to$ of (10.3.16) we get

$$\psi_n(\bar{r}) = e^{i\bar{K}_o \cdot \bar{r}} J_n(\bar{K}_o \cdot \bar{\alpha}_o) + \sum_N \frac{e^{iK_N r}}{r} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_o | b) f^{(N)}(\bar{K}_o \to \bar{K}_N) \tag{10.3.25}$$

where the N-photon radiative scattering amplitude for the linear polarization case is

$$f^{(N)}(\bar{K}_o \to \bar{K}_N) = \sum_{\substack{j,j' \\ n,n'}} J_{n-N} (\bar{K}_N \cdot \bar{\alpha}_o | b) A^{(N)}_{jn,j'n'} \; J_n(\bar{K}_o \cdot \bar{\alpha}_o | b) \tag{10.3.26}$$

with,

$$A^{(N)}_{jn,j'n'} = - 8\pi (i)^{l_{j'} - l_j} Y_{l_j m_j}(\hat{K}_N) j_{l_j}(K_N \alpha_o) [\bar{Q}']_{jn,j'n'} j_{l_j}(K_N \alpha_o) Y^*_{l_{j'} m_{j'}}(\hat{K}_o) \tag{10.3.27}$$

where $[\bar{Q}^{1}]_{jn,j'n'}$ is the inverse of the discrete-matrix $[Q]_{jn,j'n'}$ defined by (10.3.20).

This result should be compared with an alternative expression in the linear polarization case to be found in the literature[16]. Note that in the zero-field limit ($\alpha_0=0$)

$$L_{nj}^{n'j'} \rightarrow \delta_{j,j'} \quad \delta_{n,N} \quad \delta_{n',N}$$

$$[Q]_{jn,j'n'} \rightarrow 2iK_N \, h1_j^{(1)} \, (K_N \alpha_0) \, j1_j(K_N r_0) \, \delta_{j,j'} \quad \delta_{n,N} \quad \delta_{n',N}$$

and

$$A_{jn,j'n'} \rightarrow \delta_{j,j'} \quad \delta_{n,N} \quad \delta_{n',N} \quad 4\pi \frac{i}{K_N} \, \frac{j1_j(K_o r_o)}{h1_j^{(1)}(K_o r_o)}$$

Therefore, for $\alpha_0 = 0$

$$f^{(N)} \, (\bar{K}_o \rightarrow \bar{K}_N) = \delta_{N,o}[\sum_{1_j} \frac{i}{K_o}(21_j+1) \frac{j1_j(K_o r_o)}{h1_j^{(1)}(K_o r_o)} \, P1_j(\mathrm{Cos}\,\theta)] \tag{10.2.28}$$

which is the exact hard sphere scattering amplitude[14] in the absence of the field, as it should be.

10.4. Solution of the general finite-range separable Potential Model with a Laser Field.

The separable potential model can be defined by[17,18]

$$\hat{v} = \sum_{j=1}^{J} |Y_{1_j m_j}(\hat{r}) \, U_j \, (r) >< \, V_j(r) \, Y_{1_j m_j}(r)| \tag{10.4.1}$$

when J is an arbitrarily fixed integer. This potential supports J number of bound states of, in general, J-different angular momentum symmetry (1_j, m_j), j = 1,2,3,...J, and all the partial waves continua and phase shifts δ_j, the first J of which are in general non-zero. We shall give exact solutions of radiative amplitudes and the spectrum of ATI for this system of potentials plus a laser field. The Hamiltonian of the system of "electron + potential + laser field" is given by (1.2). The corresponding Floquet-Schrödinger equation is (see eq. (2.6)) :

$$[E-H_n^o]|\psi_n(\bar{r})> = \sum_{j=1}^{J} |Y_{l_j m_j}(\hat{r}) U_j(r)>< V_j Y_{l_j m_j}|\psi_{n'}> \qquad (10.4.2)$$

Circular Polarization:

We consider the case of circular polarization first.

Solution of the Radiative Scattering Problem:

Using the Green's function (5.3) we write

$$|\psi_n(\bar{r})> = \phi_n^o(\bar{r}) + \sum_{j'=1}^{J} \sum_{n'=-\infty}^{\infty} G_{nn'}^o(\bar{r},\bar{r}'|E)|Y_{l_j m_j}(\hat{r}') U_{j'}(r')> C_{j'}(n'|E) \qquad (10.4.3)$$

where we have defined

$$< V_j(r) \; Y_{l_j m_j}(\hat{r})|\psi_n(\bar{r})> = C_j(n|E) \qquad (10.4.5)$$

And

$$\phi_n^o(\bar{r}) = e^{i\bar{K}_o\cdot\bar{r}} J_n(K_o^{\perp}\alpha_o)\, e^{in\phi_o} \qquad (10.4.6)$$

is the incident Floquet-Volkov wave.

Projecting on (10.4.3) with $< V_j(r) \; Y_{l_j m_j}(\hat{r})|$ we get

$$C_j(n|E) = \tilde{\phi}_j^o(n|\bar{K}_o) + \sum_{j'=1}^{J} S_{jj'}(n,n-m_j+m_{j'}|E) C_{j'}(n-m_j+m_{j'}|E) \qquad (10.4.7)$$

where

$$S_{jj'}(n,n'|E) = \int dK K^2 \int_0^{\pi} d\theta_K \sin\theta_K \; g_{jn,j'n'}(K,\theta_K|E) \qquad (10.4.8)$$

with

$$g_{jn,j'n'}(K,\theta_K|E) = \frac{1}{(2\pi)^3} \sum_{N=-\infty}^{\infty} \Theta_{l_j m_j}(\theta_K)\tilde{V}_j(K)\frac{1}{E-K^2/2-N\omega-2\delta_\varepsilon+io}$$

$$J_{n-N}(K_N^{\perp}\alpha_o)\, J_{n'-N}(K_N^{\perp}\alpha_o)\tilde{U}_{j'}(K)\; \Theta_{l_{j'} m_{j'}}(\theta_K)\cdot \delta_{n',n-m_j+m_{j'}} \qquad (10.4.9)$$

and

$$\left.\begin{aligned} \tilde{U}_j(K) &= < U_j(r)|4-i^{l_j} j_{l_j}(Kr)> \\ \tilde{V}_j(K) &= < V_j(r)|4-i^{l_j} j_{l_j}(Kr)> \end{aligned}\right\} \qquad (10.4.10)$$

Also,

$$\tilde{\phi}^{o}_{j}(N|\bar{K}_o) = \tilde{V}_j(K_o)\, Y^{*}_{1_j m_j}(\hat{K}_o)\, J_n(K_o^{\perp}\alpha_o)\, e^{in\phi_o} \qquad (10.4.11)$$

and

$$\bar{K}_o = (K_o, \theta_o, \phi_o) = (K_z, K^{\perp}, \phi_K)$$

$$K_o = \sqrt{2E}$$

Making the useful transformation $n = p + m_j$ for any integer p, n and m_j in (10.4.7), we get the closed system of J algebraic equations

$$\sum_{j'=1}^{J} [\delta_{jj'} - S_{jj'}(p+m_j,\, p+m_{j'}|E)]\, C_{j'}(p+m_{j'}|E) = \tilde{c}^{o}_{j}(p+m_j|\bar{K}_o) \qquad (10.4.12)$$

Solution of this equation is, by Cramer's rule,

$$C_j(p+m_j|E) = A_j(p+m_j|\bar{K}_o)/Dp(E) \qquad (10.4.13)$$

where

$$Dp(E) = \det|\, \delta_{jj'} - S_{jj'}(p+m_j,\, p+m_{j'}|E)| \qquad (10.4.14)$$

is the determinant of the JxJ coefficient matrix on the left hand side of (10.4.12)and $A_j(p + m_j|\bar{K}_o)$ is the determinant obtained by replacing the j^{th} Column of (10.4.14) by the right hand side Column of (10.4.12).

Hence

$$\psi_n(\bar{r}) = \phi^{o}_{n}(\bar{r}) + \sum_{j'=1}^{J} \sum_{p=-\infty}^{\infty} G^{o}_{nn'}(\bar{r},\bar{r}';E)|U_{j'}(r')\, Y_{1_{j'} m_{j'}}(\hat{r}')> \cdot \frac{A_{j'}(p+m_{j'}|\bar{K}_o)}{Dp(E)} \qquad (10.4.15)$$

To abtain the radiative amplitude $f^{(N)}(\bar{K}_o \to \bar{K}_N)$ for N-photon Bremsstrahlung ($N > 0$) and inverse Bremsstrahlung ($N < 0$) processes, we take as usual the limit $r \to \infty$ and identify the coefficient of the out-going Floquet-Volkov state in the N^{th} channel. Thus

$$\psi_{p+m_j}(\bar{r}) \underset{r\to\infty}{=} \phi^{o}_{p+m_j}(\bar{r}) + \sum_{N=-\infty}^{\infty} \frac{e^{iK_N r}}{r}\, J_{p-m_j-N}(K_N^{\perp}\alpha_o)\, e^{i(p+m_j)\phi_{K_N}} \cdot f^{(N)}(\bar{K}_o \to \bar{K}_N) \qquad (10.4.16)$$

and

$$f^{(N)}(\bar{K}_o \to \bar{K}_N) = -\frac{1}{2\pi} \sum_{j'=1}^{J} \sum_{p'=-\infty}^{\infty} J_{p'+m_{j'}-N}(K_N^{\perp}\alpha_o) e^{-i(p'+m_{j'})\phi_{K_N}}$$

$$\tilde{U}_{j'}(K_N)\, Y_{l_{j'}m_{j'}}(\hat{K}_N) \frac{A_{j'}(p'+m_{j'}|\bar{K}_o)}{D_{p'}(E)} \qquad (10.4.17)$$

The associated cross section for radiative scattering is given by (6.13).

<u>Rate of N-photon ATI-Process:</u>

For ATI amplitude defined with respect to the initial dressed Volkov-packet associated with the initial bound state $\phi_i(\bar{r})$ (see section 9), in equation (10.4.15) we have

$$\phi_n^o(\bar{r}) = \sum_{\bar{K}} e^{i\bar{K}\cdot\bar{r}}\, \tilde{\phi}_i^*(\bar{K})\, J_n(K^{\perp}\alpha_o) e^{in\phi_K} \qquad (10.4.18)$$

and hence

$$\tilde{\phi}_i^o(n) \equiv \langle V_j(r) Y_{l_j m_j}(\hat{r}) | \phi_n^o(\bar{r}) \rangle$$

$$= \sum_{\bar{K}} \tilde{V}_j(K)\, Y^*_{l_j m_j}(\hat{K})\, \tilde{\phi}_i^*(\bar{K}) J_n(K^{\perp}\alpha_o)\, e^{in\phi_K} \qquad (10.4.19)$$

Hence absorption of N-photons can be obtained exactly by the technique of the previous section. The result is

$$f^{(N)}(\bar{K}_N) = -\frac{1}{2\pi} \sum_{j'=1}^{J} \sum_{p'=-\infty}^{\infty} J_{p'+m_{j'}-N}(K_N^{\perp}\alpha_o)\, e^{-i(p+m_{j'})\phi_{K_N}} \,.$$

$$\tilde{U}_{j'}(K_N)\, Y_{l_{j'}m_{j'}}(\hat{K}_N) \frac{A_{j'}(p'+m_{j'})}{D_{p'}(E)} \qquad (10.4.20)$$

where the determinant $D_{p'}(E)$ is given by (10.4.14) and $A_{j'}(p'+m_{j'})$ is obtained by placing the j'th column of $D_{p'}(E)$ by $\tilde{\phi}^o_{j'}(p'+m_{j'})$ obtained from the definition (10.4.19).

The differential rate of N-photon ATI-transition is simply

$$\frac{dW^{(N)}}{d\hat{K}_N} = v_N\, |f^{(N)}(\bar{K}_N)|^2 \qquad (10.4.21)$$

where $v_N = K_N$ (a.u.) is the final velocity of the electron.

Solution of the Eigenvalue Problem:

The fundamental solution of the eigenvalue problem is obtained from the condition of existence of solutions of the homogeneous system corresponding to (10.4.15) without the first term on the right hand side and hence from the secular equation

$$D_p(E) = \det |\delta_{jj'} - S_{jj'}(p+m_j, p+m_{j'}|E)| = 0 \qquad (10.4.22)$$

we observe that the irreducible Floquet-eigenvalue $E = E_{\lambda_j}$ and the associated eigenvectors $C_j(p+m_j|\ E = E_\lambda)$ satisfy the Floquet twin-transformation[1,17]

$$\begin{aligned} &E_\lambda \rightarrow E_\lambda + m\omega \\ &C_j(p+m_j|E_\lambda) \rightarrow C_j(p+m_j+m\ |\ E_\lambda + m\,\omega) \end{aligned} \qquad (10.4.23)$$

This is established by the observation that the system of equations, (10.4.12), with the right hand side set equal to zero, remains satisfied under the transformation (10.4.23) and on changing the arbitrary integer p to p+m and at the same time modifying the summation index N of (10.4.9) defining $S_{jj'}(p+m_j, p+m_{j'}|E)$ in eq. (10.4.8) and (10.4.22), by N+m.

Hence to every irreducible eigenvalue E_{λ_j}, in a given Floquet-zone[1], there exists an infinite set of eigenvalues $E_{\lambda_j}+m\omega$, $m = 0, \pm 1, \pm 2, \ldots$ which occur in all the Floquet-zones. The eigenvector $C_j(p+m_j|E = E_{\lambda_j})$ can be determined to within a normalization constant for every p from the set of algebraic eqs. (10.4.12), with the right hand side put equal to zero. The total state vector at the dressed energy E_{λ_j} can now be constructed explicitly as

$$|\psi E_{\lambda_j}> = \sum_{j'=1}^{J} \sum_{p=-\infty}^{\infty} |p+m_j> F^{(p)}_{jj'}(\bar{r})\ C_{j'}(p+m_{j'}|E_{\lambda_{j'}}) \qquad (10.4.24)$$

where

$$F^{(p)}_{jj'}(\bar{r}) = \sum_{\bar{K},N} e^{i\bar{K}\cdot\bar{r}}\ J_{p+m_j-N}(K^\perp\alpha_0) J_{p+m_{j'}-N}(K^\perp\alpha_0)$$

$$\cdot \frac{1}{E_{\lambda_j} - K^2/2 - N\omega - \delta_\varepsilon} \cdot 4\pi(i)^{l_{j'}} \tilde{U}_{j'}(K) Y_{l_j m_j}(\hat{K}) \qquad (10.4.25)$$

We observe that both the scattering and the eigenvalue problems can be solved[18] using the "length-gauge" interaction, if desired, in an analogous way.

