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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

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COLLEGE ON ATOMIC AND MOLECULAR PHYSICS:
PHOTON ASSISTED COLLISIONS IN ATOMS AND MOLECULES

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COLLISION DYNAMICS OF COLD COLLISIONS

J. WEINER

University of Maryland
Department of Chemistry
Maryland, U.S.A.

Collision Dynamics of Cold Collisions

1. *Review of Absorption & Emission of Radiation*

2. *Interaction of Radiation and Matter*

3. *Review of Classical & Quantum Collision Theory*

4. *Cooling and Trapping*

Why are cold-atom collisions interesting?

1. Nuclear motion quantal at long range e.g. deBroglie wavelength 400 a.u for Na @ 1mK

2. Doppler width very narrow:
   kT=20 MHz @ 1mK
   kT=2 GHz @ 300K: precise free-bound spectroscopy possible.

3. Long-range potentials and couplings control collision dynamics--orientation of colliding partners very important.

4. Light field directly modifies collision dynamics--potential curves become dressed states

Properties of the Interaction of Radiation & Matter

A Classical Relations

In a bulk dielectric medium the energy density w is related to the polarization $\vec{P}$ of the medium and an external field $\vec{E}_0$ by

$$w = -\frac{1}{2}\vec{P}\cdot\vec{E}(\vec{r},t)$$

The force on the dielectric medium is given by

$$\vec{F} = -\vec{\nabla} w = +\frac{1}{2}\vec{P}\cdot\vec{\nabla}\cdot\vec{E}_0 \cos(\omega t + \phi(r))$$

$$\vec{F} = +\frac{1}{2}\vec{P}\cdot\left[\nabla E_0 \cos(\omega t + \phi(r)) - E_0 \nabla\phi \sin(\omega t + \phi(r))\right]$$

Now, $\vec{P}(t)$, the polarization can be expressed in terms of the susceptibility of the medium and the field

$$\vec{P}(t) = Re(\epsilon_0 \chi E_0 e^{i\omega t}) \;;\; \chi = \chi' - i\chi'' \text{ is the complex susceptibility}$$

$$\vec{P}(t) = Re\left[\epsilon_0(\chi' - i\chi'')E_0(\cos\omega t + i\sin\omega t)\right]$$

$$\vec{P}(t) = \underbrace{\epsilon_0 E_0 \chi' \cos\omega t}_{\text{in phase with } \vec{E}(r,t)} + \underbrace{\epsilon_0 E_0 \chi'' \sin\omega t}_{\text{in quadrature with } \vec{E}(r,t)}$$

Now $\vec{F} = +\frac{1}{2}\epsilon_0 E_0\left[\chi'\cos\omega t + \chi''\sin\omega t\right]\left[\nabla E_0 \cos(\omega t) - E_0\nabla\phi \sin\omega t\right]$

note that $\phi(r) \equiv 0$ although $\nabla\phi \neq 0$ necessarily.

Now we want to average the force over a period in the field oscillation: The cross terms vanish and we note that $\int_0^{\pi}\cos^2(x)\,dx = \int_0^{\pi}\sin^2(x)\,dx = \pi$

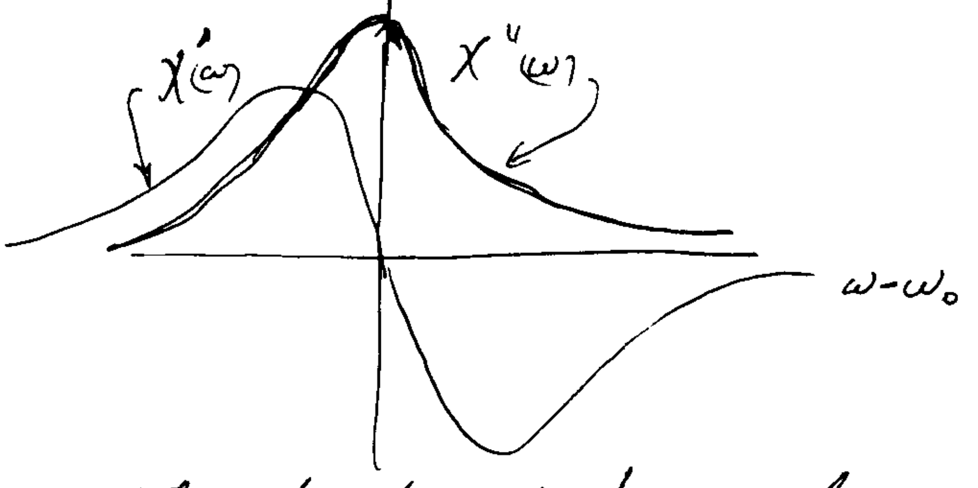
Note also that $\frac{1}{T}\int_0^T \cos^2\omega_L t = \frac{1}{2}$ because $\frac{1}{T} = \nu$ & $\omega = 2\pi\nu$

so: $\vec{F} = +\frac{1}{4}\epsilon_0 E_0\left[\chi'\vec{\nabla}E_0 - \chi''E_0\vec{\nabla}\phi\right]$

term proportional to gradient of the field — term proportional to gradient of the phase

Now χ' & χ'' are frequency dependent. If the dielectric medium has a resonance at ω_0

$$\chi'(\omega) \sim \frac{\omega_0-\omega}{(\omega-\omega_0)^2+\frac{\Gamma^2}{4}+\frac{\Omega_1^2}{2}} \; ; \; \chi''(\omega) \sim \frac{-\Omega_1^2/2}{(\omega-\omega_0)^2+\frac{\Gamma^2}{4}+\frac{\Omega_1^2}{2}}$$



Note that the gradient force changes sign through the resonance. $\vec{F}$ & $\vec{\nabla}E$ are in the same direction (strong-field seeking) when $(\omega-\omega_0) < 0$ (red detuning). But force is in opposite direction on blue side of the resonance.

