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"Interfacial Instabilities: Part A - Steady Cellular Convection"

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INTERFACIAL INSTABILITIES

PART A – STEADY CELLULAR CONVECTION

1. – Introduction.

Bénard convection or, more appropriately, Bénard-Marangoni convection refers here to the buoyancy-thermocapillary flow induced in a liquid layer heated from below when its upper boundary is a free deformable surface open to the ambient air. Earlier work by numerous authors [1-37] has elucidated the salient features of this problem. In the following notes we analyse the evolution of the liquid layer with special reference to the evolution of the liquid-air interface (for an illustration see fig. 1).

We consider the case of a horizontal liquid layer confined between planes located at heights $z = 0$ and $z = d$, respectively. For simplicity we limit ourselves to the case $d \ll L$, where L is either of the two horizontal scales of the layer. Later on we shall even restrict consideration to a two-dimensional geometry thus disregarding one of the two horizontal scales. The layer is assumed to be heated or cooled from below. In the simplest Newtonian-Boussinesquian approximation the evolution of the liquid layer is governed by the following balance equations [7, 38]:

$$\partial v_i/\partial x_i = 0\,, \tag{1.1}$$

$$\varrho^*(\partial v_i/\partial t + v_j \partial v_i/\partial x_j) = -\partial p/\partial x_i + \eta\, \nabla^2 v_i + \varrho g e_i\,, \tag{1.2}$$

$$\varrho^* c_v(\partial T/\partial t + v_j \partial T/\partial x_j) = -\partial (J_e)_i/\partial x_i \tag{1.3}$$

together with the equation of state

$$\varrho = \varrho^*[1 + \alpha(T^* - T)]\,. \tag{1.4}$$

The summation convention over repeated indices is assumed. Here an asterisk denotes some constant reference value taken, say, at the undisturbed interface. Besides, ϱ denotes density and p, pressure. v_i and T denote the i-th component of the velocity field and temperature, respectively. η is the viscosity,

(*) Lectures delivered by M. G. VELARDE.

Fig. 1. – A top view, magnified some 25 times, shows the hexagonal convection pattern in a layer of silicone oil one millimetre deep that is heated uniformly below and exposed to ambient air above. Light reflected from aluminium flakes shows liquid rising at the centre of each cell and descending at the edges. The exposure time is ten seconds, whereas fluid moves across the cell, the deformed liquid-air interface, from the centre to the edge in two seconds. *a*) Bénard convection—a few polygonal cells; *b*) Bénard convection—the hexagonal cell.

$\eta = \varrho\nu$ with ν the kinematic viscosity, c_p the specific heat at constant pressure. g is the acceleration due to gravity and e the unit normal vector along z. α is the coefficient of thermal expansion. In eq. (1.3) we have also used the simplest caloric equation of state between energy and temperature. To a first approximation the heat (J_e) flux is

$$(J_e)_i = -\lambda\,\partial T/\partial x_i\,, \tag{1.5}$$

λ is the thermal conductivity.

As we consider the liquid layer open to the ambient air, we use the following equation of state for the surface tension, σ, at the open interface:

$$\sigma = \sigma^* + (\partial\sigma/\partial T)(T - T^*)\,. \tag{1.6}$$

As a consequence of the thermal gradient across the layer, and for moderately low values of ΔT, the fluid establishes itself in a motionless steady state denoted by the superscript s:

$$v_i^s = 0\,, \tag{1.7}$$

$$T^s = T^\bullet - (\Delta T/d)\,z\,, \tag{1.8}$$

$$p^s = p^* - \varrho^* g(z-d) - \varrho^* g\alpha\Delta T(z-d)^2/2d \tag{1.9}$$

with $\Delta T = T^\bullet - T^*$, where the superscript $\bullet$ denotes values at $z = 0$.

We are interested in the instability of the motionless steady state when the thermal constraint takes higher and higher, albeit controlled values. A reasonable assumption is that under increased constraint the thermohydrodynamic fields would tend to depart from their steady values eq. (1.7)-(1.9) and the upper surface would not remain level. The latter would move to a height $z = d + \zeta(x, t)$ which shows inhomogeneity along the horizontal. For simplicity we restrict consideration to a two-dimensional problem (x, z) and disregard any y-dependence along the horizontal. Thus we exclude explicit consideration of hexagonal tessellation at the upper surface in the Bénard layer. The remaining quantities are $v_i = 0 + \delta v_i$ $(i = 1, 2)$, $T + \delta T$ and $p + \delta p$, where all disturbances depend on x, z and t. These disturbed fields also obey the original thermohydrodynamic equations. We have

$$\partial v_i/\partial x_i = 0\,, \tag{1.10}$$

$$\partial v_i/\partial t + v_j(\partial v_i/\partial x_j) = -(\varrho^*)^{-1}\partial\,\delta p/\partial x_i + \nu\nabla^2 v_i + \alpha\,\delta T g e_i\,, \tag{1.11}$$

$$\partial\,\delta T/\partial t + v_j(\partial\,\delta T/\partial x_j) = \varkappa\nabla^2\delta T + (\Delta T/d)\,v_i e_i\,. \tag{1.12}$$

On the other hand, eq. (1.6) becomes

$$\sigma = \sigma^* + (\partial\sigma/\partial T)[\delta T - \zeta(\Delta T/d)] \tag{1.13}$$

with $\varkappa = \lambda/\varrho c_p$ being the thermal diffusivity of the liquid and $e = (0, 1)$. If the problem (1.10)-(1.13) together with the appropriate boundary conditions (b.c.) has nontrivial solutions, then the steady state (1.7)-(1.9) will be unstable.

We shall consider the following b.c. The lower surface located at $z = 0$ is taken mechanically rigid, that is

$$v_i = 0 \qquad \text{at } z = 0\,. \tag{1.14}$$

Here heat is assumed to flow across the boundary following Newton's law of cooling (Robin, Biot or mixed condition). We have

$$\lambda\,\partial\,\delta T/\partial z = q^{0}\,\delta T \qquad \text{at } z = 0, \tag{1.15}$$

where q^{0} is a parameter that accounts for the transfer characteristics of the boundary. The limits q^{0} going to zero and to infinity, respectively, correspond to the cases of a poor and a perfectly conducting surface. At the upper surface we have an interface liquid-air, and we assume that it follows the velocity field (no cavitation exists). Thus we have

$$\partial\zeta/\partial t = N v_i n_i\,. \tag{1.16}$$

We also have continuity in the pressure (stress) field

$$(-p^{a} - \delta p + p^{*})\,n_i + \eta(\partial v_i/\partial x_j + \partial v_j/\partial x_i)\,n_j = k\sigma n_i + (t_j\,\partial\sigma/\partial x_j)\,t_i\,, \qquad i, j = 1, 2\,. \tag{1.17}$$

Note that writing eq. (1.17) we have tacitly assumed that the ambient air at the upper level is some kind of a large reservoir and that the interface liquid-air although deformable is mechanically ideal. Thus we have

$$-n_i\,\delta p + n_i[\varrho^{*} g\zeta + \varrho^{*} g\alpha\Delta T\zeta^{2}/2d] + \eta n_j(\partial v_i/\partial x_j + \partial v_j/\partial x_i) = k\sigma n_i + (t_j\,\partial\sigma/\partial x_j)\,t_i \tag{1.18}$$

with σ given by eq. (1.13). We have introduced the following notation. $\boldsymbol{n}$ is the outward unit normal vector to the liquid-air surface, given by

$$\boldsymbol{n} = (-\,\partial\zeta/\partial x, 1)/N \tag{1.19}$$

and the unit tangent vector $\boldsymbol{t}$ is

$$\boldsymbol{t} = (1, \partial\zeta/\partial x)/N\,. \tag{1.20}$$

The curvature k

$$k = N^{-3}\,\partial^{2}\zeta/\partial x^{2} \tag{1.21}$$

with N a normalization factor given by

$$N = [1 + (\partial\zeta/\partial x)^{2}]^{\frac{1}{2}}\,. \tag{1.22}$$

We also assume that temperature disturbances follow, at $z = d + \zeta(x, t)$, similar b.c. to (1.15). That is, for the temperature the b.c. becomes

$$\lambda n_i\,\partial(T + \delta T)/\partial x_i = -\,\lambda\,\Delta T/d - q^{*}(\delta T - \zeta\,\Delta T/d)\,. \tag{1.23}$$