Linear Polarization:

In this case the Floquet-Schrödinger wave functions $\psi_n(\bar{r})$ are

$$\psi_n(\bar{r}) = \phi_n^o(\bar{r}) + \sum_{j'=1}^{J} \sum_{n'=-\infty}^{\infty} \int d\bar{r}' G^o_{nn'}(\bar{r},\bar{r}'|E) U_{j'}(r') Y_{1_j m_j}(\hat{r}') C_{j'n'}(E) \quad (10.4.26)$$

where G^o_{nn} is now given by (5.2) and we have defined

$$C_{j'n'}(E) = \int d\bar{r}\, V_{j'}(r) Y_{1_{j'} m_{j'}}(\hat{r}) \psi_{n'}(\bar{r}) \quad (10.4.27)$$

and

$$\phi_n^o(\bar{r}) = e^{i\bar{K}_o\cdot\bar{r}}\, J_n(\bar{K}_o\cdot\bar{\alpha}_o|b)\ ,\ b = \frac{\delta_\varepsilon}{2\omega} \quad (10.4.28)$$

Multiplying (10.4.26) with $V_j(r) Y^{*}_{1_j m_j}(\hat{r})$ integrating with respect to $d\bar{r}$ and simplifying we get in a similar way as before

$$C_{jn}(E) = \tilde{\phi}^o_{jn}(\bar{K}_o) + \sum_{j'=1}^{J} \sum_{n'=-\infty}^{\infty} S_{jn,j'n'}(E)\, C_{j'n'}(E) \quad (10.4.29)$$

where

$$S_{jn,j'n'}(E) = \int_0^\infty dK K^2 \int_0^\pi d\theta_K\ \sin\theta_K\ g_{jn,j'n'}(K,\theta_K) \quad (10.4.30)$$

where

$$g_{jn,j'n'}(K,\theta_K) = \frac{1}{(2\pi)^3} \sum_{N=-\infty}^{\infty} \tilde{V}_j(K)\, \Theta_{1_j m_j}(\theta_K)\, J_{n-N}(\bar{K}\cdot\bar{\alpha}_o|b)\cdot$$

$$\cdot\ \delta_{m_j;m_{j'}}\cdot \frac{1}{E-K^2/2-N\omega-\varepsilon_\xi+i0}\, J_{n'-N}(\bar{K}\cdot\bar{\alpha}_o|b)\, \Theta_{1_{j'} m_{j'}}(\theta_K) \tilde{U}^{*}_{j'}(K) \quad (10.4.31)$$

where

$$\tilde{\phi}^o_{jn}(\bar{K}_o) = \tilde{V}_j(K_o)\, Y^{*}_{1_j m_j}(\hat{K}_o)\, J_n(\bar{K}_o\cdot\bar{\alpha}_o|\ b) \quad (10.4.32)$$

and $\tilde{V}_j(K)$ and $\tilde{U}_j(K)$ are given by (10.4.10).

It should be noted that the system of equations (10.4.29) like eq. (10.4.12), constitutes a countable set of algebraic equations inspite of the fact that the Floquet-Schrödinger equation accounts for the continuum motion of the electron completely. This should be contrasted with an attempt to obtain the solution by eigenfunction expansion with atomic eigenstates which will lead to a non-countable system of equations (a system of e.g. integral equations) for the coefficients of such an expansion (due to the presence of the continuum). Presently, however, one may solve (10.4.29) by discrete matrix algebra only. We take the asymptotic limit $r \to \infty$ in (10.4.26) and obtain, as usual,

$$\psi_n(\bar{r}) \underset{r\to\infty}{=} \phi_n^o(\bar{r}) + \sum_{N=-\infty}^{\infty} \frac{e^{iK_N r}}{r} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_0 | b)\ f^{(N)}(\bar{K}_0 \to \bar{K}_N) \tag{10.4.33}$$

with

$$f^{(N)}(\bar{K}_0 \to \bar{K}_N) = -\frac{1}{2\pi} \sum_{j=1}^{J} \sum_{n=-\infty}^{\infty} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_0 | b) \cdot$$

$$\cdot \tilde{\hat{U}}_j(K_N)\ Y_{l_j m_j}(\hat{K}_N)\ C_{jn}(E) \tag{10.4.34}$$

Using the solution of the system of algebraic equation (10.4.29) for $C_{j'n'}(E)$ one finally gets the result for the linear-polarization case,

$$f^{(N)}(\bar{K}_0 \to \bar{K}_N) = -\frac{1}{2\pi} \sum_{jn,j'n'} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_0 | b)\ A^{(N)}_{jn,j'n'}(E)\ J_n(\bar{K}_0 \cdot \bar{\alpha}_0 | b) \tag{10.4.35}$$

where

$$A^{(N)}_{jn,j'n'}(E) = \tilde{\hat{U}}_j(K_N) Y_{l_j m_j}(\hat{K}_N)\ [\bar{W}^{-1}(E)]_{jn,j'n'} Y^*_{l_{j'} m_{j'}}(\hat{K}_0)\ \hat{V}^*_{j'}(K_0) \tag{10.4.36}$$

Rate of N-photon ATI-process:

We may obtain the multiphoton ionization rate in a strong linearly polarized field in a similar way by choosing the initial state in the field to be the Volkov-packet state (see section 9):

$$\phi_n^o(\bar{r}) = \sum_{\bar{K}} e^{i\bar{K}\cdot\bar{r}}\ \tilde{\phi}^*_i(\bar{K})\ J_n(\bar{K} \cdot \bar{\alpha}_0 | b) \tag{10.4.37}$$

where $\tilde{\phi}_i(\bar{K}) = \langle \bar{K} | \phi_i(\bar{r}) \rangle$ is the Fourier transform of the initial bound state $\phi_i(\bar{r})$. Proceeding exactly similarly as above for the radiative scattering amplitude we obtain for the ATI-amplitude

$$f^{(N)}(\bar{K}_N) = -\frac{1}{2\pi} \sum_{\substack{jn \\ j'n'}} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_0 | b)\ \tilde{\hat{U}}_j(K_N)\ Y_{l_j m_j}(\hat{K}_N)$$

$$[\bar{W}^{-1}(E)]_{jn,j'n'}\ \tilde{\phi}^o_{j'}(n') \tag{10.4.38}$$

where

$$\tilde{\phi}^{0}_{j'}(n') = \sum_{\bar{K}} \tilde{V}^{*}_{j'}(K) Y^{*}_{l_{j'} m_{j'}}(\hat{K}) \tilde{\phi}_{i}(\bar{K}) J_{n'}(\bar{K}\cdot\bar{\alpha}_0|b); \; b = \frac{\delta_\varepsilon}{2\omega} . \tag{10.4.39}$$

The differential rate of ATI transition is

$$\frac{dW^{(N)}}{d\hat{K}_N} = v_N \, |f^{(N)}(\bar{K}_N)|^2 \tag{10.4.40}$$

where $v_N = K_N$(a.u.) is the final velocity of the ejected electron.

11. Exact Solution of the Resolvent and the Floquet Greens Function for the General Separable Potential Model with a Laser Field.

Let us write the general separable potential model as

$$\hat{v} = \sum_{j} | U_j(\bar{r}) >< V_j(\bar{r}) | \tag{11.1}$$

where $U_j(\bar{r})$ and $v_j(\bar{r})$ are defined in the form shown in eq. (10.4.1). The total resolvent of the system of "electron+potential+laser field" in the product space can be expanded in the number states as

$$G = \sum_{n,n'} | n > G_{nn'}(\bar{r},\bar{r}'|E) < n' | \tag{11.2}$$

The total Floquet-resolvent or Green's function $G_{nn'}(\bar{r},\bar{r}'|E)$ satisfies the equation

$$[E-H^{0}_{n} - \hat{v}] \, G_{nn'}(\bar{r},\bar{r}'|E) = \delta(\bar{r}-\bar{r}') \, \delta_{n,n'} \tag{11.3}$$

where H^{0}_{n} is given by eqs. (2.7) or (2.8).

Explicit Resolvent: Linear Polarization:

We use the potential free Floquet-Green's function $G^{0}_{nn'}$ which satifies

$$(E-H^{0}_{n}) \, G^{0}_{nn'}(\bar{r},\bar{r}'|E) = \delta(\bar{r}-\bar{r}')\delta_{n,n'} \tag{11.4}$$

to write the equation of the total Green's function $G_{nn'}$ as

$$G_{nn'} = G^0_{nn'} + \sum_{n_1=-\infty}^{\infty} G^0_{nn_1} \hat{v} G_{n_1 n'} \tag{11.5}$$

or more explicitly

$$G_{nn'}(\bar{r},\bar{r}'|E) = G^0_{nn'}(\bar{r}\bar{r}'|E) + \sum_{j_1=1,n_1=-\infty}^{J,\infty} G^0_{nn_1}(\bar{r},\bar{r}_1|E)|U_{j_1}> C_{j_1 n_1}(n'|\bar{r}') \tag{11.6}$$

where we have defined

$$C_{j_1 n_1}(n'|\bar{r}') = < v_{j_1}(\bar{r}_1)|G_{n_1,n'}(\bar{r}_1,\bar{r}'|E) \tag{11.7}$$

Projecting (11.6) with $<V_j|$ throughout, we get

$$C_{jn}(n'|\bar{r}') = C^0_{jn}(n'|\bar{r}') + \sum_{j_1 n_1} S_{jj_1}(n_1 n_1|E)\, C_{j_1 n_1}(n'|\bar{r}') \tag{11.8}$$

where

$$C^0_{jn}(n'|\bar{r}') \equiv < V_j(\bar{r})|G^0_{nn'}(\bar{r},\bar{r}'|E) \tag{11.9}$$

and

$$S_{jj_1}(n, n_1|E) \equiv < V_j(\bar{r})|\, G^0_{nn_1}(\bar{r},\bar{r}_1|E)|\, U_{j_1}(\bar{r}_1) > \tag{11.10}$$

Hence eq. (11.3) can be solved as

$$G_{nn'}(\bar{r},\bar{r}'|E) = G^0_{nn'}(\bar{r},\bar{r}'\, E) + \sum_{\substack{j_1=1\\ j_2=1}}^{J} \sum_{\substack{n_1=-\infty\\ n_2=-\infty}}^{\infty} G^0_{nn_1}(\bar{r},\bar{r}_1|E)|U_{j_1}(\bar{r}_1)> \cdot$$

$$\cdot\ [\bar{W}^1(E)]_{j_1 n_1, j_2 n_2} < v_{j_2}(\bar{r}_2)|G^0_{n_2 n'}(\bar{r}_2,\bar{r}'|E) \tag{11.11}$$

where $[\bar{W}^1(E)]$ is the inverse of the matrix $[W(E)]$ defined by

$$[W(E)]_{j_1 m_1, j_2 n_2} = [\delta_{j_1,j_2}\delta_{n_1,n_2} - S_{j_1 j_2}(n_1,n_2|E)] \tag{11.12}$$

This result, eq. (11.11), can be used to obtain the line shape spectrum or the probability density of any process involving a transition between a given pair of initial and final states of interest.

The N-photon Electron Ejection Spectrum: Linear Polarization:

We now give a systematic method of obtaining the line shape for N-photon electron ejection or ATI spectrum. In this case we consider a transition from the initial product state

$$|\phi_i> = \phi_i(\bar{r})|0> \tag{11.13}$$

where $\phi_i(\bar{r})$ is a bound atomic state and no photon is emitted or absorbed, $|n_i> \equiv |0>$, into the final Volkov-state

$$|\phi_f> = \sum_{n=-\infty}^{\infty} e^{i\bar{K}_f\cdot\bar{r}} J_{n-N_f}(\bar{K}_f\cdot\bar{\alpha}_0|b)|n> ; \; b = \frac{\delta_\varepsilon}{2\omega} \tag{11.14}$$

in which the electron has the kinetic energy $K_f^2/2$ and $N_f$photons are absorbed. Note that the total energy of the "electron+field" system in state $|\phi_i>$ is$E_i = \varepsilon_i + o\omega$ where ε_i is the eigenvalue associated with $\phi_i(\bar{r})$ and in the state $|\phi_f>$, $E_f = K_f^2/2 + N_f\omega + \delta_\varepsilon$. The matrix element of the transition of interest is

$$<\phi_f|G|\phi_i> = \sum_{nn'} <\phi_f|n> G_{nn'} <n'|\phi_i> \tag{11.15}$$

Substituting for $G_{nn'}(\bar{r},\bar{r}')$ from (11.11),

$$<\phi_f|G|\phi_i> = \sum_{n=-\infty}^{\infty} <\phi_n^f(\bar{r})|G^0_{n,o}(\bar{r},\bar{r}'|E)|\phi_i(\bar{r}')>$$

$$+ \sum_{n=-\infty}^{\infty} \sum_{\substack{j_1,n_1\\ j_2,n_2}} <\phi_n^f(\bar{r})|G^0_{nn_1}(\bar{r},\bar{r}_1|E)|U_{j_1}(\bar{r}_1)>[\bar{W}^{-1}(E)]_{j_1n_1,j_2n_2} <v_{j_2}(\bar{r}_2)|G^0_{n_2o}(\bar{r}_2,\bar{r}'|E)|\phi_i(\bar{r}')> \tag{11.16}$$

where

$$\phi_n^f(\bar{r}) = e^{i\bar{K}_f\cdot\bar{r}} J_{n-N_f}(\bar{K}_f\cdot\bar{\alpha}_0|b) \tag{11.17}$$

Substituting further from (5.3) for $G^0_{no}(\bar{r},\bar{r}')$ in the first part of the right hand side of (11.16) we get

$$\sum_{\bar{K},N}\sum_n J_{n-N_f}(\bar{K}_f\cdot\bar{\alpha}_0|b)<\bar{K}_f|\bar{K}>\cdot\frac{J_{n-N}(\bar{K}\cdot\bar{\alpha}_0|b)J_{-N}(\bar{K}\cdot\bar{\alpha}_0|b)<\bar{K}|\phi_i>}{E-K^2/2-N\omega-\delta_\varepsilon+io}$$

$$=\sum_N\Big[\sum_n J_{n-N_f}(\bar{K}_f\cdot\bar{\alpha}_0|b)\frac{J_{n-N}(\bar{K}_f\cdot\bar{\alpha}_0|b)J_{-N}(\bar{K}_f\cdot\bar{\alpha}_0|b)\tilde{\phi}_i(\bar{K}_f)}{E-K_f^2/2-N\omega-\delta_\varepsilon+io} \tag{11.18}$$

where $\tilde{\phi}_i(\bar{K}_f)$ is the fourier transform of the initial atomic state $\phi_i(\bar{r})$.