The first term is called the <u>gradient force</u>. Note that the gradient force is <u>zero</u> for a plane wave.
The second term is called the <u>dissipative force</u> because it has a line shape identical to absorption.

Suppose we take a plane wave $E(r,t) = \mathcal{E}_0 e^{i(\omega t - kz)}$

$\vec{\nabla}\phi = -\vec{k}$ so

$$\vec{F} = +\frac{1}{4}\epsilon_0 \mathcal{E}_0^2 \chi'' k = +\frac{1}{4}\epsilon_0 \mathcal{E}_0^2 k \frac{\Omega_1^2/2}{(\omega-\omega_0)^2 + \frac{\Gamma^2}{4} + \frac{\Omega_1^2}{2}}$$

$\Omega_1 = -\frac{N\langle\mu\rangle \mathcal{E}_0}{\hbar}$ = a constant for dipole moment $\langle\mu\rangle$ and fixed field

Significance of Ω_1 discussed later.
Note that the <u>gradient force</u> does <u>not</u> change sign and that $\vec{F}$ and $\vec{k}$ (propagation vector) are in the same direction.

Now consider interaction of radiation & matter at the <u>atomic</u> level.

Force on an atom is given by, $\vec{F} = \sum_i \langle\mu_i\rangle \vec{\nabla}\vec{E}(r,t)$
where $\langle\mu_i\rangle$ is the time average dipole induced in the atom by the field

Now it can be shown by analysis of the Optical Bloch equation that

$$\langle\mu_i\rangle = 2\vec{\mu}\,[u_{st}\cos\omega_L t - v_{st}\sin\omega_L t]$$

where $u_{st} = \frac{\delta}{\Omega_1}\left(\frac{s}{1+s}\right)$; $v_{st} = \frac{\Gamma}{2\Omega_1}\left(\frac{s}{1+s}\right)$

$s = \frac{\Omega_1^2/2}{\delta^2 + \Gamma^2/4}$; $\Omega_1 = -\frac{\vec{\mu}\,\mathcal{E}_0}{\hbar}$; $\vec{E}(r,t) = \vec{\mathcal{E}}_0\cos(\omega t + \phi)$, then $F_i = 2\vec{\mu}\left[\frac{\delta}{\Omega_1}\left(\frac{s}{1+s}\right)\cos\omega t - \frac{\Gamma}{2\Omega_1}\left(\frac{s}{1+s}\right)\sin\omega t\right] \times$

$\nabla_i \mathcal{E}_0^i \cos(\omega t + \phi)\Big]$

s is called the saturation parameter $= \frac{\Omega_1^2/2}{\delta^2 + \Gamma^2/4}$; $\delta = \omega - \omega_0$

Ω_1 is called the Rabi frequency

$\Gamma = \frac{1}{\tau}$ is the inverse of the radiative lifetime of the atom

$$F_i = 2\vec{\mu}\left(\frac{s}{1+s}\right)\frac{1}{\Omega_1}\left\{\delta\cos\omega t - \frac{\Gamma}{2}\sin\omega t\right]\left[\nabla\mathcal{E}\cos\omega t - \mathcal{E}_0\sin\omega t\nabla\phi\right]$$

where we take $\phi(\vec{r}) = 0$: average over an optical periode for the $\cos^2\omega t$ & $\sin^2\omega t$ terms

$$\langle F_i\rangle = \vec{\mu}\left(\frac{s}{1+s}\right)\frac{1}{\Omega_1}\Big[\underbrace{\delta\nabla\mathcal{E}}_{\text{gradient field force}} + \underbrace{\frac{\Gamma\mathcal{E}_0}{2}\nabla\phi}_{\text{gradient phase force}}\Big]$$

Note first term, gradient field force, changes sign just as the weak susceptibility did

phase force for a plane wave $\mathcal{E}_0 e^{i(\omega t - kz)}$

$\nabla\phi = -k$

$\langle F\rangle$ for the gradient of the phase $= \vec{\mu}\left(\frac{s}{1+s}\right)\frac{1}{\Omega_1}\frac{\Gamma\vec{\mathcal{E}}_0}{2}\vec{k}$

note that $\Omega_1 = \frac{-\vec{\mu}\cdot\vec{\mathcal{E}}_0}{\hbar}$

$$\boxed{\langle F\rangle_{phase} = \hbar\frac{\Gamma}{2}\left(\frac{s}{1+s}\right)\vec{k}}$$

Collisions

Review of Classical Collision Theory

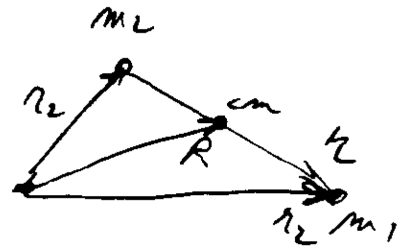
$m_2 \qquad m_1 \qquad \vec{F}_1 = -\vec{F}_2$

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_1$$
$$m_2 \ddot{\vec{r}}_2 = \vec{F}_2$$
} Newton's 3rd Law

Introduce a coordinate change

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$



Inverse Transformation

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2} \vec{r}$$

Now
$$m_2 m_1 \ddot{\vec{r}}_1 = m_2 \vec{F}_1$$
$$m_1 m_2 \ddot{\vec{r}}_2 = m_1 F_2$$

$$\ddot{\vec{r}} (m_1 m_2) = \vec{F}_1 (m_1 + m_2)$$

$$\vec{F}_1 = \left(\frac{m_1 m_2}{m_1 + m_2} \right) \ddot{\vec{r}} \quad ; \quad \text{Define } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\underline{\mu \ddot{\vec{r}} = \vec{F}}$$ Equation of motion for relative vector + reduced mass.