For universality in the argument we now rescale the variables using the following units: d for length, $d^{2}/\varkappa$ for time, $\varkappa/d$ for velocity, $\eta\varkappa/d^{2}$ for pressure, ΔT for temperature and denote $\theta = \delta T/\Delta T$. With these units the system (1.10)-(1.12) becomes in dimensionless form

$$\partial v_i/\partial x_i = 0\,, \tag{1.24}$$

$$\Pr^{-1}(\partial v_i/\partial t + v_j\,\partial v_i/\partial x_j) = \partial\tau_{ij}/\partial x_j + \mathrm{Ra}\,\theta e_i\,, \tag{1.25}$$

$$\partial\theta/\partial t + v_j(\partial\theta/\partial x_j) = \nabla^{2}\theta + w \tag{1.26}$$

with $\boldsymbol{v} = (u, w)$ and the stress tensor of the liquid $\tau_{ij} = -\,p\delta_{ij} + (\partial v_i/\partial x_j + \partial v_j/\partial x_i)$. δ_{ij} is the Kronecker delta. The following dimensionless groups have been introduced:

$$\text{Rayleigh number:} \quad \mathrm{Ra} = \frac{\alpha g d^{3}\Delta T}{\varkappa\nu}\,; \tag{1.27}$$

$$\text{Prandtl number:} \quad \Pr = \nu/\varkappa\,, \tag{1.28}$$

At $z = 0$, the b.c. (1.14) and (1.15)

$$u = w = 0\,, \tag{1.29}$$

$$\partial\theta/\partial z = \mathrm{Bi}^{0}\theta\,, \tag{1.30}$$

whereas at $z = 1 + \xi$ $(\xi = \zeta/d)$, we have, from eq. (1.16),

$$\partial\xi/\partial t = N v_i n_i\,. \tag{1.31}$$

From the continuity of stress at the interface, eq. (1.18), we have

$$\tau_{ij} n_j = -\,(\mathrm{Bo}/C)\left(\xi + \frac{d_T}{2}\xi^{2}\right) n_i + (K/C)[1 - MC(\theta - \xi)]\,n_i - \{t_j\,\partial[M(\theta - \xi)]/\partial x_j\}\,t_i\,. \tag{1.32}$$

For the temperature field, the boundary condition at the free surface, eq. (1.23), becomes

$$n_i\,\partial\theta/\partial x_i = -\,\mathrm{Bi}^{*}(\theta - \xi) + (1 - N)/N\,. \tag{1.33}$$

The dimensionless curvature of the interface, which appears in eq. (1.32), is given by

$$K = N^{-3}\,\partial^{2}\xi/\partial x^{2} \tag{1.34}$$

and

$$\boldsymbol{n} = (-\,\partial\xi/\partial x, 1)/N\,, \tag{1.35}$$

$$\boldsymbol{t} = (1, \partial\xi/\partial x)/N\,, \tag{1.36}$$

$$N^{2} = 1 + (\partial\xi/\partial x)^{2}\,. \tag{1.37}$$

Also, we have introduced the dimensionless groups

(1.38) Bond number: $\mathrm{Bo} = \varrho^* g d^2/\sigma^*$;

(1.39) Capillary number: $C = \eta\varkappa/\sigma^* d$

and

(1.40) Marangoni number: $M = -\dfrac{\partial\sigma}{\partial T}\dfrac{\Delta T d}{\eta\varkappa}$.

The heat transfer groups (Bi° and Bi^*, with $\mathrm{Bi} = qd/\lambda$) are all nonnegative.

Another useful group is the ratio of the Bond and the capillary numbers

(1.41) Galileo number: $G = \mathrm{Bo}/C = gd^3/\varkappa\nu$.

Lastly, we note that the Boussinesquian approximation demands that

$$d_T \equiv \alpha\Delta T = (\varrho^* - \varrho^\circ)/\varrho^* = \mathrm{Ra}\, C/\mathrm{Bo} \ll 1 .$$

For moderately high thermal gradients this condition would be fulfilled even for horizontally elongated cells.

To help the reader we give now the b.c. (1.18) or (1.32) in explicit form. For the normal component

(1.42) $$p - G(\xi + d_T\xi^2/2) + [1/C - M(\theta - \xi)](\partial^2\xi/\partial x^2)/N^3 = \\ = 2[(\partial u/\partial x)(\partial\xi/\partial x)^2 - (\partial u/\partial z + \partial w/\partial x)(\partial\xi/\partial x) + (\partial w/\partial z)]/N^2$$

and for the tangential component

(1.43) $$M[\partial\theta/\partial x - \partial\xi/\partial x + (\partial\xi/\partial x)(\partial\theta/\partial z)] = \\ = -\{(\partial u/\partial z + \partial w/\partial x)[1 - (\partial\xi/\partial x)^2] + 2(\partial w/\partial z - \partial u/\partial x)(\partial\xi/\partial x)\}/N$$

with $u \equiv v_1$ and $w = v_3$.

2. – Scaling analysis of the evolutionary problem.

When the heat transfer across the horizontal boundaries of the liquid layer is small enough, it is known, according to experiment and to earlier linear stability analyses [13, 38], that the pattern at the onset of convective instability tends to be of long horizontally elongated cells which dissipate less. Xe shall take advantage of this fact and set

(2.1) $$\mathrm{Bi}^\circ = \varepsilon^2 \quad \text{and} \quad \mathrm{Bi}^* = 0 .$$

which defines a scaling and smallness parameter. Thus we now redefine the units of time and length in the appropriate manner. We set

(2.2) $$\tau = \varepsilon^2 t ,$$

(2.3) $$X = \varepsilon^{\frac12} x$$

and

(2.4) $$Z = z .$$

Then we assume that

(2.5) $$G = g/\varepsilon ,$$

i.e. we assume $Gd_T = O(\varepsilon^0)$. This is not the unique scaling that can be introduced for the gravitational acceleration. An alternative to (2.5) is

(2.6) $$G = \varepsilon^0 g .$$

We shall later on comment on the use of (2.6), while for the present analysis we shall explore the relevance of (2.5) only.

Together with (2.2)-(2.5) we now assume the following expansions:

(2.7) $$\xi = \varepsilon\xi_1 + \varepsilon^2\xi_2 + \dots ,$$

(2.8) $$u = \varepsilon^{\frac12}[u_0 + \varepsilon u_1 + \dots] \equiv \varepsilon^{\frac12}\tilde u ,$$

(2.9) $$w = \varepsilon[w_0 + \varepsilon w_1 + \dots] \equiv \varepsilon\tilde w ,$$

(2.10) $$p = p_0 + \varepsilon p_1 + \dots ,$$

(2.11) $$\theta = \theta_0 + \varepsilon\theta_1 + \dots ,$$

(2.12) $$\mathrm{Ra} = R_0 + \varepsilon R_1 + \dots$$

and

(2.13) $$M = M_0 + \varepsilon M_1 + \dots .$$

Note that we do not consider here any non-Boussinesquian effect thus restricting consideration to small values of d_T.

Then the disturbance equations (1.24)-(1.26) become

(2.14) $$\partial\tilde u/\partial X + \partial\tilde w/\partial Z = 0 ,$$

(2.15) $$\varepsilon^2\partial\tilde u/\partial\tau + \varepsilon\tilde u\,\partial\tilde u/\partial X + \varepsilon\tilde w\,\partial\tilde u/\partial Z = \\ = -\mathrm{Pr}\,\partial p/\partial X + \mathrm{Pr}(\varepsilon\,\partial^2\tilde u/\partial X^2 + \partial^2\tilde u/\partial Z^2) ,$$

(2.16) $$\varepsilon^2\partial\tilde w/\partial\tau + \varepsilon^2\tilde u\,\partial\tilde w/\partial X + \varepsilon^2\tilde w\,\partial\tilde w/\partial Z = \\ = -\mathrm{Pr}\,\partial p/\partial Z + \mathrm{Pr}(\varepsilon^2\partial^2\tilde w/\partial X^2 + \varepsilon\,\partial^2\tilde w/\partial Z^2) + \mathrm{Pr}\,\mathrm{Ra}\,\theta$$

and

(2.17) $$\varepsilon^2\partial\theta/\partial\tau + \varepsilon\tilde{u}\,\partial\theta/\partial X + \varepsilon\tilde{w}(\partial\theta/\partial Z - 1) = \varepsilon\,\partial^2\theta/\partial X^2 + \partial^2\theta/\partial Z^2\,.$$

i) *Zeroth-order problem (linear stability analysis).* Inserting the expansion (2.7)-(2.13) in these equations and keeping in mind that ε is an *ordering* parameter, the evolution equations must be satisfied identically whatever the value of ε. To the zeroth-order approximation we have