Note that the quantity in the square brackets above is δ_{NN_f} by the completeness relation of Table 1. We then get, for this part, simply

$$\frac{1}{E-K_f^2/2-N_f\omega-\delta_\varepsilon+io}\; J_{-N_f}(\bar{K}_f\cdot\bar{\alpha}_0\,|\,b)\,\phi_i(\bar{K}_f) \tag{11.19}$$

The second part of (11.16) can be simplified in a similar way. We get

$$\frac{1}{E-K_f^2/2-N_f\omega-\delta_\varepsilon+io}\;\sum_{\substack{j_1n_1\\j_2n_2}} J_{n_1-N_f}(\bar{K}_f\cdot\bar{\alpha}_0\,|b)\,\tilde{U}_{j_1}(\bar{K}_f)[\bar{W}^1(E)]_{j_1n_1,j_2n_2}C^{(i)}_{j_2n_2}(E) \tag{11.20}$$

where

$$C^{(i)}_{j_2n_2}(E) = \sum_{\bar{K},N} \tilde{v}^*_{j_2}(\bar{K})\,\frac{J_{n_2-N}(\bar{K}\cdot\bar{\alpha}_0\,|\,b)J_{-N}(\bar{K}\cdot\bar{\alpha}_0\,|b)\;\tilde{\phi}_i(\bar{K})}{E-K^2/2-N\omega-\delta_\varepsilon+io} \tag{11.21}$$

Combining (11.19) and (11.21) we have

$$<\phi_f\,|\,G\,|\,\phi_i> = \frac{1}{E-E_f+io}\;A^{(N_f)}_{i\to f}\;(E) \tag{11.22}$$

where $E_f = K_f^2/2 + N_f\omega + \delta_\varepsilon$, and the stationary transition amplitude of the process in which the electron is ejected with momentum $\bar{K}_f$ in the direction between $\hat{K}_f$ & $\hat{K}_f + d\hat{K}_f$ and N_f photons are absorbed ($N_f< 0$) from the field, is

$$\begin{aligned} A^{(N_f)}_{i\to f}(E_f) &= J_{-N_f}(\bar{K}_f\cdot\bar{\alpha}_0\,|b)\;\phi_i(\bar{K}_f) \\ &+ \sum_{\substack{j_1n_1\\j_2n_2}} J_{n_1-N_f}(\bar{K}_f\cdot\bar{\alpha}_0\,|b)\;\tilde{U}_{j_1}(\bar{K}_f)\;[\bar{W}^1(E_f)]_{j_1n_1,j_2n_2}\,\cdot \\ &\qquad\cdot\;C^{(i)}_{j_2n_2}\;(E_f) \end{aligned} \tag{11.23}$$

The probability of transition associated with this transition is

$$P^{(N_f)}_{i\to f} = \sum_{\bar{K}_f}\;|\,A^{(N_f)}_{i\to f}\;(E_f)\;|^2 \tag{11.24}$$

where the summation is over a group of final continum states around the momentum $\bar{K}_f$ and $\bar{K}_f + d\bar{K}_f$ and energy E_f and $E_f + dE_f$. Hence the associated double differential probability density spectrum is

$$\frac{d^2P^{(N_f)}_{i\to f}}{d\hat{K}_f dE_f} = |A^{(N_f)}_{i\to f}\;(E_f)|^2\;\rho\,(E_f) \tag{11.25}$$

where $\rho(E_f) = \frac{K_f}{(2\pi)^3}$ is the density of the final states. The individual probability of exchange of N_f photons is obtained by integrations:

$$P_{i\to f}^{(N_f)} = \int\int d\hat{K}_f \, dE_f \, | A_{i\to f}^{(N_f)} (E_f) |^2 \rho(E_f) \tag{11.26}$$

Since in the usual electron ejection or ATI-experiments the number of photons exchanged are not detected either separately or in coincidence with the electron energy, but only the electron energy is measured, the total probability density spectrum of the measured electron energy is obtained from the sum of (11.25) over the final photon numbers and integration over angles of ejection:

$$\frac{dS(E_f)}{dE_f} = \sum_{N_f} \int d\hat{K}_f \, |A_{i\to f}^{(N_f)} (E_f) |^2 \rho(E_f) \tag{11.27}$$

Explicit Resolvent: Circular Polarization:

We may proceed as before but using now the reference Green's function $G^0_{nn'}(\bar{r},\bar{r}')$ given by (5.4). Here we quote simply the final results:

$$G = \sum_{n,n'} | n > G_{nn'}(\bar{r},\bar{r}'; E) < n' | \tag{11.28}$$

with

$$G_{nn'}(\bar{r},\bar{r}', E) = G^0_{nn'}(\bar{r},\bar{r}' |E) + \sum_{j_1,j_2}^{J} \sum_{p=-\infty}^{\infty} G^0_{n,p+m_{j_1}}(\bar{r},\bar{r}_1 | E) | U_{j_1}(\bar{r}_1) >$$

$$[\bar{W}^1(p)]_{j_1 j_2} < V_{j_2}(\bar{r}_2) \, | \, G^0_{p+m_{j_2},n'}(\bar{r}_2\bar{r}' |E) \tag{11.29}$$

where the matrix $[W(p)]$ is defined by

$$[W(p)]_{j_1 j_2} = [\, \delta_{j_1,j_2} - S_{j_1 j_2}(p+m_{j_1}, p+m_{j_2} |E)] \tag{11.30}$$

where $S_{j_1 j_2}(p+m_{j_1}, p+m_{j_2}|E)$ are defined according to eqs. (10.4.8)-(10.4.10). We note that, due to the planar symmetry of the circularly polarized field, in this case the matrix $[W(p)]$ is essentially diagonal in the Floquet-(or the photon index) space and can be treated separately for each p and thus its size is determined by only the rank J of the separable potential chosen. In the linear polarization case, because of the reduced, axial, symmetry, the size of the corresponding matrix $[W(E)]$, eq. (11.12), is determined by the size of the Floquet-space as well.

The N-photon Electron Ejection Spectrum: Circular Polarization:

The corresponding matrix element of the total resolvent between the initial bound state in the product space,

$$|\Phi_i> = \phi_i(\bar{r})\,|0>, \quad E_i = \varepsilon_i + 0\,\omega \tag{11.31}$$

and the final Volkov-state

$$|\Phi_f> = \sum_n e^{i\bar{K}_f\cdot\bar{r}}\, J_{n-N_f}(K_f^{\perp}\alpha_0)\, e^{in\phi_f}|n> \tag{11.32}$$

where,

$$\bar{K}_f = (K_f, \theta_f, \phi_f), K_f^{\perp} = K_f\sin\theta_f, \; E_f = \frac{K_f^2}{2} + N_f\,\omega + 2\delta_\varepsilon ,$$

is

$$<\Phi_f|G|\Phi_i> = -\frac{1}{E-K_f^2/2-N\omega_f-2\delta_\varepsilon+io}\, A_{i\to f}^{(N_f)}(E) \tag{11.33}$$

with the amplitude

$$A_{i\to f}^{(N_f)}(E) = J_{-N_f}(K_f^{\perp}\alpha_0)\,\tilde{\phi}_i(\bar{K}_f) +$$

$$+\sum_{j_1,j_2=1}^{J}\sum_{p=-\infty}^{\infty} J_{p+m_{j_1}-N_f}(K_f^{\perp}\alpha_0)e^{i(p+m_{j_1})\phi_{K_f}}\,\tilde{U}_{j_1}(\bar{K}_f)\cdot$$

$$\cdot[\bar{W}^{1}(p)]_{j_1j_2}\, C_{j_2}^{(i)}(p+m_{j_2}|E) \tag{11.34}$$

where

$$C_{j_2}^{(i)}(p+m_{j_2}|E) = \sum_{\bar{K},N}\tilde{V}_{j_2}^{*}(\bar{K})\, J_{p+m_{j_2}-N}(K^{\perp}\alpha_0)e^{i(p+m_{j_2})\phi_K}$$

$$\cdot\frac{1}{E-K^2/2-N\omega-2\delta_\varepsilon+io}\, J_{-N}(K^{\perp}\alpha_0)\,\tilde{\phi}_i(\bar{K}) \tag{11.35}$$

The total electron ejection or ATI-spectrum is given by

$$\frac{dS(E_f)}{dE_f} = \sum_{N_f}\int d\hat{K}_f\,|A_{i\to f}^{(N_f)}(E_f)|^2\,\rho(E_f) \tag{11.36}$$

where $A_{i\to f}^{(N_f)}(E_f)$ is given by (11.34) with

$$E = E_f = \frac{K_f^2}{2} + N_f\omega + 2\delta_\varepsilon \tag{11.37}$$

Before concluding this section we also give the simple expression for the amplitude $A_{i\to f}^{(N_f)}(E)$ for the commonly occuring case of electron ejection from a single bound s-state corresponding to a potential of rank one. From the exact general result (11.34), for J=1, j=1,$(p,l_1,m_1)=(0,0,0)$, we get:

$$A_{i\to f}^{(N_f)}(E) = J_{-N_f}(K_f^\perp\alpha_0)\,[\tilde{\phi}_i(\bar{K}_f)+\tilde{U}_1(\bar{K}_f)\cdot\frac{1}{1-S_{11}(0,0|E)}\,C_1^{(i)}(0|E)] \tag{11.38}$$

where

$$S_{11}(0,0|E) = \sum_{\bar{K},N}\frac{\tilde{V}_1(K)J_N^2(K^\perp\alpha_0)\tilde{U}_1^*(K)}{E-K^2/2-N\omega-2\delta_\varepsilon+io}\quad, \tag{11.39}$$

$$C_1^{(i)}(0|E) = \sum_{\bar{K},N}\frac{\tilde{V}_1(K)J_N^2(K^\perp\alpha_0)\tilde{\phi}_i(\bar{K})}{E-K^2/2-N\omega-2\delta_\varepsilon+io}\quad. \tag{11.40}$$

Detailed quantitative studies of this and related models have revealed a number of interesting phenomena such as the mechanism of ATI peak suppression, peak-disappearance, the presence of additional counter-shift of the spectrum (which partly compensates the usual shift due to the quiver energy) and the existence of Wigners threshold-cusps in the ATI-spectrum(at the thresholds of opening of higher photon channels in the spectrum of lower photon channels.) These will be reported elsewhere[19].

We should observe in the context of this general solution of the separable potential model, that the important problem of radiatve processes in the coulomb potential can also be solved[17] by the separable potential. Thus letting

$$\left.\begin{array}{l} U_j(\bar{r}) = -\frac{Z}{r}\phi_j(\bar{r}) \\ \text{and} \\ v_j(\bar{r}) = \phi_j(\bar{r}) \end{array}\right\} \tag{11.41}$$

(z is the charge of the coulomb center) where $\phi_j(r)$ are the so-called radial sturmian functions[20]

$$\phi_j(\bar{r}) \equiv \phi_{n_j l_j}(r)\,Y_{l_j m_j}(\hat{r}) \tag{11.42}$$

with

$$\phi_{n_j l_j}(r) = N_j \cdot e^{-x_j r} r^{l_j} {}_1F_1(l_j+1-n_j, 2l_j+2; 2x_j r) \tag{11.43}$$

where

$$N_j \equiv \frac{(2x_j)^{l_j+1}}{(2l_j+1)!} \left(\frac{(n_j+l_j)!}{(n_j-l_j-1)!}\right)^{\frac{1}{2}}, \quad x_j \equiv (-2E)^{\frac{1}{2}}$$

we obtain the <u>diagonal</u> separable expansion of the coulomb potential:

$$-\frac{Z}{r} = \sum_j \left| -\frac{Z}{r} \phi_j(\bar{r})\right\rangle \left\langle \phi_j(\bar{r}) \right| \tag{11.44}$$

This is a particular case of the general form treated here.

Similarly, the separable potential for any system which may be modelled by a finite number of bound states and the continum in a potential v(r) is obtained by replacing the original central potential v(r) by the separable potential of rank J:

$$\hat{v} = \sum_{j=1}^{J} \left| v(r) \phi_j(\bar{r})\right\rangle \left\langle \phi_j(\bar{r}) \right| \tag{11.45}$$

where $|\phi_j(\bar{r})>$ are the exact ortho-normalized eigenstates of the (field free) Schrödinger problem. This is easily established by considering the Schrödinger equation,

$$[-\frac{1}{2}\nabla^2 + \hat{v}]|\phi_j> = \varepsilon_j |\phi_j> \tag{11.46}$$

where ε_j is the energy of the eigenstate $|\phi_j>$. Putting (11.45) in (11,46), projecting with $<\phi_{j'}(\bar{r})|$, and noting that $<\phi_{j'}(\bar{r})|\phi_j(\bar{r})> = \delta_{jj'}$, one at once confirms that

$$[-\frac{1}{2}\nabla^2 + v(r)]|\phi_j(\bar{r})> = \varepsilon_j |\phi_j(\bar{r})>, \text{ for all } j = 1, 2, \ldots J, \tag{11.47}$$

as required.

12. <u>Solution of the Model 1-D Delta Potential Plus Laser Field</u>

<u>The Radiative Scattering Amplitude:</u>

This is perhaps the simplest system whose exact solution has not been obtained before. The potential is defined by

$$v(x) = -v_0\ \delta(x) \tag{12.1}$$

The Foquet-Schrödinger wave function is

$$\psi_n(x) = e^{iK_0x} J_n(K_0\alpha_0|b) - V_0 \sum_{n'=-\infty}^{\infty} \int G^0_{nn'}(x,x'|E)\delta(x')\psi_{n'}(x')dx' \tag{12.2}$$

Evaluating at x = 0 and rearranging we get

$$\sum_{n'} [\delta_{nn'} + v_0 G^0_{nn'}(0,0|E)]\ \psi_{n'} = J_n(K_0\alpha_0|b) \tag{12.3}$$

The 1-D Green's function is

$$G^0_{nn'}(x,x'|E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dK e^{iKx}\ \frac{J_{n-N}(K\alpha_0|b)J_{n'-N}(K\alpha_0|b)e^{-iKx'}}{E-K^2/2-N\omega-\delta_\varepsilon + io} \tag{12.4}$$

In the $x \to \infty$

$$G^0_{nn'}(x,x'|E) = e^{iK_Nx}\ J_{n-N}(K_N\alpha_0|b)\frac{1}{iK_N}\ J_{n'-N}(K_N\alpha_0|b)e^{-iK_Nx'}$$

Therefore

$$\psi_n(x)_{x\to\infty} = e^{iK_0x}J_n(K_0\alpha_0|b) + \sum_{N,n'} e^{iK_Nx}J_{n-N}(K_N\alpha_0|b)(-v_0)\frac{1}{iK_N}J_{n'-N}(K_N\alpha_0|b)\psi_{n'}(0) \tag{12.5}$$

with the N-photon scattering amplitude in the forward direction

$$f^{(N)}\ (K_0 \to K_N) = i\ \frac{v_0}{K_N}\ \sum_{n'} J_{n'-N}(K_N\ \alpha_0\ b)\psi_{n'}(0) \tag{12.6}$$

where

$$\psi_{n'}(0) = \sum_{n'} [W^{-1}(E)]_{nn'}\ J_{n'}(K_0\alpha_0|b) \tag{12.7}$$

where $[W^{-1}(E)]$ is the inverse of the discrete matrix defined by

$$[W(E)]_{nn'} = [\delta_{n,n'} + v_0\ G^0_{nn'}(0,0|E)] \tag{12.8}$$

An expression similar to (12.6) holds for the backward direction.

The N-photon ATI Rate:

Let us consider the strong field initial reference state to be the Volkov-like bound state (c.f. section 9)

$$|\Phi\rangle = \sum_n \phi_n^o(x)\,|n\rangle \tag{12.9}$$

where

$$\phi_n^o(x) = \frac{1}{2\pi}\int dK\,\tilde{\phi}_i^o(K)\,e^{iKx}\,J_n(K\alpha_o|b)\ ,\ b = \frac{\delta\varepsilon}{2\omega}$$

where $\tilde{\phi}_i^o(K)$ is the Fourier transform of the initial atomic bound state $\phi_i(x)$.