Now the relative particle is subject to a radial potential

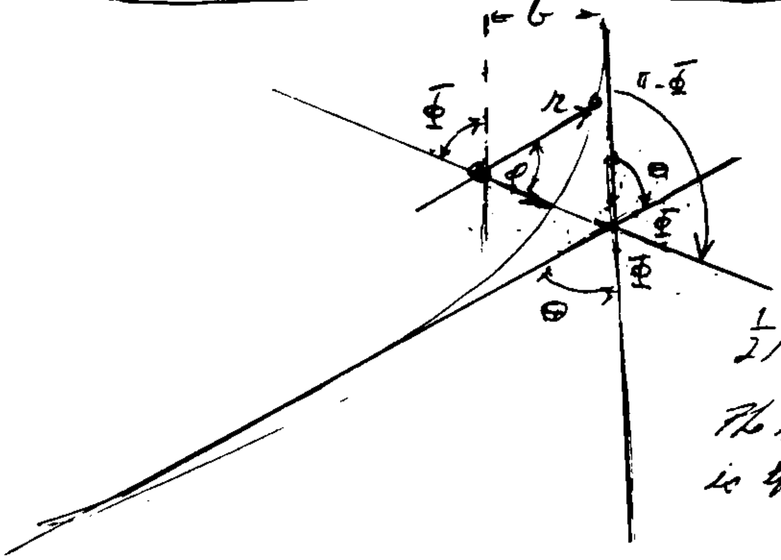
Conservation of Energy

Initial kinetic energy = radial kinetic energy + angular kinetic + pot. e.

$$\frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu v_r^2 + \frac{L^2}{2I} + V(r)$$

L = angular momentum $= \mu v_0 b$

I = moment of inertia $= \mu r^2$



$$\frac{1}{2}\mu v_0^2 = \frac{1}{2}\mu v_r^2 + \frac{\mu^2 v_0^2 b^2}{2\mu r^2} + V(r)$$

The second term on the right is the "centrifugal" potential:

$$V_{eff} = V(r) + \left(\frac{\mu v_0^2}{2}\right)\frac{b^2}{r^2}$$

Scattering angle relation

$$\Theta = \pi - 2\Phi$$

Now $\pi - \Phi = \int_{r_a}^{\infty} \frac{d\phi}{dr}\, dr$

$$-2\Phi = 2\int_{r_a}^{\infty} \frac{d\phi}{dr}\, dr - 2\pi$$

$$\Theta = -\pi + 2\int_{r_a}^{\infty} \frac{d\phi}{dr}\, dr \quad ; \quad \Theta = \pi - 2\int_{r_a}^{\infty} \frac{d\phi}{dr}\, dr$$

ATTRACTIVE POTENTIAL ; REPULSIVE POTENTIAL

Now it can be shown that $\frac{d\phi}{dr} =$

$$\frac{1}{r^2} \frac{b\, dr}{\left[1 - \frac{V(r)}{\frac{1}{2}\mu v_0^2} - \frac{b^2}{r^2}\right]^{1/2}}$$

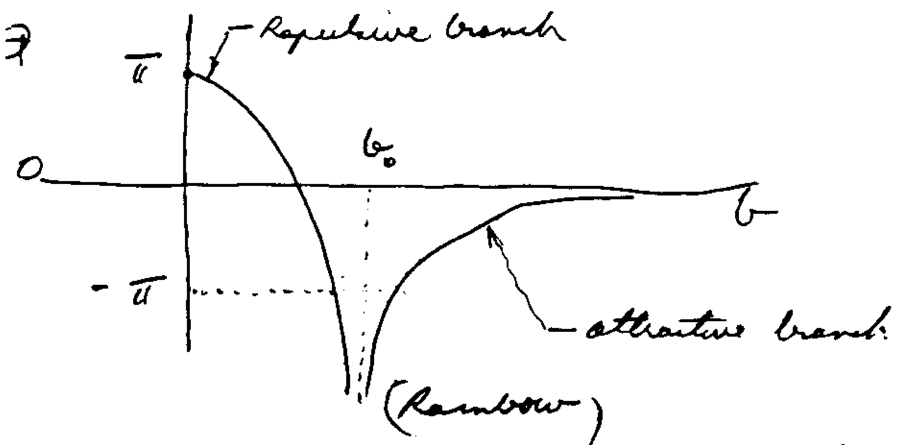
The total deflection function $\Theta = \pi - 2\int_{r_c}^{\infty} \frac{d\phi}{dr}\, dr$

$$\Theta = \pi - 2b \int \frac{1}{r^2} \frac{dr}{\left[1 - \frac{V(r)}{\frac{1}{2}\mu v_0^2} - \frac{b}{r^2}\right]^{1/2}}$$