(2.18) $$\frac{\partial u_0}{\partial X} + \frac{\partial w_0}{\partial Z} = 0\,,$$

(2.19) $$\frac{\partial p_0}{\partial X} - \frac{\partial^2 u_0}{\partial Z^2} = 0\,,$$

(2.20) $$\frac{\partial p_0}{\partial Z} - R_0\theta_0 = 0\,,$$

(2.21) $$\frac{\partial^2\theta_0}{\partial Z^2} = 0$$

together with the b.c. at $Z = 0$

(2.22) $$u_0 = w_0 = \frac{\partial\theta_0}{\partial Z} = 0$$

and at $Z = 1$ (*)

(2.23) $$w_0 = 0\,,$$

(2.24) $$p_0 = g\xi_1\,,$$

(2.25) $$M_0\frac{\partial\theta_0}{\partial X} = -\frac{\partial u_0}{\partial Z}\,,$$

(2.26) $$\frac{\partial\theta_0}{\partial Z} = 0\,.$$

(*) Note that the b.c. are at $Z = 1 + \xi$. We shall be consistent, however, with the expansion procedure introduced and take for any function

$$f(Z = 1 + \xi) = f(1 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \dots) =$$
$$= f(1) + \varepsilon\left(\frac{\partial f}{\partial Z}\right)\xi_1 + \varepsilon^2\xi_2\left(\frac{\partial f}{\partial Z}\right) + \frac{\varepsilon^2}{2}\xi_1^2\left(\frac{\partial^2 f}{\partial Z^2}\right) + O(\varepsilon^3)\,.$$

Thus incorporating the ε-expansion we are allowed to take the b.c. at $Z = 1$, according to the ordering indicated by the power of ε. Besides the following relation also holds:

$$\int_0^{1+\xi} f\,dZ = \int_0^1 f\,dZ + \varepsilon\xi_1 f(1) + O(\varepsilon^2)\,.$$

The solution of (2.21) is

(2.27) $$\theta_0 = F(X, \tau)\,.$$

Then from (2.20) and (2.24)

(2.28) $$p_0 = R_0F[Z - 1] + g\xi_1\,.$$

From (2.19) and (2.25) we get

(2.29) $$u_0 = R_0F'\left[\frac{Z^3}{6} - \frac{Z^2}{2} + \frac{Z}{2}\right] + g\xi_1'\left[\frac{Z^2}{2} - Z\right] - M_0F'Z\,,$$

where the *dash* denotes a derivative with respect to X. From (2.18) we now get

(2.30) $$w_0 = R_0F''\left[-\frac{Z^4}{24} + \frac{Z^3}{6} - \frac{Z^2}{4}\right] + g\xi_1''\left[-\frac{Z^3}{6} + \frac{Z^2}{2}\right] + M_0F''\frac{Z^2}{2}\,.$$

Using b.c. (2.23) we get

(2.31) $$\frac{g}{3}\xi_1'' + F''\left[\frac{M_0}{2} - \frac{R_0}{8}\right] = 0\,,$$

which is in fact $\left[\int_0^1 u_0\,dz\right]'$ and, as there is no net velocity of the liquid layer as a whole, the bracketed term vanishes. We have

(2.32) $$\frac{g}{3}\xi_1' + F'\left[\frac{M_0}{2} - \frac{R_0}{8}\right] = 0\,.$$

Now we note that another relation can be established between ξ_1 and F, *i.e.* between ξ_1 and θ_0. This can be obtained from the energy equation integrated over the liquid depth. It plays the role of solvability condition for the ε-hierarchy of equations. To the lowest-order approximation in ε we get

(2.33) $$F''\left[1 + \frac{M_0}{6} - \frac{R_0}{20}\right] + \frac{g}{8}\xi_1'' = 0\,,$$

that together with (2.31) yields

(2.34) $$\frac{R_0}{320} + \frac{M_0}{48} = 1\,,$$

which defines the line of neutral stability [14]. Thus the zeroth-order problem gives

$$\theta_0 = F = -\chi\xi_1 + \varphi(\tau)\,, \tag{2.35}$$

$$p_0 = \chi R_0 \xi_1[-Z+1] + g\xi_1 + R_0\varphi[Z-1]\,, \tag{2.36}$$

$$u_0 = \chi\xi_1'\left[R_0\left(-\frac{Z^3}{6}+\frac{Z^2}{2}-\frac{13}{20}Z\right)+48Z\right]+g\xi_1'\left(\frac{Z^2}{2}-Z\right) \tag{2.37}$$

and

$$w_0 = \chi\xi_1''\left[R_0\left(\frac{Z^4}{24}-\frac{Z^3}{6}+\frac{13}{40}Z^2\right)-24Z^2\right]+g\xi_1''\left(-\frac{Z^3}{6}+\frac{Z^2}{2}\right) \tag{2.38}$$

with

$$\chi \equiv \frac{g}{72}\,\frac{1}{1-R_0/120}\,.$$

ii) ***First-order problem.*** The next approximation in the ε-expansion is

$$\frac{\partial u_1}{\partial X}+\frac{\partial w_1}{\partial Z}=0\,, \tag{2.39}$$

$$\frac{\partial^2 u_1}{\partial Z^2}-\frac{\partial p_1}{\partial X}=-\frac{\partial^2 u_0}{\partial X^2}+\frac{1}{\mathrm{Pr}}\left[u_0\frac{\partial u_0}{\partial X}+w_0\frac{\partial u_0}{\partial Z}\right], \tag{2.40}$$

$$\frac{\partial p_1}{\partial Z}-R_0\theta_1 = R_1\theta_0+\frac{\partial^2 w_0}{\partial Z^2}\,, \tag{2.41}$$

$$\frac{\partial^2\theta_1}{\partial Z^2}=-\frac{\partial^2\theta_0}{\partial X^2}+u_0\frac{\partial\theta_0}{\partial X}-w_0 \tag{2.42}$$

together with the b.c. At $Z=0$

$$u_1 = w_1 = \partial\theta_1/\partial Z = 0 \tag{2.43}$$

and at $Z=1$

$$w_1 = \xi_1' u_0 - \xi_1\frac{\partial w_0}{\partial Z}\,, \tag{2.44}$$

$$\frac{\partial u_1}{\partial Z} = -M_0\frac{\partial\theta_1}{\partial X}+M_0\xi_1' - M_1\frac{\partial\theta_0}{\partial X}-\xi_1\frac{\partial^2 u_0}{\partial Z^2}-\frac{\partial w_0}{\partial X}\,, \tag{2.45}$$

$$p_1 = g\xi_2 + 2\frac{\partial w_0}{\partial Z}-\xi_1\frac{\partial p_0}{\partial Z} \tag{2.46}$$

and

$$\frac{\partial\theta_1}{\partial Z}=0\,. \tag{2.47}$$

We see that the left-hand sides of (2.39)-(2.42) and (2.18)-(2.21) are identical and will be so at all orders in ε. Thus instead of the energy relation (2.33) to solve the problem we could have used Fredholm's alternative. Note also that in (2.46) we have ξ_2 as in (2.24) we had ξ_1.