The Floquet-Schrödinger wave function of the system is

$$\psi_n(x) = \phi_n^o(x) - v_o\sum_{n'}\int G_{nn'}^o(x,x'|E)\,\delta(x')\,\psi_{n'}(x')dx'$$

substituting X = 0 and rearranging

$$\sum_{n'}\left[\delta_{n,n'} + v_o\,G_{nn'}^o(0,0\ E)\right]\psi_{n'}(0) = \phi_n^o(0) \tag{12.10}$$

The asymptotic behaviour of $\psi_n(x)$ is

$$\psi_n(x)\underset{x\to\infty}{} = \phi_n^o(x) + \sum_N e^{iK_Nx}\,J_{n-N}(K_N\alpha_o\,b)\cdot\sum_{n'}\left(\frac{-v_o}{iK_N}\right)J_{n'-N}(K_N\alpha_o|b)\,\psi_{n'}(0) \tag{12.11}$$

Identifying the coefficient of the outgoing Floquet-Volkov state on the right hand side for the N^{th} channel we get for the N-photon bound-free transition amplitude in the forward direction of electron ejection

$$f_{i\to f}^{(N)}(K_N) = \left(i\frac{v_o}{K_N}\right)\sum_{n,n'} J_{n-N}(K_N\alpha_o|b)\,[\bar{W}^1(E)]_{nn'}\,\phi_{n'}^o(0) \tag{12.13}$$

where [W(E)] is defined by (12.8) and

$$\phi_{n'}^o(0) = \frac{1}{2\pi}\int dK\,\tilde{\phi}_i^*(K)\,J_{n'}(K\alpha_o|b).$$

An expression similar to (12.13) holds for the backward direction of ejection. The rate of ATI by N-photon absorption is thus given by

$$W_{i\to f}^{(N)} = v_N\,|f_{i\to f}^{(N)}(K_N)|^2 \tag{12.14}$$

where $v_N = K_N = \sqrt{2(E-N\omega-\delta_\varepsilon)}$ is the final velocity of the electron. We note that the ATI-line shape spectrum can be obtained similar to the previous cases from the exaxt resolvent desired above.

The Exact Resolvent:

$$G_{nn'}(x,x'|E) = G^0_{nn'}(x,x'|E)$$

$$+ \sum_{n_1=-\infty}^{\infty} \int dx_1\, G^0_{nn_1}(x,x'|E)\,(-v_0)\,\delta(x_1)\, G_{n_1n}(x_1,x'|E) \qquad (12.15)$$

Integrating over dx_1 and putting $x = 0$.

$$\sum_{n_1=-\infty}^{\infty} [\delta_{n,n_1} + v_0 G^0_{nn_1}(0,0|E)]\, G_{n_1,n'}(0,x'|E) = G^0_{nn'}(0,x'|E)$$

or

$$G_{nn'}(0,x'|E) = \sum_{n_1=-\infty}^{\infty} [\bar{W}^{1}(E)]_{nn_1}\, G^0_{nn'}(0,x'|E) \qquad (12.16)$$

where $[\bar{W}^{1}(E)]$ is the inverse of the discrete matrix defined by

$$[W(E)]_{nn'} = [\delta_{n,n'} + v_0 G^0_{nn'}(0,0|E] \qquad (12.17)$$

Hence from (12.15) we get the exact solution of the total Floquet resolvent for the model delta potential plus the laser field:

$$G_{nn'}(x,x'|E) = G^0_{nn'}(x,x'|E) + \sum_{n_1,n_2=-\infty}^{\infty} G^0_{nn_1}(x,0|E)$$

$$\cdot\, (-v_0)\, [\bar{W}^{1}(E)]_{n_1n_2}\, G^0_{n_2n'}(0,x'|E) \qquad (12.18)$$

where the unperturbed resolvent $G^0_{nn'}$ is given by (12.4), and the product-space total resolvent is

$$G = \sum_{nn'} |n\rangle\, G_{nn'}(x,x'|E)\langle n| \qquad (12.19)$$

13. The Delta-Shell Potential Plus Laser Field:

This potential is defined by

$$v(r) = - v_0\ \delta(r-r_0) \tag{13.1}$$

where r_0 is the radius of the shell and v_0 is the 'strength'.

Linear Polarization:

The Floquet-Schrödinger wave function is

$$\psi_n(\bar{r}) = \phi_n^0(\bar{r}) + \sum_{n'} \int G^0_{nn'}(\bar{r},\bar{r}'|E)\ (-v_0)\ \delta(r'-r_0)\,\psi_{n'}(\bar{r}')d\bar{r}' \tag{13.2}$$

where the initial Floquet-Volkov-packet in the field

$$\phi_n^0(\bar{r}) = \begin{cases} e^{i\bar{K}_0\cdot\bar{r}}\ J_n(\bar{K}_0\cdot\bar{\alpha}_0\,|\,b) & \text{(for radiative scattering)} \quad (13.3)\\ \qquad \text{or} & \\ \sum_{\bar{K}} e^{i\bar{K}\cdot\bar{r}}\ \tilde{\phi}_i^*(\bar{K})\ J_n(\bar{K}\cdot\bar{\alpha}_0\,|\,b) & \text{(for ionization)} \quad (13.4)\end{cases}$$

where $\tilde{\phi}_i^*(\bar{K}) = \langle\bar{K}|\phi_i(\bar{r})\rangle$ is the Fourier transform of the initial bound-state $\phi_i(\bar{r})$.

Performing the integration over $d\bar{r}'$ and taking the limit $r\to\infty$ in (13.2) we get

$$\psi_n(\bar{r}) = \phi_n^0(\bar{r}) + \sum_N \frac{e^{iK_N r}}{r}\ J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0\,|\,b)\ f^{(N)} \tag{13.5}$$

with

$$f^{(N)} = -\frac{1}{2\pi} \sum_{n'} J_{n'-N}(\bar{K}_N\cdot\bar{\alpha}_0\,|\,b)\ (-v_0 r_0^2)\int e^{-i\bar{K}_N\cdot\bar{r}_0}\psi_{n'}(\bar{r}_0)d\hat{r}_0 \tag{13.6}$$

Introducing the partial wave expansion

$$\psi_n(\bar{r}_0) = \sum_{l_j m_j} \psi_j\ (n\,|\,r_0)\ Y_{l_j m_j}(\hat{r}_0)\ ;\ j \equiv (l_j, m_j) \tag{13.7}$$

$$e^{i\bar{K}_N\cdot\bar{r}_0} = \sum_{l_j m_j} 4\pi\ (i)^{l_j}\ j_{l_j}(K_N r_0)\ Y_{l_j m_j}(\hat{r}_0)\ Y^*_{l_j m_j}(\hat{K}_N) \tag{13.8}$$

we get

$$f^{(N)} = -\frac{1}{2\pi} \sum_{n'j} J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0\,|\,b)\ 4\pi\ (i)^{-l_j}\ j_{l_j}(K_N r_0)\ Y_{l_j m_j}(\hat{K}_N)$$

$$(-\ v_0 r_0^2)\ \psi_j(n\,|\,r_0) \tag{13.9}$$

To determine $\psi_{j'}(n'|ro)$ we project on to (13.2) with $\langle Y_{l_j m_j}(\hat{r})|$, substitue the partial wave expansion (5.5) for $G^o_{nn'}(\bar{r}\bar{r}'|E)$, multiply with δ (r-ro) and integrate over r^2dr and simplify, to get

$$\psi_j(n|r_o) = \phi^o_j(n|r_o) + \sum_{j'n'} S_{jn',j'n'}(E)\psi_{j'}(n'|r_o) \tag{13.10}$$

where

$$S_{jn,j'n'}(E) = \sum_{N=-\infty}^{\infty}(i)^{l_j-l_{j'}} L^{n'j'}_{nj}(K_N\alpha_o)[iK_N v_o]\cdot$$
$$\cdot [j_{l_j}(K_N r_o)h^{(1)}_{l_{j'}}(K_N r_o)+h^{(1)}_{l_j}(K_N r_o)j_{l_{j'}}(K_N r_o)]\cdot\psi_{j'}(n' r_o) \tag{13.11}$$

with

$$L^{n'j'}_{nj}(K_N\alpha_o) = \int_0^{\pi} d\theta_K \; \sin\theta_K \; J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_o|b)\Theta_{l_j m_j}(\theta_K) \; \Theta_{l_{j'}m_{j'}}(\theta_K)$$
$$\cdot J_{n'-N}(\bar{K}_N\cdot\bar{\alpha}_o|b) \; \delta_{m_j m_{j'}} \tag{13.12}$$

where

$$\phi^o_j(n|r_o) = \begin{cases} 4\pi i^{l_j} j_{l_j}(K_o r_o) Y^*_{l_j m_j}(\hat{K}_o) J_n(\bar{K}_o\cdot\bar{\alpha}_o|b) & \text{(radiative scattering)} \quad (13.13) \\ \text{or} & \\ \sum_{\bar{K}} 4\pi(i)^{l_j} j_{l_j}(Kr_o)Y^*_{l_j m_j}(\hat{K})\hat{\phi}_i(\bar{K}) J_n(\bar{K}\cdot\bar{\alpha}_o|b) & \text{(multiphoton ionization)} \quad (13.14) \end{cases}$$

with

$$b \equiv \frac{\delta\varepsilon}{2\omega}.$$

Thus the solution of (13.10) is

$$\psi_{j'}(n'|r_o) = \sum_{j''n''}[\bar{W}^{-1}(E)]_{j'n',j''n''}\phi^o_{j''}(n''|r_o) \tag{13.15}$$

where [W(E)] is a discrete-matrix defined by

$$[\bar{W}(E)]_{j'n',j''n''} = [\delta_{j',j''}\ \delta_{n',n''} - S_{jn,j'n'}(E)] \tag{13.16}$$

Finally substituting (13.15) and (13.13) in (13.9) we get the N-photon radiative scattering amplitude

$$f^{(N)}(\bar{K}_0 \to \bar{K}_N) = \sum_{jn,j'n'} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_0 | b) A^{(N)}_{jn,j'n'}\ J_{n'}(\bar{K}_0 \cdot \bar{\alpha}_0 | b) \tag{13.17}$$

where

$$A^{(N)}_{jn,j'n'} = 8\pi (i)^{l_{j'}-l_j}\ j_{l_j}(K_N r_0)\ Y_{l_j m_j}(\hat{K}_N)\ (v_0 r_0^2)\ \cdot$$

$$[\bar{W}^{-1}(E)]_{jn,j'n'}\ j_{l_{j'}}(K_0 r_0) Y^*_{l_{j'} m_{j'}}(\hat{K}_0) \tag{13.18}$$

The corresponding radiative differential scattering cross section is given by

$$\frac{d\sigma^{(N)}}{d\hat{K}_N} = \frac{K_N}{K_0}\ |f^{(N)}(\bar{K}_0 \to \bar{K}_N)|^2 \tag{13.19}$$

Rate of N-photon ATI-process:

The above-threshold-ionization (ejection) amplitude is, similarly, obtained as

$$f^{(N)}(\bar{K}_N) = \sum_{jn,j'n'} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_0 | b)\ 4\pi (i)^{l_j}\ j_{l_j}(K_N r_0) Y_{l_j m_j}(\hat{K}_N)\ \cdot$$

$$\cdot\ (v_0 r_0^2)\ [\bar{W}^{-1}(E)]_{jn,j'n'}\ \phi^0_{j'}(n' | r_0)$$

where $\phi^0_{j'}(r' | r_0)$ is given by (13.14). The differential rate of ATI is now given by

$$\frac{dW^{(N)}(\bar{K}_N)}{d\hat{K}_N} = v_N\ |f^{(N)}(\bar{K}_N)|^2 \qquad (v_N = K_N,\ \text{a.u.}) \tag{13.20}$$

Circular Polarization Case:

This case can be solved similarly. In this case the Floquet-Schrödinger wave function $\psi_n(\bar{r})$, given by (13.2) with $G^0_{nn'}$ given by (5.3) asymptotically becomes

$$\psi_n(\bar{r}) = \phi_n^o(\bar{r}) + \sum_N \frac{e^{iK_N r}}{r} J_{n-N}(K_N^{\perp}\alpha_o)e^{in\phi_{K_N}} f^{(N)} \tag{13.21}$$

with

$$f^{(N)} = -\frac{1}{2\pi}\sum_{n'} J_{n'-N}(K_N^{\perp}\alpha_o)e^{-in'\phi_{K_N}}(-v_o)\langle e^{i\bar{K}_N\cdot\bar{r}_o}|\psi_{n'}(\bar{r}_o)\rangle \tag{13.22}$$

and

$$\phi_n^o(\bar{r}) = \begin{cases} e^{i\bar{K}_o\cdot\bar{r}} J_n(K_o^{\perp}\alpha_o)e^{in\phi_o} \quad \text{(radiative scattering)} & (13.23) \\ \text{or} & \\ \sum_{\bar{K}} e^{i\bar{K}\cdot\bar{r}}\tilde{\phi}_i^{*}(\bar{K}) J_n(K^{\perp}\alpha_o)e^{in\phi_K} \quad \text{(multiphoton ionization)} & (13.24) \end{cases}$$

We also expand

$$\psi_n(\bar{r}_o) = \sum_j \psi_j(n|r_o)Y_{l_j m_j}(\hat{r}_o) \tag{13.25}$$

and proceed as before to find,

$$\psi_j(n|r_o) = \phi_j^o(n|r_o) + \sum_{jj'n} S_{jj'}(n,n-m_j+m_{j'}|E)\psi_{j'}(n-m_j+m_{j'}|r_o) \tag{13.26}$$

where

$$S_{jj'}(n,n'|E) = (i)^{l_j-l_{j'}}\sum_{N=-\infty}^{\infty} B_{nj}^{n'j'}(K_N\alpha_o)[iK_N v_o]\cdot[j_{l_j}(K_N r_o) \cdot h_{l_{j'}}^{(1)}(K_N r_o) + h_{l_j}^{(1)}(K_N r_o)j_{l_{j'}}(K_N r_o)] \tag{13.27}$$

with

$$B_{nj}^{n'j'}(K_N\alpha_o) = \int_0^{\pi} d\theta_K \sin\theta_K J_{n-N}(K_N^{\perp}\alpha_o)e^{i(n-n')\phi_{K_N}} J_{n'-N}(K_N^{\perp}\alpha_o) \Theta_{l_j m_j}(\theta_K)\Theta_{l_{j'}m_{j'}}(\theta_K)\delta_{n',n-m_j+m_{j'}} \tag{13.28}$$

with

$$K_N^{\perp} = K_N \sin\theta_{K_N}$$

and

$$\phi_j^o(n|r_o) = \begin{cases} 4\pi(i)^{l_j}\, j_{l_j}(K_o r_o)\, Y^*_{l_j m_j}(\hat{K}_o)\, J_n(K_o^{\perp}\alpha_o)e^{in\phi_o} & (13.29) \\ \text{(radiative scattering)} & \\ \text{or} & \\ \sum_{\bar{K}} 4\pi(i)^{l_j}\, j_{l_j}(Kr_o)\, Y^*_{l_j m_j}(\hat{K})\, \mathcal{F}^*_j(\bar{K}) J_n(K^{\perp}\alpha_o)e^{in\phi_K} & (13.30) \end{cases}$$

Finally, putting $n = p+m_j$ and $n' = p+m_{j'}$ we get for the radiative scattering amplitude

$$f^{(N)}(\bar{K}_o \to \bar{K}_N) = \sum_{j,j'=1}^{J}\sum_{p=-\infty}^{\infty} J_{p+m_j-N}(K_N^{\perp}\alpha_o)e^{-i(p+m_j)\phi_{K_N}}$$

$$\cdot A_{jj'}(p;N)\, J_{p+m_{j'}}(K_o^{\perp}\alpha_o)e^{i(p+m_{j'})\phi_o} \qquad (13.31)$$

with

$$A_{jj'}(p;N) = 8\pi(i)^{l_{j'}-l_j}\, j_{l_j}(K_N\alpha_o)\, Y_{l_j m_j}(\hat{K}_N)(v_o r_o^2)$$

$$[\bar{W}^{1}(p)]_{jj'}\, j_{l_{j'}}(K_o r_o)\, Y^*_{l_{j'} m_{j'}}(\hat{K}_o) \qquad (13.32)$$

with the discrete matrix $[W(p)]$ defined by

$$[W(p)]_{jj'} = [\delta_{j,j'} - s_{j,j'}(p+m_j, p+m_{j'}|E)] \qquad (13.33)$$

Rate N-photon ATI-process:

Similarly, the ATI-amplitude in this case is found to be

$$f^{(N)}(\bar{K}_N) = \sum_{j,j'=1}^{J}\sum_{p=-\infty}^{\infty} J_{p+m_j-N}(K_N^{\perp}\alpha_o)e^{-i(p+m_j)\phi_{K_N}}$$

$$\cdot 4\pi(i)^{-l_j}\, j_{l_j}(K_N r_o)\, Y_{l_j m_j}(\hat{K}_N)\,(v_o r_o^2)$$

$$[W^{-1}(p)]_{jj'}\ \phi^{o}_{j'}(p+m_{j'}|r_o) \qquad (13.34)$$

with $\phi^{o}_{j'}(p+m_{j'}|r_o)$ given by (13.30).