$E_{tot} \equiv \frac{1}{2}\mu v_0^2$

Note: when denominator goes to zero $\Theta \to \infty$; the orbiting condition:

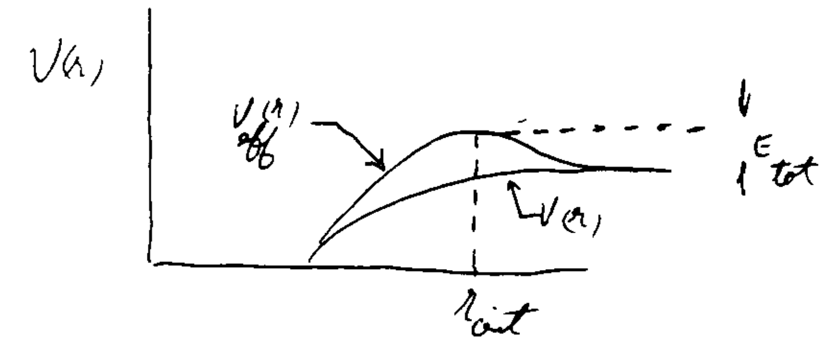
Plot deflection function vs impact parameter



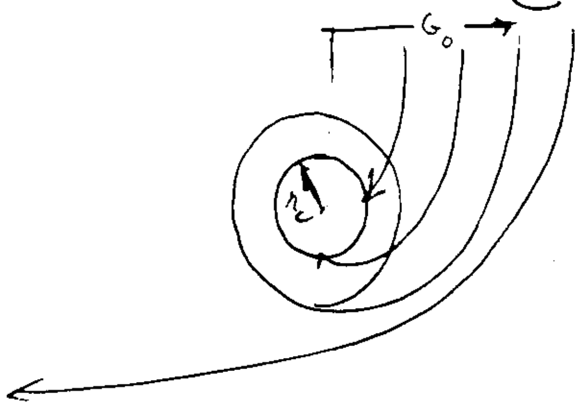
Note that at orbiting position, the radial velocity is zero

$$E_{tot} = V_{eff} = V(r_{crit}) + E_{TOT} \frac{b_0^2}{r_{crit}^2}$$

or $E_{TOT}\left(1-\frac{b_0^2}{r_{crit}^2}\right)=V(r_{crit})$



LANGEVIN MODEL: BELOW A CRITICAL RADIUS, R_C, THE "COLLISION" (INELASTIC PROCESS) OCCURS WITH UNIT PROBABILITY. For a fixed E_{TOT} + R_C, there is a max b (b_0) at which the incoming particle will reach r_C



b_0 corresponds the zero of the denominator in the deflection function ($\Theta \to \infty$)

or the the point at which the radial energy → zero

$$V_{eff}=E_{tot}\frac{b^2}{r^2}-\frac{C_n}{r^n}$$

at maximum, $\frac{dV_{eff}}{dr}=0=-2E_{tot}\frac{b_0^2}{r_c^3}+n\frac{C_n}{r_c^{n+1}}$

general relation between r_c & b_0:

$$\boxed{b_0=\left(\frac{nC_n}{2E_{tot}}\right)\cdot\frac{1}{r_c^{n-2}}}$$

for the case of $n=4$ (From two conditions: (1) extremum in V_{eff} and (2) $\Theta\to\infty$)

$$\sigma=\pi b_0^2=2\pi\left(\frac{C_4}{E_{tot}}\right)^{\frac{1}{2}}$$

Rate Constant Definition -- Velocity averaged cross section

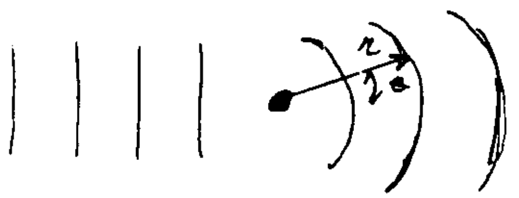
$$k = \int_0^\infty \sigma(v)\, v\, f(v)\, dv$$

For the case of ion-molecule reaction

$$\sigma(v) = \frac{2\pi}{\left(\frac{\mu}{2}\right)^{\frac{1}{2}} v} (C_4)^{\frac{1}{2}}$$

$$k = \int_0^\infty \frac{2\pi\, C_4^{\frac{1}{2}}}{\left(\frac{\mu}{2}\right)^{\frac{1}{2}} v} \cdot v\, f(v)\, dv = \frac{2\pi\, C_4^{\frac{1}{2}}}{\left(\frac{\mu}{2}\right)^{\frac{1}{2}}} \quad \text{(independent of energy)}$$

Quantum Theory of Scattering - The Scattering of Waves



Wave function must have the form $\psi \simeq e^{ikz} + \frac{e^{ikr}}{r} f(\theta)$

First term, the incoming plane wave, e^{ikz}
Second term, scattered spherical wave, $\frac{e^{ikr}}{r}$

The $f(\theta)$ modulate the amplitude of the wave w.r.t. the scattering angle

ψ has cylindrical symmetry and can be written as

$$\psi = \sum_{l=0}^{\infty} A_l P_l(\cos\theta) L_l(r) \quad \text{(sum of partial waves)}$$

A_l arbitrary constant $L_l(r)$ a solution of the radial equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dL_l}{dr}\right)+\left(k^2-U(r)-\frac{l(l+1)}{r^2}\right)L_l^{(r)}=0$$