A similar procedure to the method sketched for the zeroth-order problem, using again the energy equation to the appropriate order in ε, after elementary but lengthy algebra, provides

$$\theta_1 = \chi^2(\xi_1')^2\left[\left(\frac{Z^6}{120}-\frac{Z^4}{60}+\frac{Z^3}{120}\right)R_0 - 3Z^4+4Z^3\right]+ \\ +\chi\xi_1''\left[R_0\left(-\frac{Z^6}{720}+\frac{Z^5}{300}-\frac{Z^4}{480}\right)+\frac{3}{5}Z^5-Z^4+\frac{Z^2}{2}\right]+\Gamma(X,\tau)\,, \tag{2.48}$$

$$p_1 = \chi^2(\xi_1')^2\left[\left(\frac{Z^6}{720}-\frac{Z^5}{300}+\frac{Z^4}{480}\right)R_0^2+R_0\left(-\frac{3Z^5}{5}+Z^4\right)\right]+ \\ +\chi\xi_1''\left[\frac{R_0^2}{120}\left(\frac{-Z^7}{42}+\frac{Z^6}{15}-\frac{Z^5}{20}\right)+ \right. \\ \left. +R_0\left(\frac{Z^5}{10}-\frac{Z^4}{5}+\frac{Z^3}{3}-\frac{Z^2}{5}+\frac{Z}{20}\right)-36Z^2+24Z\right]+ \\ +R_0\Gamma Z+\Pi(X,\tau)+R_1\varphi(\tau)Z-R_1\chi\xi_1 Z\,, \tag{2.49}$$

$$u_1 = \chi^2[(\xi_1')^2]'\left[\left(\frac{Z^8}{8\cdot7\cdot720}-\frac{Z^7}{7\cdot25\cdot72}+\frac{Z^6}{14\,400}\right)R_0^2+R_0\left(-\frac{Z^7}{70}+\frac{Z^6}{30}\right)\right]+ \\ +\chi\xi_1''\left[R_0^2\left(-\frac{-Z^9}{9\cdot8\cdot7\cdot720}+\frac{Z^8}{7\cdot14\,400}-\frac{Z^7}{7\cdot14\,400}\right)+\right. \\ \left.+R_0\left(\frac{Z^8}{8\cdot70}-\frac{Z^7}{210}+\frac{Z^6}{40}-\frac{Z^4}{30}+\frac{Z^3}{60}\right)-6Z^4+8Z^3\right]+ \\ +\frac{\chi^2}{\mathrm{Pr}}[(\xi')^2]'\left[\left(\frac{Z^9}{7\cdot32\cdot72}-\frac{Z^7}{42\cdot120}+\frac{37Z^6}{144\,000}-\frac{Z^5}{6\,000}+\frac{Z^4}{19\,200}\right)R_0^2+\right. \\ \left.+R_0\left(-\frac{Z^7}{28}+\frac{29}{300}Z^6-\frac{11}{100}Z^5+\frac{Z^4}{20}\right)+\frac{36Z^6}{5}-\frac{72Z^5}{5}+12Z^4\right]+ \\ +\mathrm{Ra}\,\Gamma''\frac{Z^3}{6}+\Pi'\frac{Z^2}{2}-R_1\xi_1'\frac{Z^3}{6}+VZ \tag{2.50}$$

and

$$w_1 = \chi^2[(\xi_1')^2]''\left[\left(\frac{-Z^9}{72\cdot7\cdot720}+\frac{Z^8}{7\cdot72\cdot200}-\frac{Z^7}{7\cdot72\cdot200}\right)R_0^2+\right. \\ \left.+R_0\left(\frac{Z^8}{560}-\frac{Z^7}{210}\right)\right]+\chi\xi_1''\left[R_0^2\left(\frac{Z^{10}}{72\cdot7\cdot7\,200}-\frac{Z^9}{18\cdot7\cdot7\,200}+\frac{Z^8}{16\cdot7\cdot7\,200}\right)+\right. \\ \left.+R_0\left(-\frac{Z^9}{72\cdot70}+\frac{Z^8}{24\cdot70}-\frac{Z^6}{240}+\frac{Z^5}{150}-\frac{Z^4}{240}\right)+\frac{6}{5}Z^6-2Z^4\right]+ \tag{2.51}$$

$$+\frac{\chi^2}{\mathrm{Pr}}[(\xi')^2]''\left(\frac{-Z^9}{7\cdot 4\cdot 72\cdot 72}+\frac{Z^8}{7\cdot 8\cdot 720}-\frac{37Z^7}{7\cdot 72\cdot 2000}+\frac{Z^6}{72\cdot 500}-\frac{Z^5}{96\,000}\right)R_0^2+$$
$$+R_0\left(\frac{Z^8}{7\cdot 32}-\frac{29}{7\cdot 300}Z^7+\frac{11}{600}Z^6-\frac{Z^5}{100}\right)-\frac{36}{7\cdot 5}Z^7+\frac{12}{5}Z^6-\frac{12}{5}Z^5\Bigg]-$$
$$-R_0\Gamma''\frac{Z^4}{24}-\Pi''\frac{Z^3}{6}+R_1\chi\xi_1''\frac{Z^4}{24}-V'\frac{Z^2}{2}$$

together with an equation for ξ_1 which comes from applying the solvability condition using the integrated energy equation (2.17) to order ε^4. This equation is

$$\frac{\partial\xi}{\partial\tau}+\xi_1^{\mathrm{iv}}\left[-\frac{241}{11\cdot 27\cdot 35}q^2+\frac{19\cdot 2}{9\cdot 105}q+\frac{1}{15}\right]+ \tag{2.52}$$
$$+\xi_1''\left[\frac{1-q}{\chi}+\left(\frac{M_1}{48}+\frac{R_1}{320}\right)\right]+\xi_1+[\xi_1^2]''[1+q]+$$
$$+\chi[(\xi_1')^2]''\left[-\frac{4}{7\cdot 9}q^2+\frac{397}{7\cdot 90}q-\frac{13}{20}\right]+\frac{\chi}{\mathrm{Pr}}[(\xi_1')^2]''\left[\frac{-8}{7\cdot 5\cdot 9}q^2+\frac{11\cdot 19}{9\cdot 7\cdot 20}q-\frac{1}{5}\right]+$$
$$+\chi^2[(\xi_1')^3]'\left[-\frac{16\cdot 8}{35\cdot 81}q^2+\frac{8}{35\cdot 3}q-\frac{48}{35}\right]=0$$

with $q=R_0/320$. Thus, in order to obtain the temperature, velocity and pressure fields, we must solve eq. (2.52) which is in fact the *nonlinear* evolution equation for the deformable liquid-air interface.

3. – Nature of the bifurcation of the nonlinear steady convective state.

The nonlinear evolution equation of the liquid-air interface, eq. (2.52), can be written in a more compact and clear form by rescaling the space coordinate. Taking

$$\tilde{X}=\left[-\frac{241}{11\cdot 27\cdot 35}q^2+\frac{38}{945}q+\frac{1}{15}\right]^{-\frac{1}{2}}X, \tag{3.1}$$

eq. (2.52) becomes

$$\frac{\partial\xi_1}{\partial\tau}+\xi_1^{\mathrm{iv}}+2m\xi_1''+\xi_1+D[(\xi_1)^2]''+E[(\xi_1')^2]''+H[(\xi_1')^3]'=0 \tag{3.2}$$

with

$$m=\frac{1}{2}\left[\frac{1-q}{\chi}+\left(\frac{M_1}{48}+\frac{R_1}{320}\right)\right]\left[\frac{-241}{11\cdot 27\cdot 35}q^2+\frac{38}{945}q+\frac{1}{15}\right]^{-\frac{1}{2}}, \tag{3.3}$$

which plays the role of the control parameter. Note indeed that neither M_1 nor R_1 taken separately are the relevant control parameters but rather the

combination of both as in the bracketed term. $S=M/48+\mathrm{Ra}/320$ defines the critical line. The critical state, for $\mathrm{Bi}^0=0$, corresponds to $S_0=1$, eq. (2.34), whereas $S_1=M_1/48+R_1/320$ accounts for the departure from the critical value. Then S_1 gives the first-order correction when Bi^0 is very small albeit nonzero. The other quantities in eq. (3.2) are

$$D\equiv[1+q]\left[-\frac{241}{11\cdot 27\cdot 35}q^2+\frac{38}{945}q+\frac{1}{15}\right]^{-\frac{1}{2}}, \tag{3.4}$$

$$E\equiv\chi\left\{-\frac{4}{63}q^2+\frac{397}{630}q-\frac{13}{20}+\frac{1}{\mathrm{Pr}}\left[-\frac{8}{315}+\frac{209}{1260}q-\frac{1}{5}\right]\right\}\cdot \left[-\frac{241}{11\cdot 27\cdot 35}q^2+\frac{38}{945}q+\frac{1}{15}\right]^{-1} \tag{3.5}$$

and

$$H\equiv\chi^2\left[-\frac{128}{35\cdot 81}q^2+\frac{8}{105}q-\frac{48}{35}\right]\left[-\frac{241}{11\cdot 27\cdot 35}q^2+\frac{38}{945}q+\frac{1}{15}\right]^{-1}. \tag{3.6}$$

To study the onset of the convective branch we now assume infinitesimal disturbances. We set

$$\xi_1\sim\exp[ik\tilde{X}+\omega\tau]. \tag{3.7}$$

Thus, neglecting the nonlinear terms, we obtain the dispersion relation

$$\omega-2mk^2+k^4+1=0. \tag{3.8}$$

At the neutral states $\omega=0$. Then when $\mathrm{d}\omega/\mathrm{d}k=0$ we have the critical state. It follows from eq. (3.8)

$$m_c=1 \quad\text{and}\quad k_c=1. \tag{3.9}$$

For a slight departure of Bi^0 from zero, these are the corrections to the critical line, eq. (2.34), and to the wave number, respectively.