These exact amplitudes indicate the following qualitative picture of the N-photon radiative processes. First, the electron is dressed by the field into a superposition of n-photon component states each with an amplitude proportional to J_n. Second, the electron in each of these component states interacts with the potential which leads to an exchange of N-photons into the corresponding component states with (n-N) photons. Third, the electron propagates in these final component states with the associated amplitude J_{n-N}. The total amplitude of the observed N-photon process is the coherent sum of the products of the individual amplitudes for these three stages over all n.

Note also that in the limit of weak field strength, or low-frequency, or high kinetic energy, the interaction amplitude with the potential in the second stage of the process tends to take the form of the ordinary scattering amplitude $f(E-n\omega)$ evaluated at an energy, $E-n\omega$, shifted from the initial energy E by the energy of the component waves, $n\omega$, for each n. This leads to the so-called "low-frequency approximation"[21-25] of the radiative scattering amplitude:

$$\sum_{n=-\infty}^{\infty} J_{n-N}(\bar{K}_N\cdot\bar{\alpha}_0|b)\ f(E-n\,\omega)\ J_n(\bar{K}_0\cdot\bar{\alpha}_0|b) \qquad (13.35)$$

(linear polarization)

$$f^{(N)}(\bar{K}_0\to\bar{K}_N) \cong$$

$$\sum_{n=-\infty}^{\infty} J_{n-N}(K_N^{\perp}\alpha_0)e^{in\phi_{K_N}}\ f(E-n\,\omega)\ J_n(K_0^{\perp}\alpha_0)e^{-in\phi_0} \qquad (13.36)$$

(circular polarization)

Analogous expressions in the same limits hold for the ATI amplitudes as well.

We obtain,

$$f^{(N)}_{ATI}(\vec{K}_N) \simeq \begin{cases} \sum\limits_{\bar{K}} J_{n-N}(\bar{K}_N \cdot \bar{\alpha}_o | b)\; f(\bar{K} \to \bar{K}_N | E-n\omega) J_n(\bar{K} \cdot \bar{\alpha}_o ; b)\; \tilde{\phi}_i(\bar{K}) & (13.37) \\ \text{(linear polarization)} & \\ \sum\limits_{\bar{K}} J_{n-N}(K_N^{\perp} \alpha_o) e^{in(\phi_{K_N} - \phi_o)} f(\bar{K} \to \bar{K}_N | E-n\omega)\, J_n(K^{\perp}\alpha_o)\, \tilde{\phi}_i(\bar{K}) & (13.38) \\ \text{(circular polarization)} & \end{cases}$$

where $f(\bar{K} \to \bar{K}_N | E-n\omega)$ is the analytically continued scattering amplitude for $\bar{K} \to \bar{K}_N$ at the off-shell energy $E-n\omega$.

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New Class of Resonance in the $e+H^+$ Scattering in an Excimer-Laser Field

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(Received 30 December 1986)

We report on the numerical evidence of a new class of low-energy $e+H^+$ scattering resonances which dominate the field-modified elastic (Rutherford) and the inelastic (inverse bremsstrahlung) scattering cross sections in the presence of a strong excimer laser.

PACS numbers: 34.80.Qb, 32.80.-t

During the last few years much interest has developed[1-8] in the study of electron scattering in the presence of strong laser fields. One of the basic problems in this context is to understand how the $e+H^+$ scattering cross section, described exactly by the well-known Rutherford formula, changes in the presence of a strong laser field.

Up to now this apparently simple scattering problem has defied exact solution. This is primarily due to the analytical difficulties associated with the long-range Coulomb potential *and* with the increased dimension of the Schrödinger equation in the presence of the radiation field.

Recently this problem has been discussed by Gavrila and Kaminski,[6] using a high-frequency approximation in which only the static part of the "dressed" potential seen by the electron was retained. Their high-frequency approximation presumably provides only average information for the low-energy (electron energy less than the photon energy) radiative Coulomb scattering; it did not reveal a whole class of Rydberg resonances, which are reported in this Letter. These resonances are found to dominate both the elastic and the inelastic radiative Coulomb scattering.

In view of the lack of an exact analytical solution we are led to attack the problem by direct numerical means. To this end we extend the well-known close-coupling method[9] of solution of the ordinary electron-scattering problems and incorporate the interaction of the radiation field via the Floquet representation of the scattering equations. Our numerical solution of the extended close-coupling equations for the problem reveals a new class of resonances which dominate the cross sections of the elastic (Rutherford) as well as the inelastic (inverse bremsstrahlung) processes in the presence of the field.

The Schrödinger equation of the problem is ($e=\hbar=m=1$)

$$i\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left[-\frac{1}{2}\nabla^2-\frac{\mathbf{A}(t)}{c}\cdot\hat{\mathbf{p}}+\frac{1}{2c^2}\mathbf{A}^2(t)-\frac{Z}{r}\right]\Psi(\mathbf{r},t), \tag{1}$$

where we have chosen a circularly polarized radiation field

$$\mathbf{A}(t)=A_0[\hat{\epsilon}_x\cos(\omega t+\delta)-\hat{\epsilon}_y\sin(\omega t+\delta)]. \tag{2}$$

$\hat{\epsilon}_x$ and $\hat{\epsilon}_y$ are the orthogonal unit vectors. A_0 is the peak vector potential, ω is the frequency, and δ a constant phase; $\hat{\mathbf{p}}=-i\nabla$, $Z=1$ for H^+. In the present form the interaction Hamiltonian in (1) does not fall off with increasing distance and is thus not convenient for direct application of the close-coupling asymptotics. We therefore change the representation by making the unitary transformation[10,11]

$$\Psi(\mathbf{r},t)=\exp\left[-i\int^t d\tau\left(-\frac{\mathbf{A}(\tau)}{c}\cdot\hat{\mathbf{p}}+\frac{1}{2c^2}\mathbf{A}^2(\tau)\right)\right]\Phi(\mathbf{r},t) \tag{3}$$

in (1) to obtain

$$i\partial\Phi(\mathbf{r},t)/\partial t=[-\tfrac{1}{2}\nabla^2-Z/|\mathbf{r}-\boldsymbol{\alpha}_0(t)|]\Phi(\mathbf{r},t), \tag{4}$$

where $\boldsymbol{\alpha}_0(t)=c^{-1}\int^t d\tau\,\mathbf{A}(\tau)$.

The interaction Hamiltonian in (4) has a Coulomb asymptotic behavior which can be conveniently treated within the well-known close-coupling method,[9] extended for the strong-field radiative collisions.[8] We make a Floquet plus partial-wave expansion of $\Phi(\mathbf{r},t)$,

$$\Phi(\mathbf{r},t)=\sum_{n=-\infty}^{\infty}\sum_{lm}\exp[-iEt+in(\omega t+\delta)]r^{-1}F_{nlm}(r)Y_{lm}(\hat{\mathbf{r}}), \tag{5}$$

and substitute it in (4), equate equal coefficients of $\exp[in(\omega t+\delta)]$, and project onto $Y^*_{lm}(\hat{r})$ to obtain the Floquet representation of the radial close-coupling equations for the channel functions $F_{nlm}(r)$ in the channel $i\equiv(n,L,m)$:

$$\left[\frac{d^2}{dr^2}+k_n^2-\frac{l(l+1)}{r^2}+\frac{2Z}{(r^2+a_0^2)^{1/2}}\right]F_{nlm}(r)=-2Z\sum_{\lambda=1,p=0}^{\infty,\lambda}\sum_{l'm'}\langle Y_{lm}|V_{\lambda_p}(\theta,\phi)|Y_{l'm'}\rangle F_{n-\lambda_p,l',m'}(r), \tag{6}$$

where the multipole-coupling potentials are

$$V_{\lambda_p}(\theta,\phi)=\frac{(2\lambda-1)!!}{2^\lambda\lambda!}\binom{\lambda}{p}\left(\frac{ra_0\sin\theta}{r^2+a_0^2}\right)^\lambda\frac{1}{(r^2+a_0^2)^{1/2}}\exp(i\lambda_p\phi) \tag{7}$$

with $\lambda_p=\lambda-2p$ and $k_n=[2(E-n\omega)]^{1/2}$. The asymptotic behavior of the jth solution in the ith open channel for the scattering problem is

$$F_i^{(j)}(r)|_{r\to\infty}=(k_i)^{-1/2}[\delta_{ij}\sin(\theta_i)+K_{ij}\cos(\theta_i)],\quad i,j=1,2,\ldots,n_{op}, \tag{8}$$

where $\theta_i=k_ir-l_i\pi/2+(Z/k_i)\ln(2k_ir)+\arg\Gamma(l_i+1-iZ/k_i)$. n_{op} is the number of open channels included. $K=\{K_{ij}\}$ is the real K matrix which is related to the (complex) S matrix by $S_{ij}=[(1+iK)(1-iK)^{-1}]_{ij}$. Exponentially decreasing boundary conditions are required to be satisfied by the closed channels. We can show by an extension of the standard analysis[12] that the radiative-scattering amplitude in which the momentum of the incident electron, $\mathbf{k}_0$, is in the direction $\hat{\mathbf{k}}_0$ and the final momentum, $\mathbf{k}_N$, after exchange of N photons, is in the direction $\hat{\mathbf{k}}_N$, is related to the open-channel S-matrix elements by

$$f_{0\to N}(\hat{\mathbf{k}}_0,\hat{\mathbf{k}}_N)=f_{\rm Coulomb}(\Theta)\delta_{N,0}+\sum_{l_0m_0lm}\frac{2\pi i}{(k_0k_N)^{1/2}}i^{l_0-l}\exp[i(\sigma_{l_0}+\sigma_l)]Y^*_{l_0m_0}(\hat{\mathbf{k}}_0)Y_{lm}(\hat{\mathbf{k}}_N)[\delta_{N,0}\delta_{ll_0}\delta_{mm_0}-S^{Nlm}_{0l_0m_0}]. \tag{9}$$

σ_l are the l^{th} partial-wave Coulomb phase shifts and $f_{\rm Coulomb}(\Theta)$ is the ordinary Rutherford scattering amplitude, where Θ is the angle between the incident and the scattered directions, for $N=0$. Hence from (9) the elastic cross section modified by the field is

$$\frac{d\sigma^{(0)}}{d\Omega}(\Theta)=\frac{d\sigma_{\rm Coulomb}(\Theta)}{d\Omega}+|f^{(0)}_{\rm rad}(\Theta)|^2+2\,{\rm Re}[f^*_{\rm Coulomb}(\Theta)f^{(0)}_{\rm rad}(\Theta)], \tag{10}$$

where $f^{(0)}_{\rm rad}(\Theta)$ is the term with $N=0$ in the second expression on the right-hand side of (9). Similarly for the inelastic processes of stimulated absorption ($N<0$) and emission ($N>0$) the differential cross sections are

$$\frac{d\sigma^{(N)}}{d\Omega}(\Theta)=\frac{k_N}{k_0}|f^{(N)}_{\rm rad}(\hat{\mathbf{k}}_0,\hat{\mathbf{k}}_N)|^2, \tag{11}$$

where $f^{(N)}_{\rm rad}(\hat{\mathbf{k}}_0,\hat{\mathbf{k}}_N)$ is given by the second part of (9) for $N\neq0$. Hence, from (11) the angle-integrated total cross section of N-photon absorption or emission is given by

$$\sigma^{(N)}(\hat{\mathbf{k}}_0)=\frac{4\pi^2}{k_0^2}\sum_{lm}\left|\sum_{l_0m_0}i^{l_0}\exp(i\sigma_{l_0})S^{Nlm}_{0l_0m_0}Y^*_{l_0m_0}(\hat{\mathbf{k}}_0)\right|^2. \tag{12}$$

Figure 1 shows the radial dependence of the channel-coupling potentials as a function of increasing radial distance r for $\lambda=0$, 1, 2, and 3. The ordinate is shown in units of Z/a_0 and the abscissa in $\rho=r/a_0$, where $a_0=A_0\omega/c=F_0/\omega^2$ is the classical radius of vibration of the electron in the field of peak strength F_0. It is to be noted that for $\lambda=0$, at small r the laser field lifts the usual divergence of the Coulomb potential at $r=0$ and gives a constant value $-Z/a_0$ which decreases in magnitude with increasing field strength, F_0, and/or decreasing frequency. Higher-multipole potentials are seen to decrease both in maximum strength as well as in range; this is essential for the successful implementation of the standard close-coupling method in practice. In Fig. 2 we show the real part of the angular dependence of the multipole-coupling potential (7) for $\lambda=0$ to 3. The imaginary part has a phase shift of $\phi=\pi/2$.

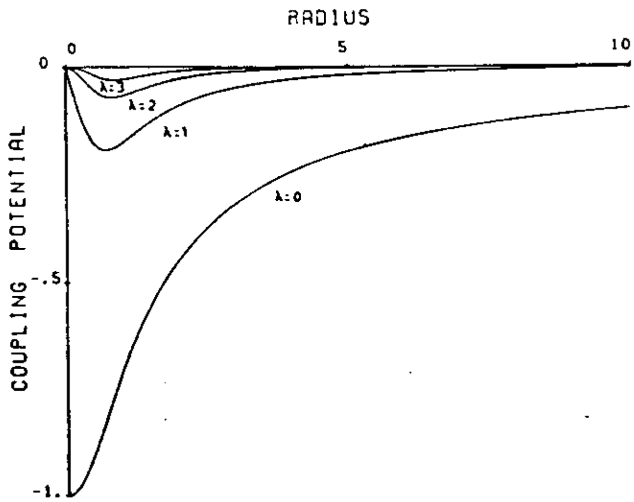

FIG. 1. Radial dependence of the channel-coupling potentials for the multiples $\lambda=0$, 1, 2, and 3. The ordinate is in units of Z/a_0 in a.u. and the abscissa is in $\rho=r/a_0$ (a.u.); $a_0=F_0/\omega^2$ (a.u.); $\phi=0$, $\theta=\pi/2$; $p=0$.

