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SPRING COLLEGE IN MATERIALS SCIENCE
ON
"CERAMICS AND COMPOSITE MATERIALS"
(17 April - 26 May 1989)

PLASTICITY AND FRACTURE OF CERAMIC MATERIALS
(Lectures V - VIII)

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Ceramics and Statistics

for ceramics

fracture strength varies – 100% and more

- with rising volume strength drops, $V\uparrow \rightarrow \bar{\sigma}\downarrow$
- with type of loading – variation in strength
  $\bar{\sigma}_{tensile} < \bar{\sigma}_{bending} < \bar{\sigma}_{pressure}$
- with processing and machining, thermal and mechanical history

- no experimental error, but systematic deviation!
- variation in results characteristically for brittle materials
- mean value $\bar{\sigma}$ of strength not sufficient – extreme values, especially the minimal values are important to characterize the safety of a construction

→ strength characterization of ceramics is directly connected to statistics

→ designer has to change his mind:
from absolute safety with metals and alloys to a reliable probability with ceramics

Weibull statistics and the problem of bending strength-measurement

fracture toughness K_{Ic} – macroscopic defects: notches, artificial sharp cracks
→ fracture mechanics

bending strength σ_B – microscopic, structural defects grain sized
→ statistics

Introduction to Weibull-statistics

It is an experimental fact that fracture stresses from 3-, 4- point bending- and tension tests are extreme-value distributed – a description by Weibull statistics is established;

the definition of the Weibull-function bases on the "weakest link" – concept (e.g. Jayatilaka, 1979):

- defects are statistically distributed, the probability to find one of certain dangerousness is for every part of the volume the same;
- defects do not affect each other
- fracture occurs, when the stress intensity K of the largest defect (critical defect) arrives at K_{Ic}; S_f = defect strength, the min. of all stresses causing fracture – the weakest link in the chain; $S_f = \frac{K_{Ic}}{\sqrt{\pi a_c}}$,
  for small defects: $Y \simeq 1$

the probability of survival is inverse proportional to the number of the links in the chain and thus the volume V (V = V for uniaxial stresses e.g. for loading in tension):

$$P_s = \exp\left[-\frac{V}{V_0}\phi(\cdot)\right]$$

$1/V_0$ volume density of strength-impairing flaws

the fracture probability $P_f = 1 - P_s$
(P_f = cumulative density function, c.d.f.),

for the unknown function $\phi(\cdot)$ Weibull assumed:

$$\phi(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m \qquad \text{for } \sigma > \sigma_u$$

$$\phi(\sigma) = 0 \qquad \text{for } \sigma \le \sigma_u$$

$$\Rightarrow \quad P_f = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m\right]$$

usual assumption $\sigma_u = 0$ → distribution two-parametric in m, σ_0:

m : Weibull modul; the higher, the more homogene the material; depends on the distribution of the defects

σ_0 : for constant m and V the mean strength $\bar{\sigma}$ rises with σ_0; usually the unknown V_0 is included into σ_0, then σ_0 gets a dimension of [stress]·[volume]$^{1/m}$

if there is no information on σ_0 and V_0 and the density of defects in the volume is low, the results are described in a sufficient way by the empirical approximate function

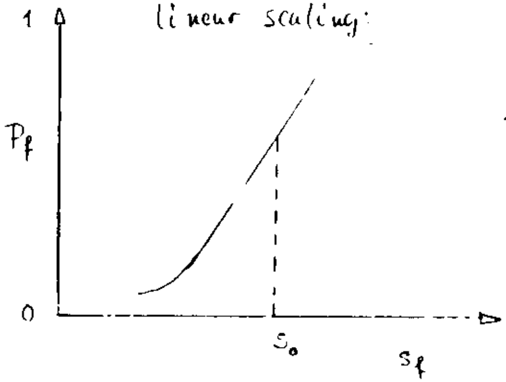
$$P_f = 1 - \exp\left[-\left(\frac{s_f}{s_0}\right)^m\right]$$