The nature of the bifurcation in the neighbourhood of the critical state, $m_c=1$, can be studied by considering small departures around m_c. We set

$$m=m_c(1+\delta^2) \tag{3.10}$$

with δ a smallness parameter yet to be determined.

To be consistent we rescale the time

$$\tilde{\tau}=\delta^2\tau. \tag{3.11}$$

We now seek solutions of eq. (3.2) in the form

$$\xi_1 = \delta\zeta_1(\tilde{\tau}, \tilde{x}) + \delta^2\zeta_2(\tilde{\tau}, \tilde{x}) + \delta^3\zeta_3(\tilde{\tau}, \tilde{x}) + O(\delta^4)\,. \tag{3.12}$$

As in the previous sections δ is an ordering parameter. To first-order approximation we have

$$\xi_1 = A_1(\tilde{\tau}) \cos \tilde{x}\,, \tag{3.13}$$

whereas to the second order

$$\zeta_2 = A_2(\tilde{\tau}) \cos(\tilde{x} + \Phi) + A_1^2(D - E)\,\frac{2}{9} \cos 2\tilde{x}\,. \tag{3.14}$$

The solvability condition (Fredholm alternative) for the third-order equation in the δ-hierarchy originated from the nonlinear eq. (3.2) implies the following relationship:

$$\frac{\mathrm{d}A_1}{\mathrm{d}\tilde{\tau}} = 2A_1 + A_1^3\left[\frac{3}{4}H + \frac{2}{9}(D - E)(D + 2E)\right], \tag{3.15}$$

which is a form of Landau equation in the slow time scale. Therefore, when the coefficient of A_1^3 is negative, the bifurcation is direct and, when it becomes positive, the bifurcation is inverted. We shall not dwell on this point any further for the sign of this coefficient depends on the actual values of the parameters in an experiment.

4. – The case of thin liquid layers and/or microgravity conditions.

Until now we have restricted ourselves to situations where $G = O(\varepsilon^{-1})$, that may correspond to conditions on Earth and arbitrarily thick liquid layers. However, for experiments conducted aboard a spacecraft like the Shuttle with thin liquid layers, G could be of order unity (ε^0). Under these circumstances, the appropriate scaling is rather (we still keep $\mathrm{Bi}^* \equiv \varepsilon^2$)

$$\xi = \sum_{i=1}^{\infty} \varepsilon^i \xi_i\,, \tag{4.1}$$

$$u = \varepsilon^{\frac{1}{2}} \sum_{i=1}^{\infty} \varepsilon^i u_i\,, \tag{4.2}$$

$$w = \varepsilon \sum_{i=1}^{\infty} \varepsilon^i w_i\,, \tag{4.3}$$

$$P = \sum_{i=1}^{\infty} \varepsilon^i P_i\,, \tag{4.4}$$

$$\theta = \sum_{i=1}^{\infty} \varepsilon^i \theta_i\,, \tag{4.5}$$

$$\mathrm{Ra} = R_0 + \varepsilon R_1 + \dots\,, \tag{4.6}$$

$$M = M_0 + \varepsilon M_1 + \dots\,. \tag{4.7}$$

Note that with respect to the former scaling (2.7)-(2.11) the main difference is that all the variables have increased one order in ε, except the surface deformation. Also the leading order of the product $G\xi$ has increased, being now $G\xi = O(\varepsilon)$, whereas formerly was of order unity.

Inserting this scaling into eqs. (1.24)-(1.26) and boundary conditions at $z = 0$ (1.29), (1.30) and at $z = 1$ (1.31)-(1.33), (1.42), (1.43), we obtain again a hierarchy of linear equations in powers of ε. Solving the first-order equation and accounting for the solvability condition to the following higher order, we get

$$(M_0/48 + R_0/320 - 1)\,2G/3 + M_0(1 - R_0/120) = 0\,, \tag{4.8}$$

which defines the critical line for the transition to steady cellular convection. As expected, in the limit of G going to infinity, it coincides with the result obtained in the previous section, eq. (2.34). Besides, for a stress-free upper surface ($M = 0$), the onset of buoyancy-driven instability is at $\mathrm{Ra} = 320$ predicted by other authors [38-40]. Note that the latter critical value does not depend on G, whereas, if buoyancy is negligible, the critical Marangoni number is

$$M_0 = 48/(1 + 72/G)\,, \tag{4.9}$$

which shows a drastic dependence on G. In particular, in the limit of vanishing G, M_0 goes to zero and a vanishing small heating suffices to make unstable the liquid layer. This was also predicted by other authors [13].

Finally, to the lowest-order approximation in the hierarchy we also obtain

$$(1 - R_0/120)\,\theta_1(\varepsilon^{\frac{1}{2}}x, \varepsilon^2 t) = -G\xi_1(\varepsilon^{\frac{1}{2}}x, \varepsilon^2 t)/72 + \varphi(\varepsilon^2 t)\,, \tag{4.10}$$

that relates the first-order temperature disturbance θ_1 to the interfacial deformation ξ_1. One is tempted to say that one quantity tightly *slaves* the other. We obtain from eq. (4.10) that θ_1 is maximum where ξ_1 is minimum provided $\mathrm{Ra} < 120$, which agrees with Bénard's finding [1, 2]. However, for higher values of the Rayleigh number (*i.e.* $\mathrm{Ra} > 120$) we have the opposite structure as predicted long time ago by JEFFREYS [41] for buoyancy-driven convection.

Solving the second-order approximation and accounting for the corresponding solvability condition, this yields the nonlinear evolution equation for the interfacial deformation

$$Q_1\,\partial\xi_1/\partial\tau + 2S_1Q_2\xi_1'' + Q_3\xi_1^{\text{iv}} + Q_4\xi_1 + \frac{1}{2}Q_5(\xi_1')^2 + Q_6[(\xi_1)^2]'' - \left(\varphi + \frac{d\varphi}{d\tau}\right)Q_7 - \varphi\xi_1''Q_8 = 0 \tag{4.11}$$

with

$$S_1 = (1 - R_0/120 + G/72)\,3M_1/2G + [(G/120 - 1)/(1 - R_0/120 + G/72)]\,R_1/192\,. \tag{4.12}$$

S_1 plays the role of the bifurcation parameter. For convenience to have a compact form (4.11) we have used

$$Q_1 \equiv (1 - 8q/3)(1 - 16q) + G(1/8 - 8q/9) + G^2/216\,, \tag{4.13}$$

$$Q_2 \equiv (1 - 8q/3 + G/72)\,G/6\,, \tag{4.14}$$

$$Q_3 \equiv [24/5 - 64\cdot 46/315q + 29\cdot 8q^2/315 + 320q^3/693 + G(1/45 + 38q/2835 - 241\cdot 64q^2/6237)]\,G/72 + (1-q)(1 - 8q/3)/3C\,, \tag{4.15}$$

$$Q_4 \equiv (G/3 - 40q)\,G/72\,, \tag{4.16}$$

$$Q_5 \equiv (1 - 8q/3)(40q - G/3)\,, \tag{4.17}$$

$$Q_6 \equiv (q-1)(1 - 8q/3)\,320q/6 + (-12 - 268q/3 + 64q^2/3)\,G/72 + (q+1)\,G^2/216\,, \tag{4.18}$$

$$Q_7 \equiv G/3 - 40q\,, \tag{4.19}$$

$$Q_8 \equiv (q-1)\,320q/3 \tag{4.20}$$

and

$$q \equiv R_0/320\,. \tag{4.21}$$

Equation (4.11) is now the analog to eq. (2.52) when $G \simeq O(\varepsilon^0)$. The condition that volume must be conserved

$$\int_0^L \xi_1\,dX = 0 \tag{4.22}$$

with periodic boundary conditions at $X = 0$ and $X = L$ determines $\varphi(\tau)$.