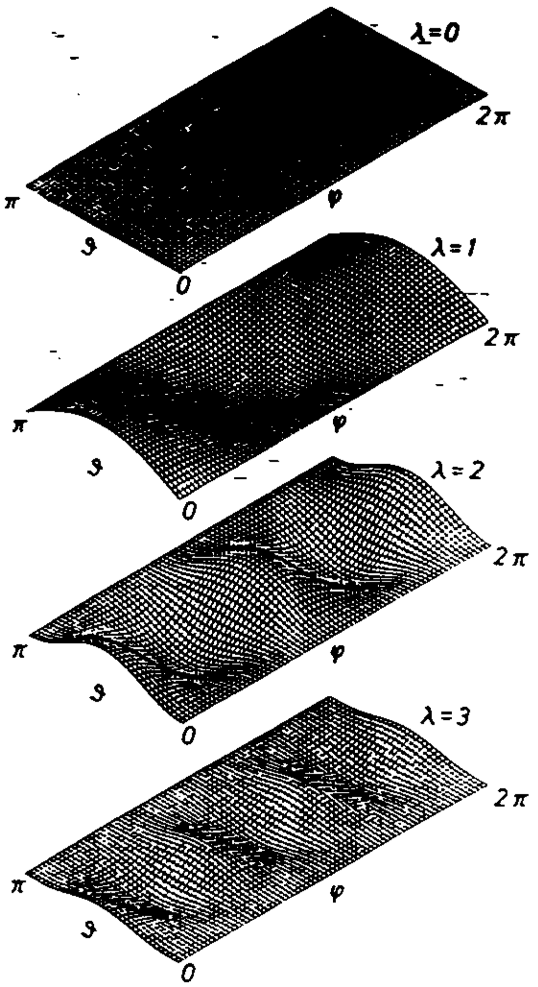

FIG. 2. The angular dependence of the real part of the channel-coupling potentials for the multiples $\lambda=0$ to 3 at a fixed radial distance $\rho \equiv r/a_0 = 1$ a.u.

We first present the result of the field-modified elastic scattering in Fig. 3, as a function of the incident electron energy. For the sake of comparison we have shown the ratio of the field-modified scattering cross section to that in the absence of the field—the latter being given by the Rutherford scattering formula. The most prominent features of this result are (a) the existence of a series of very clear resonance structures and (b) the fact that away from the resonances the field-modified elastic cross section is rather closely given by the unmodified Rutherford cross section. At a given laser frequency ($\hbar\omega = 6.419$ eV) which matches the energy difference between the incident electron energy (positive) and a Rydberg-level energy (negative) the laser field can force the electron to emit a photon and cause it to be captured temporarily in the Rydberg state until the subsequent absorption of a photon permits the electron to escape from this state into the continuum again. The delay introduced by this capture-escape episode shows up as a resonance in the scattering signal.

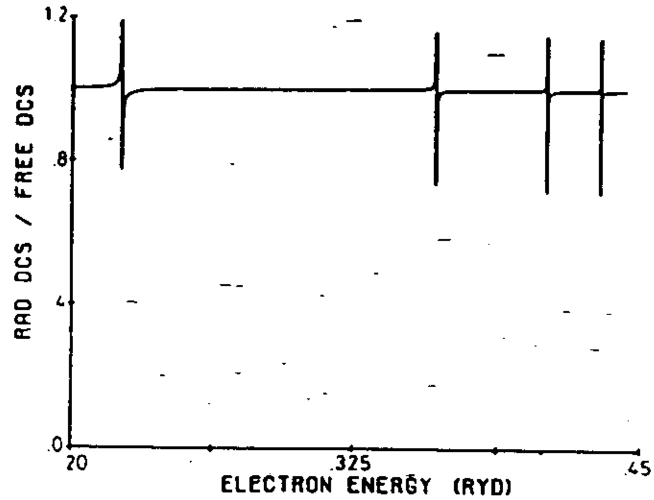

FIG. 3. The ratio of the field-modified elastic $e+H^+$ scattering cross section in a circularly polarized field of field strength $F_0=0.005$ a.u. and $\hbar\omega=0.472$ Ry $=6.419$ eV, with respect to the ordinary Rutherford cross section, as a function of incident electron energy. The incident momentum is in the direction $\Omega_0=(\Theta=90°,\Phi=0°)$ and the final momentum is in the direction $\Omega=(\Theta=90°,\Phi=90°)$. The field propagation direction is along the z axis. Note the occurrence of the capture-escape Rydberg resonances with respect to the $n=2$ to 5 states of neutral H.

Figure 3 provides the first numerical evidence of such resonances corresponding to principal quantum numbers $n=2$ to 5 of the intermediate neutral H atom. We note the interesting analogy of this resonance with the radiative $e+$atom scattering resonance which is due to the formation of a negative ion.[5,8,13–15] The numerical re-

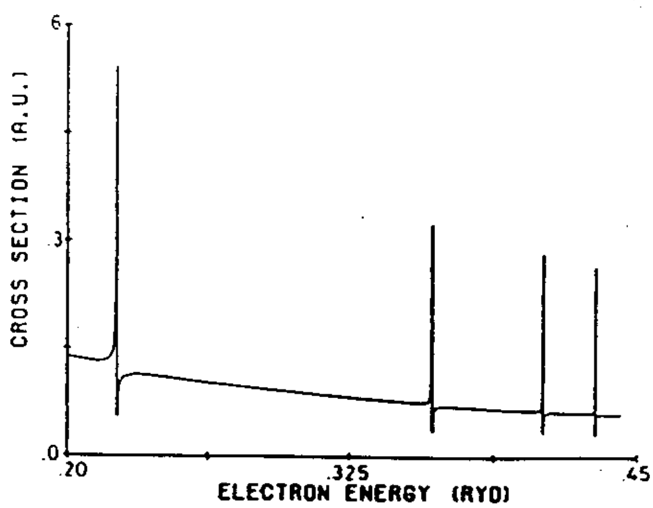

FIG. 4. The one-phonon angle-integrated total absorption cross section as a function of the incident electron energy in the presence of a circularly polarized field. Field strength $F_0=0.005$ a.u., $\hbar\omega=0.472$ Ry $=6.419$ eV. Incident electron direction $\Omega_0=(90°,0°)$. The field propagation direction is along the z axis.

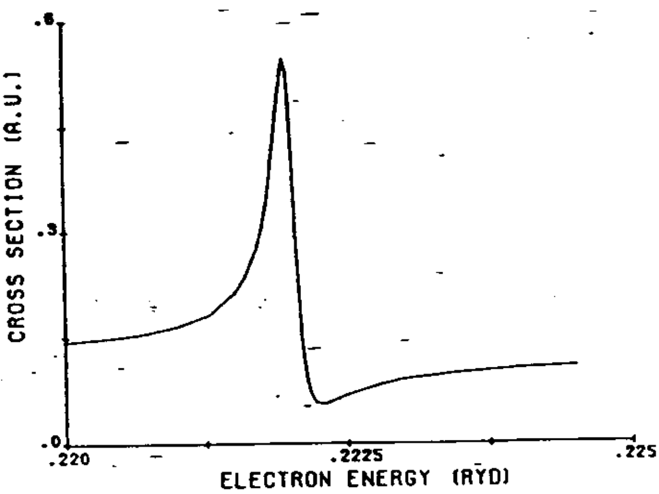

FIG. 5. Magnification of the one-phonon angle-integrated total absorption cross section for $e+H^+$ scattering in a circularly polarized field in the region of the capture-escape Rydberg resonance with the $n=2$ state of neutral H. Incident electron momentum along $\Omega_0=(90°,0°)$. Laser propagation direction along $\hat{z}$ axis. Field strength $F_0=0.005$ a.u., $\hbar\omega=0.472$ Ry $=6.419$ eV.

sults used here were found to converge rapidly within a maximum photon order $|n|=3$ and angular momentum channels up to $l=3$ for each photon order. With increasing intensity and/or decreasing frequency the number of relevant channels increases rapidly and the calculation is limited mainly by the available computer storage space.

In Fig. 4 we present the radiative inelastic (inverse bremsstrahlung) cross section for the absorption of a photon, which is also dominated by the capture-escape Rydberg resonances.

In Fig. 5 we show the $n=2$ resonance in magnification. It has a width of $\simeq 5$ meV. We note that with increasing field strength this and the other resonances for higher n tend to broaden. In particular the higher Rydberg resonances (which are weak at the given intensity) begin to appear significantly with increasing field strength. We also remark that the width of such a resonance with a given n can be thought of as a measure of the strong-field photoionization rate of that particular Rydberg state of the neutral atom.

Finally, we note that these resonances can be "tuned" either by varying the photon frequency at a fixed electron energy or by varying the electron energy at a given photon frequency. This flexibility in tuning the resonances combined with the fact that their widths can be manipulated by a change of the field intensity may prove to be useful in observing them experimentally.

This work has been supported by Deutsche Forschungsgemeinschaft under Project No. SFB-216-M2. A preliminary version of part of the results reported here has been presented at the Cortuna workshop.[16]

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ANALYTICAL ATI SPECTRUM OF AN EXACTLY SOLVABLE 3D-MODEL OF LASER-ATOM INTERACTION

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ABSTRACT

Starting from the stationary quantum Hamiltonian and using the theory of resolvent we rigorously define and analytically derive the stationary intrinsic ATI spectrum of an exactly solvable 3D-model of laser-atom interaction. The result permits us to determine the phenomena of switching, suppression and disappearence of the ATI peaks as a function of the field intensity. The dependence of the ATI spectrum on the field modified ionization-gap and the broadening of the initial bound state are investigated. The modification of these spectra due to ponderomotive acceleration is also shown. The results are discussed with graphical illustrations and their implication for ATI in general are pointed out.

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Experimental information (e.g Agostini et al. 1979, Kruit et al. 1983, Bucksbaum et al. 1986, Hippler et al. 1987) on the above-threshold-ionization (ATI) of atoms in strong laser fields are primarily obtained as energy spectra of the ejected electrons , consisting of a series of peaks spaced by the photon energy. These spectra reveal a number of dynamical phenomena like switching, suppression and disappearence of the peaks as a function of laser intensity. In order to get a physical insight into these processes, ideally one requires the knowledge of the exact solution of the Schrödinger problem. Due to their great complexity, exact solutions of the real systems are at present not possible. A number of authors (Geltman 1977, Cerjan and Kosloff 1987, Pindzola and Bottcher 1987, Javanainen and Eberly 1988, Collins and Merts 1988, Dörr and Shakeshaft 1988, Sundaram and Armstrong 1988, Kulander 1987, 1988) have therefore begun to consider numerical solutions of simple models of laser-atom interaction to investigate the energy distribution of the ATI electrons. These calculations, however, could be carried out, due to their purely numerical nature, primarily for 1D-models and generally for restricted domains of interaction time. The latter restriction tends to generate non-stationary energy spectra which often requires some adhoc procedure for selecting the cut off-time, in order to ensure certain stability to the calculated electron energy distribution. It is thus desirable to obtain the rigorously stationary spectra of exactly solvable models of ATI which would complement the short time investigations. It would be also desirable to extend the 1D-model studies to 3D-models. This letter reports on the fully stationary ATI spectrum obtained for the first time analytically from an exactly solvable 3D-model. The result permits us to determine quantitatively the nature of the phenomenon of switching, suppression and disappearence of the ATI peaks as a function of the field intensity. The change in the ionization-gap and the width of the initial bound state due to the laser interaction are also investigated. Implications of the present results for ATI in general are summarized in the end.

The exact solution of the Schrödinger problem of laser-atom interaction defined by $(e = m = \hbar = 1)$

$$i\frac{\partial}{\partial t} \mid \Psi(r) >= [\frac{1}{2}(\hat{p} - \frac{1}{c}\hat{A})^2 + \hat{v}] \mid \Psi(r) >, \tag{1}$$

where

$$\hat{v} = \sum_{j=1}^{J} \mid U_j(r)Y_{l_j m_j}(\hat{r}) >< Y_{l_j m_j}(\hat{r})V_j(r) \mid$$

a general separable potential of arbitrary rank J, and a circularly polarized field represented by the vector potential $\hat{A} = s_o[(\hat{\epsilon}_x + i\hat{\epsilon}_y)a^+ + (\hat{\epsilon}_x - i\hat{\epsilon}_y)a]$, where $s_o \equiv \left(\frac{2\pi\omega^2}{L^3 c}\right)^{\frac{1}{2}}$ and L^3 is the field normalization volume, can be obtained as shown in [Faisal,1987 a,b]. For the present purpose we restrict ourselves to the simplest case and choose the hermitian potential,

$$\hat{v} = - \mid u >< u \mid = - \mid \frac{1}{r}\phi_{1s}(\bar{r}) >< \phi_{1s}(\bar{r})\frac{1}{r} \mid \tag{2}$$

where $\phi_{1s}(\bar{r})$ is the $1s-$ wave function of the hydrogen atom. Potential (2) supports a bound-state whose energy and wave function coincide with that of the hydrogen atom in the ground state and allows for the entire free-wave continuum.

We solve the corresponding equation for the total resolvent $\hat{G}$ defined by

$$(E-H)\hat{G}=1,$$

where H is the operator on the right hand side of (1) with the potential (2), and calculate the matrix element of $\hat{G}$ between the initial and final states. As the initial state in the product space we choose

$$|\Phi_i>=|\phi_i>|0>$$

with $\phi_i=\phi_{1s}$, and no photon is emitted or absorbed. The (unperturbed) initial total energy is $E_i=-|\epsilon_{1s}|+0\omega$.

The final state is described by the dressed state corresponding to the quantum Volkov-state (see e.g. Faisal 1987 c)

$$|\Phi_f>=e^{i\overline{K}_f\cdot\overline{r}}\sum_n J_{n-N_f}(K_f\alpha_o sin\vartheta_f)e^{in\varphi_f}\ |n>.$$

The energy of this state is $E_f=\epsilon_f+N_f\omega+\frac{A_o^2}{2c^2}$, $\epsilon_f=\frac{K_f^2}{2}$ (we use the convention $N_f<0$ for absorption and $N_f>0$ for emission).

The desired matrix element of the total resolvent is then given by

$$<\Phi_f|\hat{G}|\Phi_i>=\frac{1}{E-E_f+i0}\ A^{(N_f)}(E)$$

where the amplitude of absorption of N_f photons is

$$A^{(N_f)}(E)=(2\pi)^{\frac{3}{2}}J_{-N_f}(K_f\alpha_o sin\vartheta_f)[\tilde{\phi}_i(\overline{K}_f)-\tilde{u}(\overline{K}_f)\cdot C_{1s}(E)]. \tag{3}$$

$\tilde{\phi}_i$ and $\tilde{u}$ are the Fourier transforms of ϕ_{1s} and of $u=|\frac{1}{r}\phi_{1s}>$, respectively. The coefficient $C_{1s}(E)$ is given by the formula

$$C_{1s}(E)=\frac{<u|G_{oo}^o|\phi_i>}{1+<u|G_{oo}^o|u>}, \tag{4}$$

where $G_{oo}^o(\overline{r},\overline{r}')$ is the unperturbed Floquet Green's function (Faisal 1987 c).

The stationary probability-density spectrum $\frac{dP^{(N_f)}(\epsilon_f)}{d\epsilon_f}$ for bound-free transitions involving the absorption of N_f photon is obtained rigorously from the square of the amplitude $A^{(N_f)}(E=E_f)$ in eq. (3):

$$\frac{dP^{(N_f)}(\epsilon_f)}{d\epsilon_f}=\int d\hat{K}_f\,|A^{(N_f)}(E_f)|^2\,\rho(\epsilon_f)$$

$$=\frac{2\pi}{2|N_f|+1}\cdot\frac{(K_f\alpha_o)^{2|N_f|}}{(2|N_f|)!}\ {}_1F_2\left[{}_{2|N_f|+1,\ |N_f|+\frac{3}{2}}^{|N_f|+\frac{1}{2}}\ \middle|\ -K_f^2\alpha_o^2\right]\times$$

$$\times \mid \tilde{F}_i(K_f) - \tilde{F}_u(K_f) \cdot C_{1s}(E) \mid^2 \cdot K_f, \tag{5}$$

where $\tilde{F}_i$ and $\tilde{F}_u$ are the radial parts of $\tilde{\phi}_i$ and $\tilde{u}$, respectively.