(Schrodinger equation in cylindrical coordinates)

$L_l(r)$ takes the simple form $\frac{1}{r}G_l(r)$

and then

$$\frac{d^2G_l}{dr^2}+\left[k^2-U(r)-\frac{l(l+1)}{r^2}\right]G_l=0$$

$U(r)$ is related to the scattering potential $U(r)=\frac{2m}{\hbar^2}V(r)$

$k=\frac{mv}{\hbar}$; $(\vec{p}=\hbar\vec{k})$

Note that $\lim_{r\to\infty}\left[U(r)+\frac{l(l+1)}{r^2}\right]=0$; so the asymptotic solution G_l should look like the plane-wave solution:

$G_l\sim A(\sin kr)+\text{phase shift})$

we find $\frac{G_l}{r}\sim\frac{1}{kr}\sin(kr-\frac{1}{2}l\pi+\eta_l)$

The $\frac{l\pi}{2}$ comes from centrifugal potential & η_l comes from specific nature of the radial potential.

$$\psi=\sum_{l=0}^{\infty}(2l+1)i^l e^{i\eta_l}L_l(r)P_l(\cos\theta)$$

$$f(\theta)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left[e^{2i\eta_l}-1\right]P_l(\cos\theta)$$

The definition of <u>cross</u> section for elastic scattering

$$\sigma = 2\pi \int_0^\pi |f(\theta)|^2 \sin\theta \, d\theta$$

Introduce S_{12} matrix $= e^{2i\eta_l}$

$$f(\theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)[S-1] P_l(\cos\theta)$$

$$\boxed{\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \eta_l}$$

Rate content

$$\boxed{\int \sigma(v)\, v\, f(v)\, dv = \frac{1}{Q_T} \cdot \frac{1}{\hbar} \int \Sigma (2l+1) |S_{12}|^2 e^{-E/kT} dE}$$

Cooling & Trapping

Cooling Force

We have seen that the gradient force is given by

$$F_g \propto -\frac{\Gamma \mathcal{E}_0}{2} \nabla \phi \quad ; \quad E(r,t) = \mathcal{E}_0 \cos(\omega t + \phi)$$

or

$$E(r,t) = \mathcal{E}_0 e^{i(\omega t - kr)}$$

$$\vec{F}_g \propto + \frac{\Gamma \mathcal{E}_0}{2} \vec{k}$$

Photon relations: $E = h\nu$

$$p = \hbar k = \frac{h\nu}{c} = \frac{h}{\lambda}$$

$$\langle \vec{F} \rangle_{grad} = \hbar \frac{\Gamma}{2} \left(\frac{s}{1+s} \right) \vec{k} \quad \text{where } s = \frac{\Omega_1^2/2}{\delta^2 + \Gamma^2/4}$$

This could be a decelerating force on the atoms. How long does it take to slow down a thermal Na atom?

$\langle F \rangle = ma$; take $\lambda = 5890 Å$, $m = 23$ amu (sodium)

$$a = \frac{\langle F \rangle}{m} = -10^6 \text{ m sec}^{-2}$$

$$v_i - v_f = \int_0^t a\, dt \quad ; \quad v_i = 10^3 \text{ m sec}^{-1}, \; v_f = 0, \; a = -10^6 \text{ m sec}^{-1}$$

$$v_i - v_f = at \; ; \; t = \frac{10^3 \text{ m sec}^{-1}}{10^6 \text{ m sec}^{-2}} = 1 \text{ msec}$$

what distance is required

$$x = \int \frac{dx}{dt} dt = \int_0^{10^{-3}} at\, dt = \frac{1}{2} at^2 = \frac{1}{2} \text{ meter}$$

How many photon absorptions required? At saturation, absorption and emission of one photon ≈ 32 μsec. Deceleration to zero vel. takes 1 msec

$$\#\text{photons} = \frac{1\times10^{-3}\ \text{sec}}{32\ \text{nsec/photon}} = 3100\ \text{photons}$$

note also that $\Delta v/\text{photon} = 0.03\ \text{m sec}^{-1}$

However, what about Doppler shift? We cannot pick one frequency to slow down atom, because of shifting out of resonance

Need to get spatial variation of velocity as an atom decelerates down beam path so as to calculate spatial variation of atom frequency

$$z = \int_0^z dz = \int_0^t v_0\,dt - \int_0^t a\,t\,dt = v_0 t - \tfrac{1}{2} a t^2$$

$$z = v_0\left[\frac{v_0 - v}{a}\right] - \frac{1}{2}a\left[\frac{v_0 - v}{a}\right]^2$$

$$z = \frac{1}{2a}\left(v_0^2 - v^2\right)$$

$$v^2 = v_0^2 - 2az \quad \text{or} \quad \boxed{v(z) = \left(v_0^2 - 2az\right)^{\frac{1}{2}}}$$

Now the Doppler shift: $\Delta\nu = \dfrac{v}{\lambda_0}$

$$\text{so}\quad \boxed{\Delta\nu(z) = \frac{1}{\lambda_0}\left(v_0^2 - 2az\right)^{1/2}}$$

TWO WAYS TO SOLVE PROBLEM:

1. VARY ATOM FREQ.
2. VARY LASER FREQ

} Compensation for $\Delta\nu_D(z)$ due to Doppler

FIRST SOLUTION: SPATIALLY VARYING ATOMIC ZEEMAN SHIFT.