Now we rewrite eq. (4.11) in a more universal form. We define

$$\tilde{x} = (Q_4/Q_3)^{\frac{1}{4}}\,X\,, \tag{4.23a}$$

$$\tilde{t} = \tau Q_4/Q_1\,, \tag{4.23b}$$

$$\zeta = \xi_1 Q_5 (Q_4 Q_3)^{-\frac{1}{2}}\,, \tag{4.23c}$$

and then eq. (4.11) becomes

$$\frac{\partial\zeta}{\partial\tilde{t}} + 2m\zeta'' + \zeta^{\text{iv}} + \zeta + \frac{1}{2}(\zeta')^2 + D(\zeta^2)'' - E\left(\Phi + \frac{Q_4}{Q_1}\frac{d\Phi}{d\tilde{t}}\right) - H\Phi\zeta'' = 0 \tag{4.24}$$

with

$$m \equiv S_1 Q_2 (Q_4 Q_3)^{-\frac{1}{2}}\,, \tag{4.25}$$

$$D \equiv Q_6/Q_5\,, \tag{4.26}$$

$$E \equiv Q_5 Q_7 (Q_4^3 Q_3)^{-\frac{1}{2}}\,, \tag{4.27}$$

$$H \equiv Q_8 (Q_4 Q_3)^{-\frac{1}{2}}\,. \tag{4.28}$$

Using (4.22) we obtain

$$\Phi = \exp[-Q_1\tilde{t}/Q_4]\left[\int_0^{\tilde{t}} \frac{d\tilde{t}}{2EL}\int_0^L d\tilde{x}\,Q_1(\partial\zeta/\partial\tilde{x})^2 \exp[Q_1\tilde{t}/Q_4]/Q_4 + \Phi_0\right]. \tag{4.29}$$

Equation (4.24) looks very much like eq. (3.2) and again, as in sect. 3, we study the onset of convection by considering infinitesimal disturbances. We set

$$\zeta \sim \exp[ik\tilde{x} + \omega\tilde{t}]\,. \tag{4.30}$$

Thus, neglecting the nonlinear terms, we get the same dispersion relation eq. (3.8) with critical values $m_c = 1$ and $k_c = 1$, respectively.

The nature of the bifurcation can also be studied by introducing (3.10) and (3.11),

$$m = m_c(1 + \delta^2)\,, \tag{4.31}$$

$$\bar{\tau} = \delta^2\tilde{t} \tag{4.32}$$

and seeking solutions of eq. (4.24) in the form (3.12). We set

$$\zeta = \delta\zeta_1(\bar{\tau}, \tilde{x}) + \delta^2\zeta_2(\bar{\tau}, \tilde{x}) + \delta^3\zeta_3(\bar{\tau}, \tilde{x}) + O(\delta^4)\,, \tag{4.32}$$

where again δ is an ordering parameter. To first-order approximation in δ

$$\zeta_1 = A_1(\bar{\tau})\cos\tilde{x}\,, \tag{4.33}$$

whereas to the second order

$$\zeta_2 = A_2(\bar{\tau})\cos(\tilde{x} + \Phi) + A_1^2(\bar{\tau})\left[\frac{1}{8} + D\right]\frac{2}{9}\cos 2\tilde{x}\,. \tag{4.34}$$

To obtain ζ_2 we need the solvability condition (Fredholm alternative) for the third-order equation, which as in (3.15) yields a Landau equation

$$\frac{\mathrm{d}A_1}{\mathrm{d}\tau} = 2A_1 + A_1^3\left[\left(\frac{1}{8}+D\right)(D-1)\frac{2}{9}-\frac{H}{4E}\right]. \tag{4.35}$$

Then, as in the preceding section, the sign of the coefficient of A_1^3 determines the nature of the bifurcation, direct or inverted.

5. – The simplest case in microgravity: Kuramoto-Velarde equation, bifurcations and chaos.

When either buoyancy effects are negligible ($q = 0$) or surface tension inhomogeneities are absent ($q = 1$) the last term in eq. (4.24) vanishes. Then the evolution of the interfacial deformation is given by

$$\frac{\partial\zeta}{\partial t} + 2m\zeta'' + \zeta^{\mathrm{iv}} + \zeta + \frac{1}{2}(\zeta')^2 + D(\zeta^2)'' - \frac{1}{2L}\int_0^L(\zeta')^2\,\mathrm{d}\tilde{x} = 0\,, \tag{5.1}$$

with periodic b.c.: $\zeta(\tilde{x} + L, \tilde{t}) = \zeta(\tilde{x}, \tilde{t})$.

With $q = 0$, $D = \frac{1}{2}(1 - G/36)$ and thus according to eq. (4.36) it appears that the bifurcation is direct for G smaller than 45 and inverted otherwise. In the opposite case, $q = 1$, the bifurcation is direct for G smaller than 60 and inverted otherwise.

We shall refer now to the case $q = 0$. For this case under appropriate change of scales and variables, respectively, eq. (5.1) becomes

$$u_t + 4u_{xxxx} + \alpha\left[u_{xx} + \frac{1}{2}(u_x)^2 + \gamma(uu_x)_x\right] + \beta u - (\alpha/4\pi)\int_0^{2\pi}(u_x)^2\,\mathrm{d}x = 0\,. \tag{5.2}$$

where the subscripts denote time (t) and space (x) derivatives.

It suffices to define $u = \zeta/2m$, $t = \tilde{t}\,16m^2/\alpha^2$, $x = \tilde{x}(8m\alpha)^{\frac{1}{2}}$, $\beta = \alpha^2/16m^3$ and $\gamma = 2D$.

Recently, HYMAN and NICOLAENKO [42] have carried out an extensive numerical study of eq. (5.2) with $\beta = 0$, which corresponds to large values of m, much larger than unity, the critical value for the onset of steady cellular convection. They have also taken $\gamma = 1$, which corresponds to $D = \frac{1}{2}$. A comprehensive picture of their findings is given in fig. 2. We see that at first, as expected, there is a pitchfork bifurcation into a unimodal steady cellular state. Then at the second bifurcation point a travelling wave appears. Later on this state undergoes a Hopf bifurcation into an invariant circle. This first sequence does not lead to further bifurcations and transition to chaos. Rather a bit later it undergoes a reverse homoclinic bifurcation and so on. All these calculations were done with an error tolerance of 10^{-8} per unit time step.

What is remarkable and indicates some universal properties of the K-V equation (5.2) is that CILIBERTO and RUBIO [43, 44] have found quite an identical cascade of bifurcations for the evolution of the temperature profile in Bénard-Rayleigh buoyancy-driven convection.

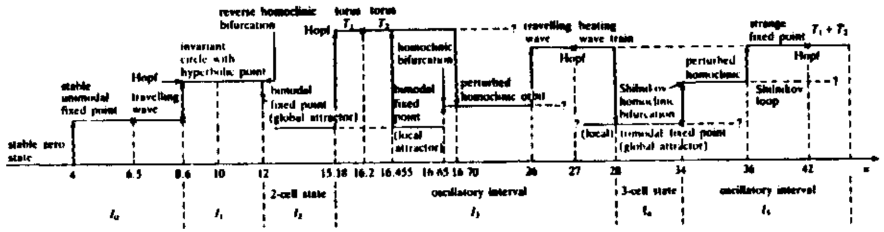

Fig. 2. – Bifurcation picture of the Kuramoto-Velarde equation for interfacial instabilities in microgravity using *alpha* as the bifurcation parameter.

6. – A glimpse at oscillatory Bénard-Marangoni convection or how to sustain gravity-capillary waves in a liquid layer under microgravity.

In the previous sections we have focused our attention only to the kind of instabilities leading to the classical Bénard convective cells, *i.e.* those which arise as stationary super- or subcritical bifurcations with vanishing imaginary part of the eigenvalue. However, the liquid layer could be destabilized *via* oscillatory modes that in the case of an open boundary may appear as surface waves, thus providing a mechanism to sustain capillary-gravity waves in the layer.

To simplify the calculations let us consider the simplest model problem: an infinitely deep liquid layer subjected to a linear gradient of temperature, covered by an infinitely high air layer of negligible density and *dynamic* viscosity with a Newton heat transfer law at the interface. Further we assume negligible thermal liquid expansion (no buoyancy, $\mathrm{Ra} = 0$) and temperature-dependent surface tension $\sigma = \sigma(t)$, $\partial\sigma/\partial T \neq 0$. Then, the dispersion relation for capillary-gravity waves is obtained by solving the linear disturbance approximation to the continuity, Navier-Stokes and energy balance equations, eqs. (1.24)-(1.26), with the appropriate boundary conditions at the interface, eqs. (1.31)-(1.33). We also assume that all disturbances decay to zero at the bottom of the liquid layer.