The full electron energy spectrum $S(\epsilon_f)$ is therefore given by the sum of(5) over all channels N_f:

$$S(\epsilon_f) \equiv \sum_{-N_f=|N_o|}^{\infty} \frac{dP^{(N_f)}(\epsilon_f)}{d\epsilon_f} \tag{6}$$

starting from $\mid N_o \mid$, the minimum number of photons required to overcome the binding energy of the atom in the field.

The coefficient $C_{1s}(E)$ is evaluated analytically. We have:

$$< u \mid G^o_{oo} \mid u >= -4 \sum_{N=-\infty}^{+\infty} \frac{(-\alpha_o^2)^{|N|}}{(2 \mid N \mid)!} B_N, \tag{7}$$

and

$$< u \mid G^o_{oo} \mid \phi_i >= -4 \sum_{N=-\infty}^{+\infty} \frac{(-\alpha_o^2)^{|N|}}{(2 \mid N \mid)!} \times$$

$$\times \frac{1}{1+K_N^2} \left\{ (\mid N \mid - \frac{1}{2}) \; {}_1F_2 \left[{}^{|N|+\frac{1}{2}}_{|N|-\frac{1}{2},\; 2|N|+1} \mid \alpha_o^2 \right] + 2B_N \right\}. \tag{8}$$

In these expressions,

$$B_N = \frac{1}{(1+K_N^2)^2} \cdot \left\{ -(1+K_N^2) \; {}_oF_1 \left[{}_{2|\bar{N}|+1} \mid \alpha_o^2 \right] \right.$$

$$+ \frac{1}{\mid N \mid + \frac{1}{2}} \; {}_1F_2 \left[{}^{|N|+\frac{1}{2}}_{|N|+\frac{3}{2},\; 2|N|+1} \mid \alpha_o^2 \right]$$

$$\left. + \frac{i}{\mid N \mid + \frac{1}{2}} (-1)^{|N|} K_N^{2|N|+1} \; {}_1F_2 \left[{}^{|N|+\frac{1}{2}}_{|N|+\frac{3}{2},\; 2|N|+1} \mid -K_N^2 \alpha_o^2 \right] \right\}, \tag{9}$$

$K_N = \sqrt{2(E - N\omega - \frac{Ao^2}{2c^2})}$ is the wave number in the Nth channel, and ${}_oF_1$ and ${}_1F_2$ are generalized hypergeometric functions (e.g.. Erdélyi,1953).

Expressions (3) - (9) give for the first time an exact analytic solution of the problem of ATI-spectrum for a 3D model system.

Figure 1 shows the ATI spectra for four different intensities corresponding to the frequency of the well-known high power excimer laser (e.g. Rhodes, 1985) $\omega = 6.419eV(\lambda = 193nm)$. Note that for convenience of presentation the scale of the spectrum at the top is chosen to be different from that of the remaining ones. The considered range of kinetic energy of the ATI-electrons is shown divided into intervals or *zones* of ω, labeled by the zone index S, starting with S=0 for the zone between 0 and ω. The ATI peaks can be thus identified by S, according to the zone in which they lie. It is, however, more informative to identify them also with respect to the absolute number of absorbed photons, $(-N_f)$. The minimum value of $(-N_f)$ is of course the minimum number of photons required to overcome the *perturbed* ionization-gap.

The spectrum at the top of Figure 1 corresponds to the intensity $I = 1.0 \times 10^{15} W/cm^2$ and shows a regular behaviour , i.e., the successive ATI-peak heights decrease monotonically. This is expected from the usual perturbation theoretical intuition. Note the arrow on the energy scale which is set at the *unperturbed* energy value of the first peak. This would be the position of the S=0 peak by absorption of the minimum number of photons $(-N_f = 3)$ if the initial bound state were unperturbed. The actual S=0 peak at $I = 1.0 \times 10^{15} W/cm^2$ is clearly displaced to the red from the arrow. The red-shifts of all the peak positions in Figure 1 are due physically to the increase of the ionization-gap in the field. This energy-gap can be determined from any of the ATI-peaks in the spectra of Figure 1 using the relation:

ionization-gap $= \mid N_f \mid \omega$ $-$ position of the $(-N_f)$th peak.

We now consider the non-perturbative phenomenon of peak-switching, in which, for example, a peak in a given zone becomes shorter than that in the next one. This can be seen in Figure 1 in the second spectrum, which is obtained at $I = 1.5 \times 10^{15} W/cm^2$. On the next lower spectrum, corresponding to $I = 2.5 \times 10^{15} W/cm^2$, the first peak (S=0,$-N_f = 3$) is seen to be strongly *suppressed*. However, it has not yet completely disappeared, as can be confirmed formally by noting that the position of the $(-N_f = 4)$ peak can still be found in the S=1 zone. Since the energy difference between neighbouring peaks must be ω (a consequence of the quantum nature of the photons), the $(-N_f = 3)$ peak at this intensity, greatly suppressed as it is, must still lie in the S=0 zone. The disappearence of this peak below the positive energy threshold occurs in fact at a still higher intensity; in the present case at $I_d = 2.9 \times 10^{15} W/cm^2$. The last spectrum in this figure corresponds to $I = 3.5 \times 10^{15} W/cm^2$ and the observed peak in the S=0 zone arises from the red-shift of the peak $(-N_f = 4)$, which could be seen in the S=1 zone at intensities lower than I_d. The lowest energy peak S=0 in this spectrum therefore can be designated unambiguously as the $(-N_f = 4)$ peak.

It is important to note that these spectra are generated *inside* the field, assumed to be of constant amplitude in the region of photon-atom interaction. The change of such an 'intrinsic' spectrum at the boundary of the macroscopically extended field may be obtained by translating the spectrum along the energy axis by an amount equal to the kinetic energy due to the ponderomotive acceleration experienced by the outgoing electron. For sufficiently long pulses this is given by $U_p = \frac{8\pi}{c} \cdot \frac{I}{\omega^2}$ where I is the peak intensity of the pulse. This is consistent with the argument originally made by Muller et al.(1983). The 'extrinsic' spectra thus obtained from those of Figure 1 are shown in Figure 2. Note that in this (extrinsic) representation, the ATI peaks appear as blue shifted with respect to the expected unperturbed positions, while in fact they are red-shifted in the "intrinsic" representation. In the case of pulses shorter than the passage time of the ATI electron through the field gradient, the kinetic energy of ponderomotive acceleration must be smaller than the maximum U_p. This would cause the peak-positions of the observed (extrinsic) spectra to depend on the pulse lengths. Such dependence has in fact been observed in a number of experiments using short pulses (Luk et al., 1987; Agostini et al., 1987; Freeman et al., 1987; Muller et al., 1988). It is thus clear that the observed (extrinsic) ATI spectra require to be deconvoluted with respect to the macroscopic effect of the ponderomotive acceleration, before the intrinsic positions of the ATI peaks, which are directly determined by the atom-field interaction dynamics,can be ascertained.

In Figure 3 we show the actual dependence of the *increase* of the ionization-gap , which is an intrinsic quantity, on the field intensity. The zero on the ordinate corresponds to the value of the unperturbed ionization-gap, $| \epsilon_{1s} |$. The exactly calculated intensity dependence (continuous line) is highly accurately reproducible by a linear fit with a slope $m = 1.98[eV/(10^{15}W/cm^2)]$. This behaviour is similar to the linear dependence of the well-known "quiver energy", $\delta_\epsilon = \frac{A_0^2}{2c^2}$, on the intensity. Despite this similarity, the exact result (for the entire range of intensity considered) is found to be *not* equal to but less than the quiver-energy (dotted line). This difference between the quiver-energy and the actual increase of the ionization-gap may be attributed to the shift of the initial bound state. We should note that conceptually the least ambiguous quantity is the ionization-gap, rather than the shift of the bound state and the so-called "continuum-shift". The linear dependence on the intensity of the exact result in Figure 3 strongly suggests that in the intensity range considered (and at $\omega = 6.419$ eV) the change of the ionization-gap and hence the change in the positions of the ATI-peaks, follow the lowest order perturbative (LOPT) behaviour in intensity (in this case the usual second order AC-stark shift). The actual heights and line shape of ATI however, behave, as seen above, highly non-perturbatively. This implies that there could exist intensity domains in which one aspect of the ATI-spectrum exhibits LOPT-behaviour (e.g. peak positions), while another aspect behaves highly non-perturbatively (e.g. peak-heights). It would be interesting to test this conjecture further in future.

Our calculations show that the FWHM of the ATI-peaks in Figure 1 are equal. This suggests that for non-resonant ATI processes the peak widths are essentially due to the field induced broadening of the initial bound state. The actual dependence of the line width on the laser intensity is shown in Figure 4. This dependence is clearly non-linear in intensity. The initial rise of the curve is found to be proportional to I^λ, where $\lambda \simeq 2$. Since the width of the bound state is a measure of the total rate of ionization, this rate in the present case does *not* follow the lowest order perturbation theory, which would predict $\lambda = 3$.

Finally, the calculated ATI-spectrum is found to be practically free from a background so long as it has a 'regular' or 'perturbative-like' behaviour (see e.g. the top spectrum in Figure 1). At higher intensities the ATI-spectra clearly develop a non-zero background between the peaks.

We conclude this report by briefly indicating the implication of the present results for ATI in general

(a) The fundamental characteristics of ATI-spectra (e.g. the switching, suppression , disappearence, shift and broadening of the ATI peaks), can be understood qualitatively invoking the single-electron hypothesis of laser-atom interaction.

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Figure Captions :

Fig. 1. The "intrinsic" ATI-spectra for four different intensities at $\omega = 6.419eV(\lambda = 193nm)$. From the top: $I = 1.0\times10^{15}W/cm^2$ the spectrum is *regular*, i.e. the peak heigths decrease with increasing energy. The arrow on the energy scale indicates the position of the first unperturbed peak. Note that the actual peak positions in this and subsequent spectra are displaced to the *red*. $I = 1.5\times10^{15}W/cm^2$ exhibits 'peak-switching', i.e. the peak in the first zone (S = 0) is smaller than the peak in the next zone. $I = 2.5\times10^{15}W/cm^2$ exhibits 'peak-suppression', i.e. the peak in the first zone (S = 0) is reduced to such an extent that the first *visible* peak lies in the next zone (S = 1). $I = 3.5\times10^{15}W/cm^2$ shows that the observed peak in the S = 0 zone arises from the red-shift of the peak which was in the S = 1 zone at lower intensities.

Fig. 2. The "extrinsic" ATI-spectra for the same intensities and frequency as in Figure 1. They are obtained by taking into account the effect of the full ponderomotive acceleration experienced by the outgoing electron. Note that the ATI peaks now appear as *blue* shifted with respect to the unperturbed positions.

Fig. 3. Dependence of the *increase* of the ionization-gap on the field intensity (continuous line). The zero on the ordinate corresponds to the value of the unperturbed ionization-gap $|\epsilon_{1s}|$. This result can be accurately fitted to a straight line with a slope $m = 1.98[eV/(10^{15}W/cm^2)]$. Dependence of the "quiver"-energy on the intensity (dotted line) is also shown for comparison.

Fig. 4. Dependence of the line-width on the field intensity. The initial rise of the curve is proportional to I^λ, with $\lambda \cong 2$. Note that in this case LOPT would predict $\lambda = 3$.