$$\Delta\nu_B(z) = \left[\frac{\Delta\nu}{B_0}\right] B(z)$$

fixed for atom ↗ ↖ spatially varying field

Set $\Delta\nu_B = \Delta\nu_D$

$$\left[\frac{\Delta\nu}{B}\right] B(z) = \frac{1}{\lambda_0}\left(v_0^2 - 2az\right)^{\frac{1}{2}}$$

$$B(z) = \left(\frac{B}{\Delta\nu\,\lambda_0}\right)\left[v_0^2 - 2az\right]^{\frac{1}{2}} = \frac{B}{\Delta\nu}\frac{v_0}{\lambda_0}\left[1 - \frac{2az}{v_0^2}\right]^{\frac{1}{2}}$$

Define $\Delta\nu_0 = \frac{v_0}{\lambda_0}$ &

$$\Delta\nu_0 = \left(\frac{\Delta\nu}{B}\right) B_0$$

or $\frac{B}{\Delta\nu} = \frac{B_0}{\Delta\nu_0}$

$$= \frac{B_0}{\Delta\nu_0}\cdot\Delta\nu_0\left[1 - \frac{2az}{v_0^2}\right]^{\frac{1}{2}}$$

$$\boxed{B(z) = B_0\left[1 - \frac{2az}{v_0^2}\right]^{\frac{1}{2}}}$$

In Na $\frac{\Delta\nu_0}{B_0} = \frac{14\,GHz}{Tesla}$; comes from E_B = energy shift due to Zeeman interaction

$$E_B = \beta B g_J M_J \; ; \; \beta = \frac{e\hbar}{2mc} \text{ (Bohr magneton)}$$

$^2P_{3/2}$ — $g_{3/2} = 4/3$ $\qquad E_B^{P_{3/2}} = \beta \frac{4}{3} B M_J$

$^2S_{1/2}$ — $g_{1/2} = 2$ $\qquad E^{S_{1/2}} = \beta 2 B M_J$

$$\boxed{\Delta E_B = 2\beta B - \beta B = \beta B} = 14\,\text{GHz}/T \qquad (\beta = 9.274 \times 10^{-24}\,J/T)$$

Need 1.7 GHz for B_0 $\qquad \frac{1.7\,\text{GHz}}{14\,\text{GHz}/T} = 1200$ gauss

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VARY LASER FREQUENCY (Frequency Chirping)

Laser Cooling by Frequency Chirping - Want to get $\nu_L(t)$

Change in velocity per ~~photon absorption~~ unit time, assuming saturation:

$$\frac{\Delta P}{\Delta t} = -\frac{h\nu}{2\tau c}$$

$$\dot{v} = \frac{dv}{dt} = -\frac{h\nu_L(t)}{mc2\tau} \quad ; \quad \begin{array}{l}\tau = \text{natural life time} \\ m = \text{atom mass} \\ \nu_L(t) = \text{frequency of laser}\end{array}$$

Frequency shift due to Doppler slowing.

$$\nu_L(t) = \nu_0\left(1 - \frac{v(t)}{c}\right)$$

or

$$\frac{v(t)}{c} = 1 - \frac{\nu_L(t)}{\nu_0}$$

$$v(t) = c\left[1 - \frac{\nu_L(t)}{\nu_0}\right]$$

$$\dot{v}(t) = -\frac{\dot{\nu}_L(t)c}{\nu_0}$$

$$-\frac{\dot{\nu}_L(t)c}{\nu_0} = -\frac{h\nu_L(t)}{mc2\tau}$$

$$\dot{\nu}_L(t) - \underbrace{\frac{h\nu_0}{mc^2 2\tau}}_{\alpha}\nu_L(t) = 0$$

By inspection: $\boxed{\nu_L(t) = \nu_S \cdot e^{\alpha t}}$; ν_S starting frequency

$$\dot{\nu}_L - \alpha \nu_L = 0$$
$$\nu_L(t) = e^{\alpha t}$$
$$\dot{\nu}_L(t) = \alpha e^{\alpha t}$$

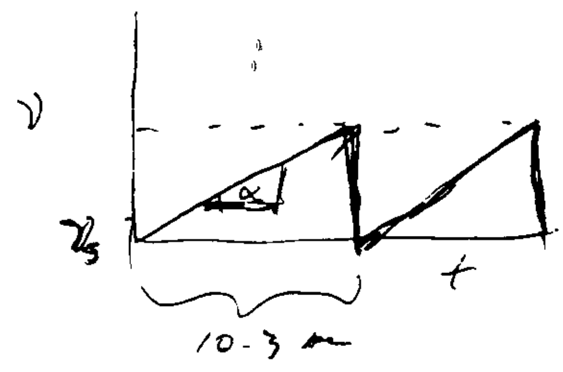
Practical size of α: $\alpha = \dfrac{h\nu_0}{mc^2 2T}$

$\nu_0 = 5.1 \times 10^{14}$ sec^{-1}
$h = 6.63 \times 10^{-27}$ erg·sec
$m = 23 \times 1.66 \times 10^{-24} = 3.81 \times 10^{-23}$
$c = 3 \times 10^{10}$
$T = 16$ nsec