For an adiabatic air-liquid interface (*i.e.* for a locally constant heat transfer flux, $Bi^* = 0$), the dispersion relation is

$$\Omega\left(\frac{\Omega}{\nu\tilde{k}^2}+2\right)^2-\frac{4m}{\tilde{k}}+\left(g+\frac{\sigma\tilde{k}^2}{\nu}\right)\frac{\Omega}{\nu^2\tilde{k}^3}+ \tag{6.1}$$
$$+\frac{\partial\sigma}{\partial T}\frac{\beta k^2}{\varrho\nu\Omega}\left[\frac{g+\sigma\tilde{k}^2/\varrho}{\nu\tilde{k}^3}\left(\frac{\nu m/\varkappa r-1}{\nu/\varkappa-1}-\frac{\tilde{k}}{r}\right)\right]+\frac{m/r-1}{\nu/\varkappa-1}\frac{2\Omega}{\tilde{k}^2}+\frac{\nu m/\varkappa r-1}{\nu/\varkappa-1}\frac{\Omega^2}{\nu k^4}=0\,,$$

where $\tilde{k}$ is the Fourier wave number, β is the vertical temperature gradient, Ω is the eigenvalue (frequency factor), whose real part determines stability, $m = (\tilde{k}^2 + \Omega/\nu)^{\frac{1}{2}}$ and $r = (\tilde{k}^2 + \Omega/\varkappa)^{\frac{1}{2}}$.

Note that, when the interfacial tension is constant, eq. (6.1) reduces to the well-known dispersion relation for capillary-gravity waves [45]. Generally, $\mathrm{Re}\,\Omega$ is negative due to the viscous damping of the liquid. However, when the surface tension changes with the temperature along the interface, the fourth term in eq. (6.1) affects this damping and under suitable conditions can induce a change of sign of $\mathrm{Re}\,\Omega$ thus amplifying an initial disturbance in the form of a surface wave.

For convenience in the discussion we now use (6.1) in dimensionless form. Using the capillary length, $l_c = (\sigma/\varrho g)^{\frac{1}{2}}$, as the space scale, l_c^2/ν as the time scale and introducing $\alpha = \Omega/\nu k^2$, $\tilde{k} = kl_c$, together with the Prandtl, capillary and Marangoni numbers, eq. (6.1) becomes

$$\alpha(\alpha+2)^2-4\alpha(1+\alpha)^{\frac{1}{2}}+\alpha(1+k^2)/PCk^2- \tag{6.2}$$
$$-\frac{M}{Pk^2\alpha}\left\{\frac{1+k^2}{PCk^2}\left[\frac{P((1+\alpha)/(1+P\alpha))^{-\frac{1}{2}}-1}{P-1}-(1+P\alpha)^{-\frac{1}{2}}\right]+\right.$$
$$\left.+\frac{[(1+\alpha)/(1+P\alpha)]^{\frac{1}{2}}-1}{P-1}2\alpha+\frac{P((1+\alpha)/(1+P\alpha))^{\frac{1}{2}}-1}{P-1}\alpha^2\right\}=0\,.$$

This dispersion relation can be solved exactly with a computer. However, one can proceed analytically to the end by just recalling that (6.2) with $M = 0$ possesses solutions for large values of $\mathrm{Im}\,\alpha$ describing damped travelling waves, *i.e.* capillary-gravity waves. Then we look for high-frequency ($\mathrm{Im}\,\alpha \equiv \omega \gg 1$) solutions of eq. (6.2) at marginal states, $\mathrm{Re}\,\alpha = 0$. Then an approximate solution to (6.2) can easily be obtained. For such a purpose an estimate of the values of the different parameters of the problem must be made. One such choice is $O(\omega^{-2})$, $M = O(\omega^{\frac{3}{2}})$ and $P = O(\omega^0) \equiv O(1)$. Then we obtain

$$4\omega^2+\frac{M}{Pk^2}\left(\frac{\omega}{2P}\right)^{\frac{1}{2}}=0\,, \tag{6.3a}$$

$$\omega^2-\frac{1+k^2}{PCk^2}=0\,, \tag{6.3b}$$

which yield $M(k)$ and $\omega(k)$. It clearly appears that the lowest value of the Marangoni number capable of sustaining capillary-gravity waves is $M_c = -7.93(P/C)^{\frac{1}{2}}$ at $k_c = \sqrt{5}/5$ and $\omega_c^2 = 6\sqrt{5}/PC$. Note that the critical Marangoni number appears *negative*, thus showing that for standard liquids, *i.e.* liquids whose surface tension decreases with increasing temperature, the heating must be from the air, to have sustained oscillations as described by LINDE [8]. There are liquids [46] that show a minimum in their surface tension as a function of temperature. For these liquids there is a possibility of sustaining capillary-gravity waves by heating the layer from the liquid side.

The solution of the problem also provides an explanation of the phenomena to be seen in an experiment. When the air is hotter than the liquid, the surface tension gradient along the interface produces convective motion in the liquid (Marangoni effect) that brings cold fluid from below. The effect at the interface is a motion in the direction of propagation of the wave and thus sustains the oscillatory state, provided the thermal gradient is large enough to overcome the viscous and heat diffusion damping. This result agrees with the experimental findings of LINDE [8]. In the opposite case, *i.e.* when the liquid is the hotter, the Marangoni effect brings warmer and warmer liquid and the effect at the interface opposes the propagation of the wave thus leading to a dynamic equilibrium in the form of steady cellular convection. The latter is a novel approach to understand the Marangoni effect in Bénard convection with a deformable air-liquid interface. Indeed, although cellular Bénard convection can appear even if we assume a negligible interfacial deformation, it is known [33, 34] that with a deformable air-liquid interface we need a higher Marangoni number for the onset of steady convective instability than in the case of a level surface.

Figure 3 shows the dependence of $\beta_c = (M_c, k_c, \omega_c)$ from the gravitational acceleration. Asymptotically, $\beta_c \sim g^{\frac{1}{2}}$ as g vanishes. Thus under microgravity conditions capillary-gravity waves and so oscillatory convection can be sus-

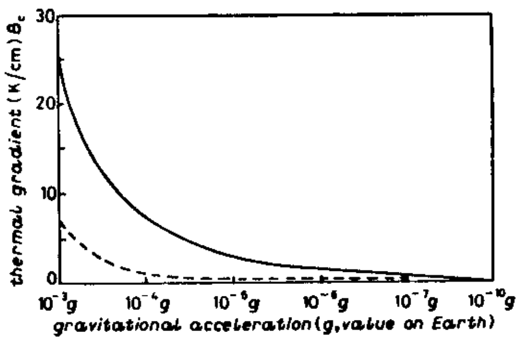

Fig. 3. – Critical thermal gradient for oscillatory convection as a function of gravity: —— water, – – – mercury; asymptotic behaviour $g^{\frac{1}{2}}$.

tained even for infinitesimal values of the thermal gradient. Table I provides estimates of the oscillation frequency for various liquids and two values of the gravitational acceleration to explicitly show the relevance of our finding for microgravity experiments.

TABLE I. – *Typical values for the relevant quantities in oscillatory instability.*

	Water		Mercury		Tin		$NaNO_3$	
β_s (K/cm)	$1.9\cdot10^3$	5.85	441	1.39	55	1.8	$5.2\cdot10^3$	16.54
period (s)	0.14	143	0.12	119.5	0.17	146	0.14	137.8
penetration depth (cm)	$1.5\cdot10^{-2}$	0.48	$0.5\cdot10^{-2}$	0.15	$6\cdot10^{-3}$	0.18	$1.8\cdot10^{-2}$	0.56
gravitational acceleration	g	$10^{-4}g$	g	$10^{-4}g$	g	$10^{-4}g$	g	$10^{-4}g$

N. B. For standard liquids the heating is from *above*.

For the case of double diffusion [47], *i.e.* the case of a binary liquid mixture with or without Soret effect, the threshold values are

$$M_T + M_s Le^{\frac{1}{2}} = -7.93(P/C)^{\frac{1}{2}} \tag{6.4}$$

and

$$M_T + M_s Le^{\frac{1}{2}}/(1 + Le^{\frac{1}{2}}) = -7.93(P/C)^{\frac{1}{2}}, \tag{6.5}$$

respectively. Here M_T denotes the (thermal) Marangoni number and M_s, by analogy, the solutal Marangoni or elasticity number. All other threshold values (wavelength, oscillation frequency) are the same. Thus, according to the signs of M_T and M_s, both Marangoni effects either compete or co-operate to sustain the waves at the open surface. For instance, an interesting experimental case is that of a benzene-methanol mixture whose Soret coefficient and thus its elasticity (solutal Marangoni) number changes sign according to the value of the benzene concentration. There are other cases similar to this [48].