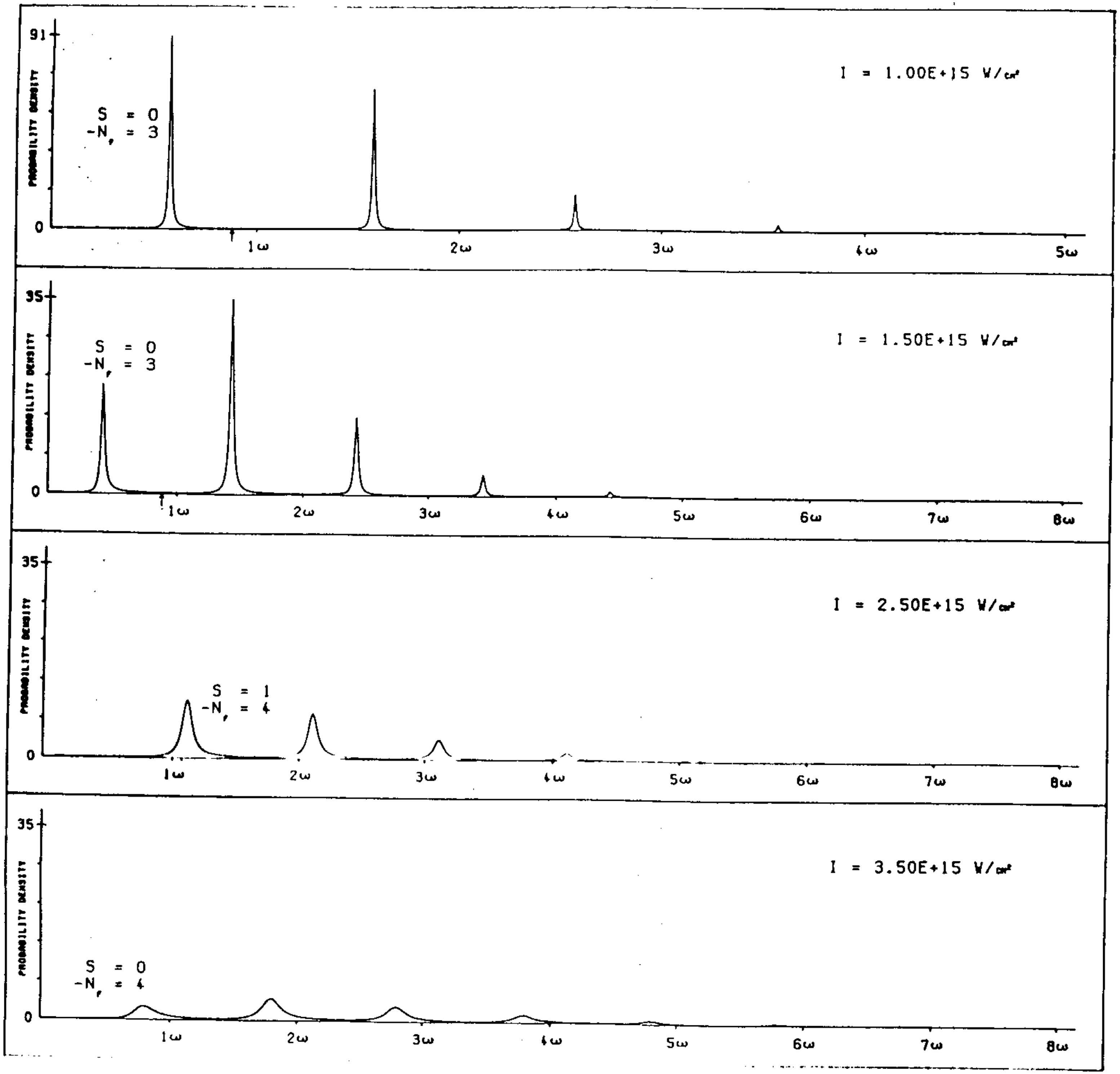
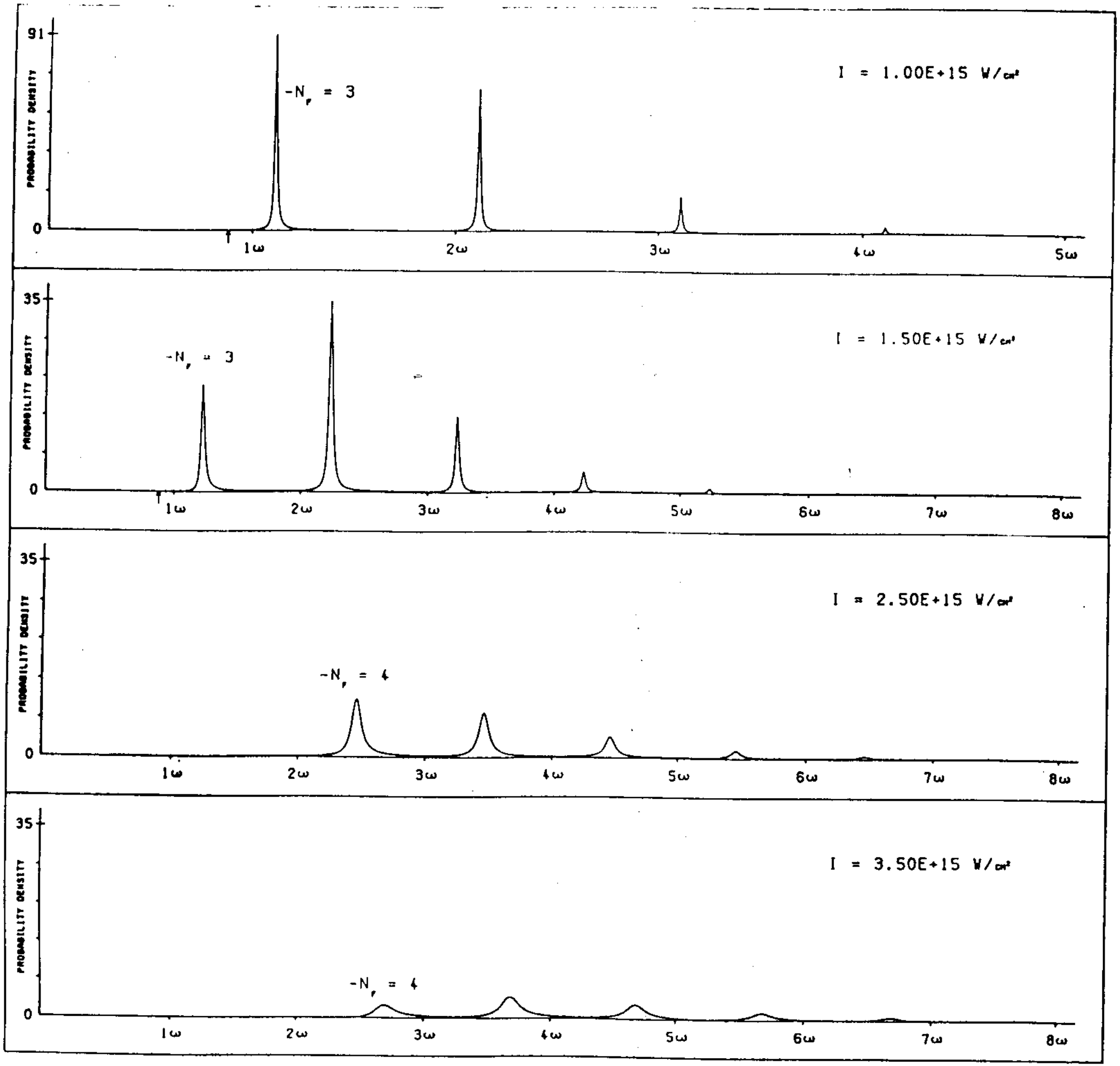
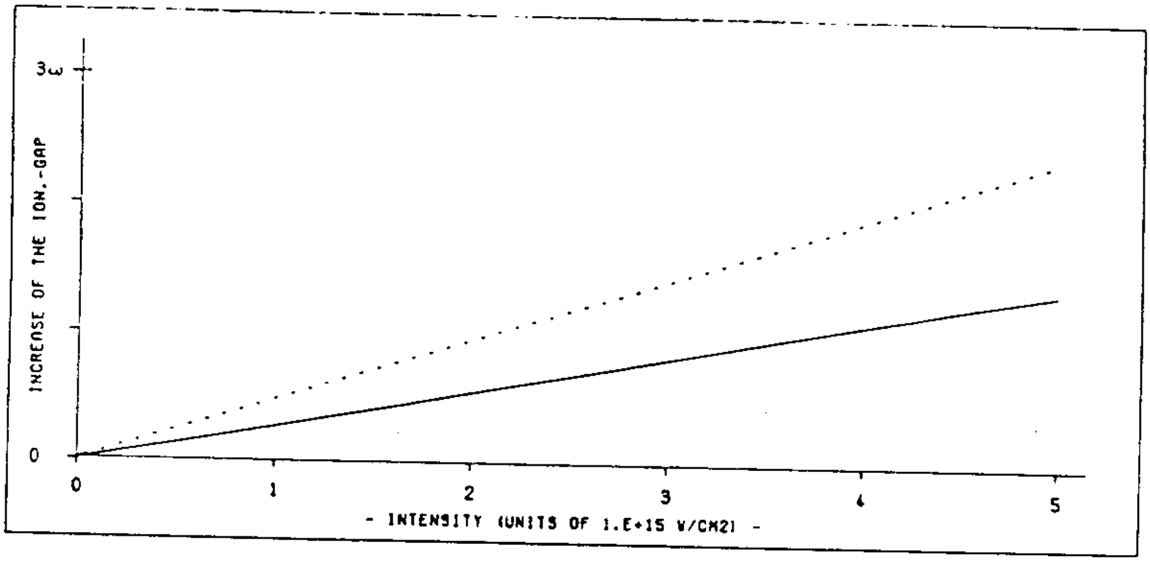

91
PROBABILITY DENSITY
0
S = 0
-N, = 3
I = 1.00E+15 W/cm²
1ω
2ω
3ω
4ω
5ω
35
PROBABILITY DENSITY
0
S = 0
-N, = 3
I = 1.50E+15 W/cm²
1ω
2ω
3ω
4ω
5ω
6ω
7ω
8ω
35
PROBABILITY DENSITY
0
S = 1
-N, = 4
I = 2.50E+15 W/cm²
1ω
2ω
3ω
4ω
5ω
6ω
7ω
8ω
35
PROBABILITY DENSITY
0
S = 0
-N, = 4
I = 3.50E+15 W/cm²
1ω
2ω
3ω
4ω
5ω
6ω
7ω
8ω

Fig. 1



I = 1.00E+15 W/cm²
-N_r = 3
I = 1.50E+15 W/cm²
-N_r = 3
I = 2.50E+15 W/cm²
-N_r = 4
I = 3.50E+15 W/cm²
-N_r = 4
PROBABILITY DENSITY
91
35
0
1ω
2ω
3ω
4ω
5ω
6ω
7ω
8ω

Fig. 2



3ω
0
INCREASE OF THE ION.-GAP
0
1
2
3
4
5
- INTENSITY (UNITS OF 1.E+15 W/CM2) -

Fig. 3

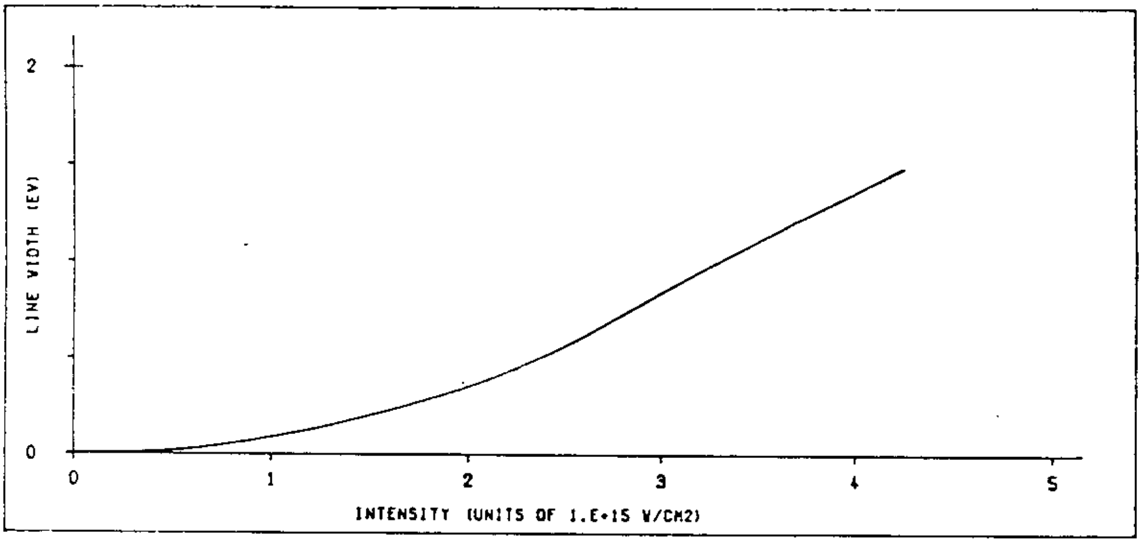


Fig. 4

Summary

We have shown that a general connection exists between different radiative scattering processes involving the motion of the electron in the continuum in a laser field. The origin of the most dominant structures in the radiative scattering cross sections and in the ATI spectra can be understood qualitatively in terms of the following two propensity rules:

1. Every singularity of the S-matrix of the unperturbed system (e.g. eigen energy, field free scattering resonances, threshold singularities or cusps etc.) in the absence of the field, tends to be *replicated* in the continuum in the presence of the field, at an interval of energy $N\omega, (N = 0, \pm 1, \pm 2, ...)$.
2. The 'replicas' in the continuum tend to appear significantly at field intensities, I, for which the quiver energy $\delta_{\epsilon} = \frac{e^2}{m} 2\pi I/\omega^2$, approaches or exceeds the energy of the photon, $\hbar\omega$ i.e. for $\delta_{\epsilon} \geq \hbar\omega$.

More specifically it is shown that physical channels which are not energetically available to ordinary electron scattering can influence the scattering cross sections dramatically in the presence of the field; typical expressions of such influences are the new resonances such as the capture-escape resonances and resonant (and also non-resonant) sub-threshold excilation processes.

We have seen how the presence of the laser field which breaks the axial symmetry of the e-atom scattering process in the absence of the field, causes modification in the angular distribution of the scattered electrons. Modification of the angular distribution can also arise due to differences in polarization of the field. Besides these "geometrical" modifications, the angular distributions are seen to be altered, by the dynamic effect of field intensity.

For a suitable description of half-scattering processes the cross-sections (or rates) are found to be inadequate parameters; they are more appropriately described by the probability-spectrum, i.e. the probability of finding the ionized (detached) electron anywhere in the continuum within a small interval of energy (between E and E + dE).

The characteristic properties of ATI spectrum, for non-resonant cases (no intermediate resonance) can be essentially understood as the dynamical effect of field intensity on the "ionization gap" and on the width of the initial-bound state. The resonant ATI process is further controlled by the splitting, shift and width of the intermediate resonances.

Finally, radiative scattering and half-scattering processes can alter the properties of the radiation field e.g. at selected incident electron energies, the incident photon field can be amplified by the scattering electrons.

Acknowledgements

Results of theoretical calculations discussed here are obtained in collaboration with Dr. L. Dimou (radiative electron scattering) and Dr. P. Scanzano and Dr. J. Zaremba (above threshold ionization).

Multiphoton ionization of atomic hydrogen in intense sub-ps-laser pulses.

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PACS 32.80 K

We report results of experiments with laserpulses of about 0.6 ns duration and wavelengths around 600 nm and intensities up to 10^{14} Wcm^{-2}.

Introduction:

When short laser pulses are used for multiphoton-ionization (MPI) of atoms two new effects are observed: 1. The energies of the detected photoelectrons are shifted towards values lower than those expected from simplest energy considerations [1, 2, 3]. This effect is reasonably well understood. 2. The "above threshold ionization" (ATI) peaks belonging to different numbers of photons absorbed in the continuum show a sharp substructure when pulses of about 0.5 ps duration are used [2]. This substructure was interpreted by Freeman et al. [2] by assuming that high lying excited states of the atom are shifted into resonance in the intense field, thus providing very efficient ionization pathways at well defined intensities. In a first approximation it was assumed that all high lying atomic states and the ionization limit shift by the same amount, given by the quiver energy of a free electron in the intense laserfield. This effect was first observed in experiments with xenon atoms [2]. The resonant states were identified on the basis of the peak energies in the photoelectron spectrum.

We have performed MPI experiments with atomic hydrogen which provide a simpler test case for comparison with theory. In addition to electron energy spectra we have measured angular distributions of the photoelectrons.
Figure 1 shows an energy diagram for atomic hydrogen with the relevant states and their intensity dependent shifts. For simplification it is assumed that all excited states shift by the same amount. This is probably not true for n = 2,3 and n = 4. The shift of the ground state is neglected. Also shown are the energies of the ground state dressed by N photons. N ranging from 5 to 9. Resonant enhancement of MPI should occur at intensities where a shifted bound state "crosses" one of these (1s + Nħω) states. As an example this situation is shown along the arrows for the n = 4 state and (1s + 7ħω). The lowest energy of photoelectrons from this resonant ionization is also shown. At wavelengths around 608 nm we have the special situation that (1s + 5ħω) and (1s + 6ħω) are nearly degenerate (resonant) with the (n = 2) and (n = 3) states at low intensities respectively. This should lead to efficient resonant ionization at relatively low intensities. Using calculated cross sections [4, 5] for this wavelength a saturation intensity of less than $5 \cdot 10^{12}$ Wcm^{-2} may be expected. As we find no signal which can be attributed to such a simple 5- and 6-photon resonant 7-photon ionization process we shall concentrate here on the higher n-states which are shifted into resonance with (1s + 7ħω) at higher intensities of up to $4 \cdot 10^{13}$ Wcm^{-2}.

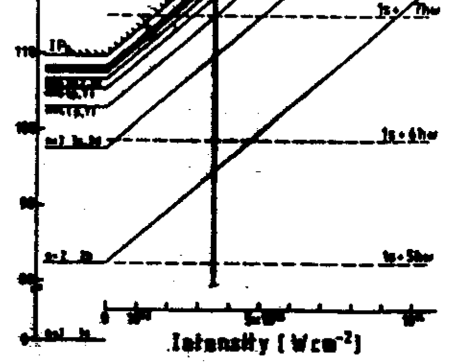

Figure 1: Simplified diagram of energy levels and their intensity dependent shifts.

(b) ATI, considered as a half-collision process, satifys the general *propensity rule* (Faisal, 1982) formulated in connection with radiative electron-atom collision processes. According to this rule all singularities of the S-matrix of the system in the absence of the field (in this case, the initial bound state) *tend* to be *replicated* in the continuum in the presence of the field at an interval of the photon energy.

(c) Significant background to ATI peaks can develop at intensities for which the widths of the bound state becomes $\gtrsim \frac{\omega}{2}$.

Details of the mathematical derivations as well as results of investigations of resonant ATI-processes will be presented elsewhere (Faisal et al., 1988).