} $\alpha = .0031$ sec^{-1}

We already know that $t = 3 \times 10^{-3}$ sec will stop Na
$\alpha t = 3.1 \times 10^{-3} \times 3 \times 10^{-6} \doteq 10^{-5}$
$\therefore$ argument of $e^{\alpha t}$ always small

$$\nu_L(t) = \nu_S e^{\alpha t} \doteq \nu_S(1 + \alpha t)$$



$$\frac{d\nu_L}{dt} = \alpha(1+\alpha t) \simeq \alpha \ \nu_S \alpha t$$

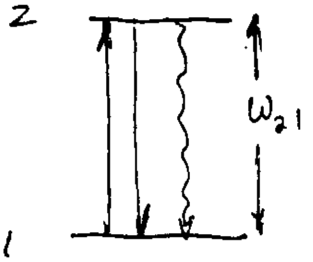
let $\nu_S \simeq 5 \times 10^{14}$ $\alpha = 3.1 \times 10^{-3}$ $t = 10^{-3}$

$$\frac{d\nu_L}{dt} = 15 \times 10^{8} = 1.5 \times 10^{9} \text{ Hz} = 1.5 \text{ GHz}$$ = Sweep Frequency per sec ✓

Review of Basic Concepts and Vocabulary

A. Absorption and Emission of Radiation

1. Einstein Coefficients



W_{12}^{i} = rate of absorption (induced by field)

W_{21}^{i} = rate of emission (induced by field)

W_{21}^{s} = rate of spontaneous emission from state 2 to state 1

$W_{12}^{i} = B_{12}\, \rho_\omega N_1$; B_{12} = Einstein "B" coefficient
ρ_ω = energy density per unit angular frequency
N_1 = population in level 1.

$W_{21}^{i} = B_{21}\, \rho_\omega N_2$

Relations: $B_{21}^{\omega} g_2 = B_{12}^{\omega} g_1$ ^ induced rate constants per state are equal. Note g_2, g_1 are level degeneracy factors

Units: ρ has units of energy density per unit angular frequency.
(joules m^{-3} Hz^{-1})
B_{12}, B_{21} (m^3 Hz / joule sec)
N_1, N_2 (unitless)

Spontaneous Emission: $A_{21} = \frac{1}{\tau}$; τ is the lifetime of the excited state
$\Gamma \equiv \frac{1}{\tau}$ (sec^{-1})

Relations between B_{12}, B_{21}, + A_{21}

$$B_{21}^{\omega} = \frac{\pi^2 c^3}{\hbar\, \omega_{21}^3} A_{21} \quad ; \quad B_{12}^{\omega} = \frac{g_2}{g_1} B_{21}^{\omega}$$

N.B. Einstein Relations assume ρ_ω is constant over the line-width.

N.B. The relation between B & A depends on the definition of ρ. For example, suppose ρ defined as energy density per unit frequency ν instead of angular frequency $\omega = 2\pi\nu$

The rate must be the same so: $B_{21}^{\omega}\rho_\omega = B_{21}^{\nu}\rho_\nu$
and $\rho_\omega d\omega = \rho_\nu d\nu$

and $d\omega = 2\pi d\nu$

so $\rho_\nu = \rho_\omega \frac{d\omega}{d\nu} = \rho_\omega 2\pi$

so $B_{21}^{\omega}\rho_\omega = B_{21}^{\nu}\rho_\omega 2\pi$

and $\frac{B_{21}^{\omega}}{B_{21}^{\nu}} = 2\pi$ — Einstein B coefficients must be consistent with definition of ρ

2. Lineshape:
If an excited state decays exponentially we can derive its lineshape by a Fourier transformation:

$$f(t) = e^{-t\Gamma}$$

$$f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-t\Gamma} e^{-i(\omega-\omega_0)t}\, dt$$

$$f(\omega) = \frac{\Gamma \cdot 2}{\pi} \cdot \frac{1}{(\omega-\omega_0)^2 + \Gamma^2} \quad \text{(Lorentzian)}$$

Then $\sigma_{12}(\omega) = B_{12}^{\omega} f(\omega)$; must use Einstein $\sigma_{12}(\omega)$ when light source is narrow compared to natural atomic linewidth

Then $W_{12}^{i} = \int_{-\infty}^{+\infty} \rho_\omega \delta(\omega-\omega_0) N_1 B_{12}(\omega)\, d\omega$

$$= \rho_{\omega_0} N_1 \int_{-\infty}^{+\infty} B_{12} f(\omega)\, d\omega$$

What is a typical natural width

$$\frac{1}{2} f(\omega_0) = \frac{1}{\pi\Gamma} = \text{half maximum}$$

$$\frac{1}{\pi\Gamma} = \frac{2\Gamma}{\pi} \cdot \frac{1}{(\omega-\omega_0)^2 + \Gamma^2}$$

$$\pi\Gamma = \frac{\pi\left[(\omega-\omega_0)^2 + \Gamma^2\right]}{2\Gamma}$$

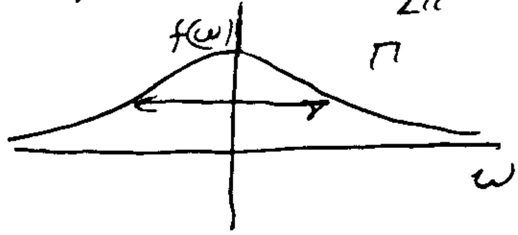
$$2\pi\Gamma^2 - \pi\Gamma^2 = \pi(\omega-\omega_0)^2_{FWHM}$$

$$\Gamma^2 = (\omega-\omega_0)^2_{FWHM}$$

$$\boxed{\Gamma = (\omega-\omega_0)_{FWHM}}$$

Thus, Γ is the full width at half max of the transition.