Note that the oscillatory motion here described demands a diverging Marangoni number as the capillary number, *i.e.* the interfacial deformation vanishes. Other oscillatory motions can be triggered in a two-component Bénard layer even when C vanishes provided there is competition of both Marangoni numbers as pointed out a long time ago [49, 50]. Moreover, for two-layer liquids with a common *underformable* interface, several authors [11-13] also pointed out another possibility of oscillatory motion provided the upper phase transfers matter and/or heat across the interface to the lower phase.

Finally, when we compare these results with the analysis developed in

the previous sections we see that what remains to be done now is the study leading to the nonlinear equation albeit with a second-order time derivative whose linear oscillatory behaviour is described by eq. (6.1) or (6.2).

(*)Already done,it is dealt with in PART B of these Trieste ICTP Space Physics Workshop Notes.

7. – Conclusion.

In these notes we have illustrated the use of singular perturbation methods to obtain evolution equations for the liquid-air interfacial deformation in Bénard-Marangoni convection. One of the crucial assumptions in our analysis is the nonzero albeit small value assumed for the heat transfer Biot number at the bottom of the liquid layer. We have used this Biot number as a smallness parameter in our problem for at vanishingly small Biot numbers the onset of convective instability is with rather large cellular wavelengths and this drastically simplifies the problem.

As a by-product of our study we have rederived various linear stability predictions already known in the literature.

We have also shown how Bénard-Marangoni convection can sustain gravity-capillary waves.

Finally, in all cases we have shown the expected predictions for microgravity conditions. The latter are of importance nowadays due to the availability of the Spacelab and other laboratory facilities aboard spacecraft. The field of interfacial instabilities is at present gaining *momentum* due to those facilities and the fashion of research in space. Besides, interfacial phenomena are of importance in many realms of science and industry.

* * *

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REFERENCES

[1] H. BÉNARD: *Rev. Gén. Sci. Pures Appl.*, **11**, 1261 (1900).
[2] H. BÉNARD: *Ann. Chim. Phys. (Paris)*, **23**, 62 (1901).
[3] H. BÉNARD: *Proceedings of the III International Congress on Applied Mechanics*, Vol. **1** (1930), p. 120.
[4] M. G. VELARDE and C. NORMAND: *Sci. Am.*, **243**, 92 (1980).
[5] E. L. KOSCHMIEDER: *Adv. Chem. Phys.*, **26**, 177 (1974).

[6] J. C. Berg, A. Acrivos and M. Boudart: *Adv. Chem. Eng.*, **6**, 61 (1966).
[7] M. G. Velarde and J. L. Castillo: in *Convective Transport and Instability Phenomena*, edited by J. Zierep and H. Oertel jr. (Braun-Verlag, Karlsruhe, 1982).
[8] H. Linde: in *Dynamics and Instability of Fluid Interfaces*, edited by T. S. Sørensen, *Lecture Notes in Physics*, Vol. **105** (Springer-Verlag, Berlin, 1979), p. 75. The rest of the volume contains more papers about the subject.
[9] M. Block: *Nature (London)*, **178**, 650 (1956).
[10] J. R. A. Pearson: *J. Fluid Mech.*, **4**, 489 (1958).
[11] C. V. Sternling and L. E. Scriven: *A. I. Ch. E. J.*, **5**, 514 (1959).
[12] L. E. Scriven and C. V. Sternling: *Nature (London)*, **187**, 186 (1960).
[13] L. E. Scriven and C. V. Sternling: *J. Fluid Mech.*, **19**, 321 (1964).
[14] D. A. Nield: *J. Fluid Mech.*, **19**, 341 (1964).
[15] D. A. Nield: *J. Appl. Math. Phys. (Z. Angew. Math. Phys.)*, **17**, 226 (1966).
[16] K. A. Smith: *J. Fluid Mech.*, **24**, 401 (1966).
[17] E. L. Koschmieder: *J. Fluid Mech.*, **30**, 9 (1967).
[18] J. W. Scanlon and L. A. Segel: *J. Fluid Mech.*, **30**, 149 (1967).
[19] S. H. Davis and L. A. Segel: *Phys. Fluids*, **11**, 470 (1968).
[20] S. H. Davis: *J. Fluid Mech.*, **39**, 347 (1969).
[21] H. J. Palmer and J. C. Berg: *J. Fluid Mech.*, **47**, 779 (1971).
[22] M. Bentwich: *Appl. Sci. Res.*, **24**, 305 (1971).
[23] T. E. Hinkenbein and J. C. Berg: *Int. J. Heat. Mass Transfer* **21**, 1241 (1978).
[24] J. Pantaloni, R. Bailleux, J. Salan and M. G. Velarde: *J. Non-Equilib. Thermodyn.*, **4**, 201 (1979).
[25] J. R. Kraska and R. L. Sani: *Int. J. Heat Mass Transfer*, **22**, 535 (1979).
[26] S. H. Davis and G. M. Homsy: *J. Fluid Mech.*, **98**, 527 (1980).
[27] E. Guyon and J. Pantaloni: *C. R. Acad. Sci., Ser. B*, **290**, 301 (1980).
[28] M. Takashima: *J. Phys. Soc. Jpn.*, **50**, 2745 (1981). See also *J. Phys. Soc. Jpn.*, **47**, 1321 (1979).
[29] G. I. Sivashinsky: *Physica (Utrecht) D*, **4**, 227 (1982).
[30] V. L. Gertsberg and G. I. Sivanshinsky: *Prog. Theor. Phys.*, **66**, 1219 (1980).
[31] S. P. Lin and H. Brenner: *J. Colloid Interface Sci.*, **85**, 59 (1982).
[32] S. Rosenblatt, S. H. Davis and G. M. Homsy: *J. Fluid Mech.*, **120**, 123 (1982).
[33] J. L. Castillo and M. G. Velarde: *J. Fluid Mech.*, **125**, 463 (1982).
[34] J. L. Castillo and M. G. Velarde: *J. Colloid Interface Sci.*, **108**, 264 (1985).
[35] A. Cloot and G. Lebon: *J. Fluid Mech.*, **145**, 447 (1984).
[36] E. L. Koschmieder and M. I. Biggerstaff: *J. Fluid Mech.*, **167**, 49 (1986).
[37] P. L. García-Ybarra and M. G. Velarde: *Phys. Fluids*, **30**, 1649 (1987). See also M. G. Velarde, P. L. García-Ybarra and J. L. Castillo: *Physicochem. Hydrodyn.*, **9**, 387 (1987).
[38] C. Normand, Y. Pomeau and M. G. Velarde: *Rev. Mod. Phys.*, **49**, 581 (1977).
[39] E. M. Sparrow, R. J. Goldstein and V. K. Jonsson: *J. Fluid Mech.*, **18**, 513 (1964).
[40] D. T. Hurle, E. Jakeman and E. R. Pike: *Proc. R. Soc. London, Ser. A*, **296**, 469 (1967).
[41] H. Jeffreys: *Q. J. Mech. Appl. Math.*, **4**, 283 (1951).
[42] J. M. Hyman and B. Nicolaenko: *Coherence and chaos in the Kuramoto-Velarde equation*, preprint LA-UR-86-1388, Los Alamos National Laboratory, N.M. 87 545 (1986).
[43] S. Ciliberto and M. A. Rubio: *Phys. Rev. Lett.*, **58**, 2652 (1987).
[44] S. Ciliberto: in *Physicochemical Hydrodynamics: Interfacial Phenomena*, edited by M. G. Velarde (Plenum Press, New York, N. Y., to appear).

[45] B. G. Levich: *Physicochemical Hydrodynamics* (Prentice-Hall, Englewood Cliffs, N. J., 1962).
[46] R. Vochten and G. Petré: *J. Colloid Interface Sci.*, **42**, 320 (1973). See also J. C. Legros, M. C. Limbourg-Fontaine and G. Petré: *Acta Astron.*, **11**, 143 (1984).
[47] M. G. Velarde and R. S. Schechter: *Phys. Fluids*, **15**, 1707 (1972).
[48] M. G. Velarde and R. S. Schechter: *Chem. Phys. Lett.*, **12**, 312 (1971).
[49] J. L. Castillo and M. G. Velarde: *Phys. Lett. A*, **66**, 489 (1978).
[50] J. L. Castillo and M. G. Velarde: *J. Nonequilib. Thermodyn.*, **5**, 111 (1980).