$\Gamma = \frac{1}{\tau}$; $\Delta\nu = \frac{\Gamma}{2\pi}$ = 10 MHz (in the case of Na)



3. The Measures of Probability of a Transition

Relation between A_{21} and transition dipole:

$$A_{21} = \frac{2\ \omega_{21}^{3}}{3\epsilon_0 \hbar c^3} \mu_{21}^{2}$$;

ω_{21} = angular frequency of transition
$\hbar$ = planck's constant
ϵ_0 = permittivity of free space
c = speed of light
μ_{21} = transition dipole

From Q.M.

$$\mu_{21} = e \langle 1 | \vec{r} | 2 \rangle$$

Line strength $S_{21} = S_{12} = g_2 \mu_{21}^2$

and $S_{21} = \frac{3\epsilon_0 \hbar c^3}{2\omega_{21}^3} g_2 A_{21}$

Oscillator strength - a measure of the "strength" of a transition compared to a classical oscillator.
Consider harmonically oscillating charge:
It is accelerated by harmonic force and therefore radiatively damped.
The time constant γ of the radiative damping:

$-\frac{dU}{dt}$ = rate of energy loss by radiation $= \frac{e^2 \omega_0^2}{12\pi\epsilon_0 c^3} A^2 e^{-\gamma t}$

A = amplitude of oscillation

$$\gamma = \frac{e^2 \omega_0^2}{6\pi\epsilon_0 c^3 m}$$

So if we were to consider the atom as a classical oscillator:

$$\gamma = \frac{e^2 \omega_{12}^2}{6\pi\epsilon_0 c^3 m_e}$$

Now Oscillator strength defined:

$$\boxed{f_{21} = -\frac{1}{3} A_{21} \gamma^{-1}}$$ (emission)

Absorption oscillator strength: $g_1 f_{12} \equiv -g_2 f_{21} \equiv gf$

$\therefore \quad f_{12} = \frac{g_2}{g_1} \, \frac{1}{3} \, A_{21} \, \gamma^{-1}$

For a $J=0 \rightarrow J=1$ transition:

$g_1 = 1 \qquad g_2 = 3$

$f_{12} = A_{21} \gamma^{-1}$. Note that A_{21} will be $\leq$ unity

Note the sum rule: $\sum_i f_{i1} = 1$ THOMAS KUHN SUM RULE

Summary: Measure of transition probability:

$A_{21}, \mu_{21}, S_{21}, f_{21}$

We have not mentioned the cross section. We give the relation between the absorption cross section and the Einstein B_{12} without proof development

$$\sigma_0 = \frac{\hbar \omega_{21} B_{12}^{\omega}}{c}$$

a table of conversion follows:

	A_{21}						
A_{21}	1	B^{ω}_{12}					
B^{ω}_{12}	$\frac{g_2}{g_1}\frac{\pi^2 c^3}{\hbar\omega_{21}^3}$	1	B^{f}_{12}				
B^{f}_{12}	$\frac{g_2}{g_1}\frac{c^3}{8\pi h f^3}$	$\frac{1}{2\pi}$	1	σ_0			
σ_0	$\frac{1}{4}\frac{g_2}{g_1}\lambda_{21}^2$	$\frac{\hbar\omega_{21}}{c}$	$\frac{h\omega_{21}}{c}$	1	f_{12}		
f_{12}	$\frac{g_2}{g_1}\frac{2\pi\epsilon_0 mc^3}{\omega_{21}^2 e^2}$	$\frac{2\epsilon_0 m\hbar\omega_{21}}{\pi e^2}$	$\frac{4\epsilon_0 m\hbar\omega_{21}}{e^2}$	$\frac{2\epsilon_0 mc}{\pi e^2}$	1	μ_{21}^2	
μ_{21}^2	$\frac{3\epsilon_0 hc^3}{2\omega_{21}^3}$	$3\frac{g_1}{g_2}\frac{\epsilon_0\hbar^2}{\pi}$	$6\frac{g_1}{g_2}\epsilon_0\hbar^2$	$3\frac{g_1}{g_2}\frac{\epsilon_0\hbar c}{\pi\omega_{21}}$	$\frac{3}{2}\frac{g_1}{g_2}\frac{\hbar e^2}{\omega_{21} m}$	1	S_{21}
S_{21}	$g_2\frac{3\epsilon_0 hc^3}{2\omega_{21}^3}$	$\frac{3g_1\epsilon_0\hbar^2}{\pi}$	$6g_1\epsilon_0\hbar^2$	$\frac{3g_1\epsilon_0\hbar c}{\pi\omega_{21}}$	$\frac{3g_1\hbar e^2}{2\omega_{21} m}$	g_2	1

Last relation very important for future discussion: the Rabi frequency

$$\Omega_1 = \frac{-\mu_{21} \mathcal{E}_0}{\hbar}$$

where $\mathcal{E}_0$ is the amplitude of a field incident on the atom

$$\vec{E}_e(\vec{r}, t) = \vec{\varepsilon}\, \mathcal{E}_0(\vec{r}) \cos[\omega_L t + \phi(\vec{r})]$$

$\vec{\varepsilon}$ is the polarization vector

$\mathcal{E}_0(\vec{r})$ the field amplitude (not necessarily constant)

ω_L the incident frequency – we think of it coming from the laser (L)

$\phi(\vec{r})$ is a phase factor indicating spatial propagation.

E.g. $\phi(\vec{r}) = \vec{k} \cdot \vec{r}$