with s_f defect strength
s_0 normalizing parameter

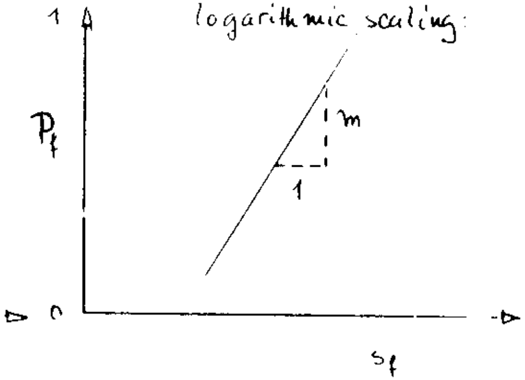
the approximate function only describes the results without any physical background!

the Weibull plot (c.d.f.):

linear scaling:

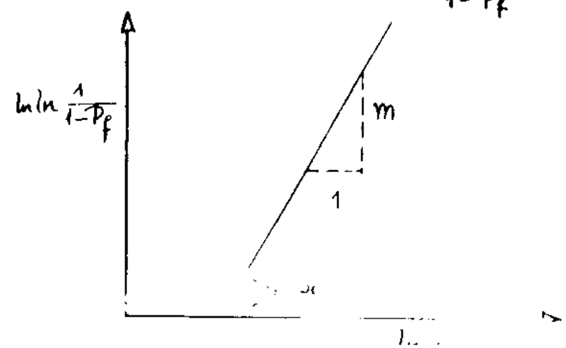


logarithmic scaling:



to get the Weibullparameters: linearize and plot

$$\ln\ln\frac{1}{1-P_f} = m\ln s_f - m\ln s_0$$



geometrical dependence:

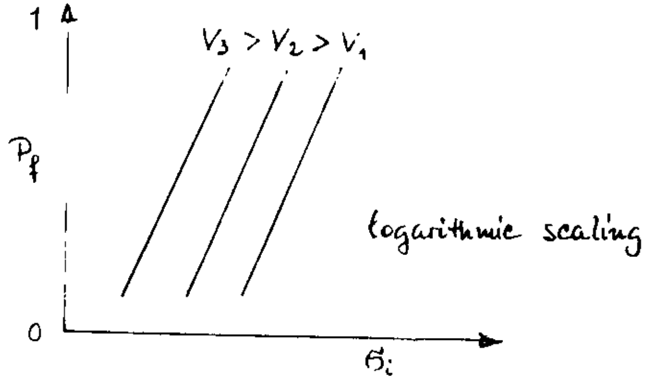
results depend on the loaded volume or on the loaded surface, in general on both;

assumption: surface defects from inadequate machining etc have no influence;

two batches of specimen with different volumes V_1 and V_2 of the same material with the same m:

$$P_f = 1 - \exp\left[-\frac{V_1}{V_0}\left(\frac{\sigma_1}{\sigma_0}\right)^m\right] = 1 - \exp\left[-\frac{V_2}{V_0}\left(\frac{\sigma_2}{\sigma_0}\right)^m\right]$$

$$\rightarrow \quad \frac{V_1}{V_2} = \left(\frac{\sigma_2}{\sigma_1}\right)^m \qquad \sigma_2 = \sigma_1 \left(\frac{V_1}{V_2}\right)^{1/m}$$



it is to expect that the strengths will rank as follows:
3-point bending > 4-point bending > tension

The problem of bending strength-measurement

up to now there exists no standard for bending strength-measurement, except the Japanese JIS R 1601-1981;
in the FRG a DIN-standard is to be published soon;
in the USA there exists a US-military standard and the problem is discussed in the ASTM C-28;
in europe with respect to the common market the problem is discussed by the Comité Européen de Normalisation (ECN);

for the DIN-standard the most important conditions will be: specimen's cross section 3×4 mm²,
flat loaded; spans 40/20 mm, loading rate such that fracture occurrs in 5-10 s;

the problems to find a universal testing procedure are numerous, three main problems will be discussed:

- the dependence on the testing device
- the dependence on the loading rate
- the influence of the surface condition

to check these points: pilot material
Al_2O_3 + 3 wt.% glassy phase

• The dependence on the testing device :

the 3-point bending as proposed for K_{Ic} - measurement, should not be applied to σ_B - measurement, because the loaded volume in 3-point bending is around one magnitude smaller than in 4-point bending;

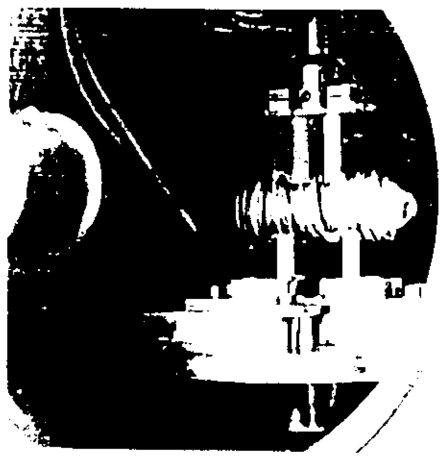
the bending device should have:

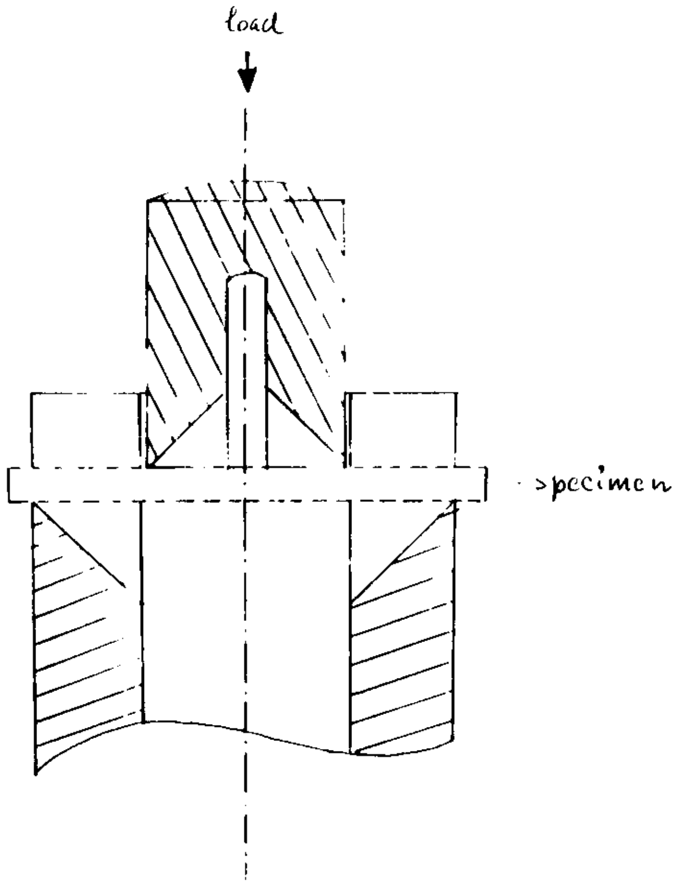
- high stiffness
- free moving loading and supporting rollers (rolling and sliding)
- at least the loading rollers should pivot around the axis of the specimen

it is to expect that these problems are difficult to solve, especially at high temperatures, when the should be e.g. from SiC - material;

three different loading systems are presented :

- a rolling, sliding and pivoting system; SiC - material; "freies Rollersystem" (Maytec)
- a fixed roller system; loading rods from alumina ϕ 10 mm, knive edges rounded to a radius of 1.5 mm; "festes Rollersystem" (MPI Stuttgart)
- a fixed roller system; concentric tubes from steel, alumina or SiC with knive edges (Bornhauser, 1984)





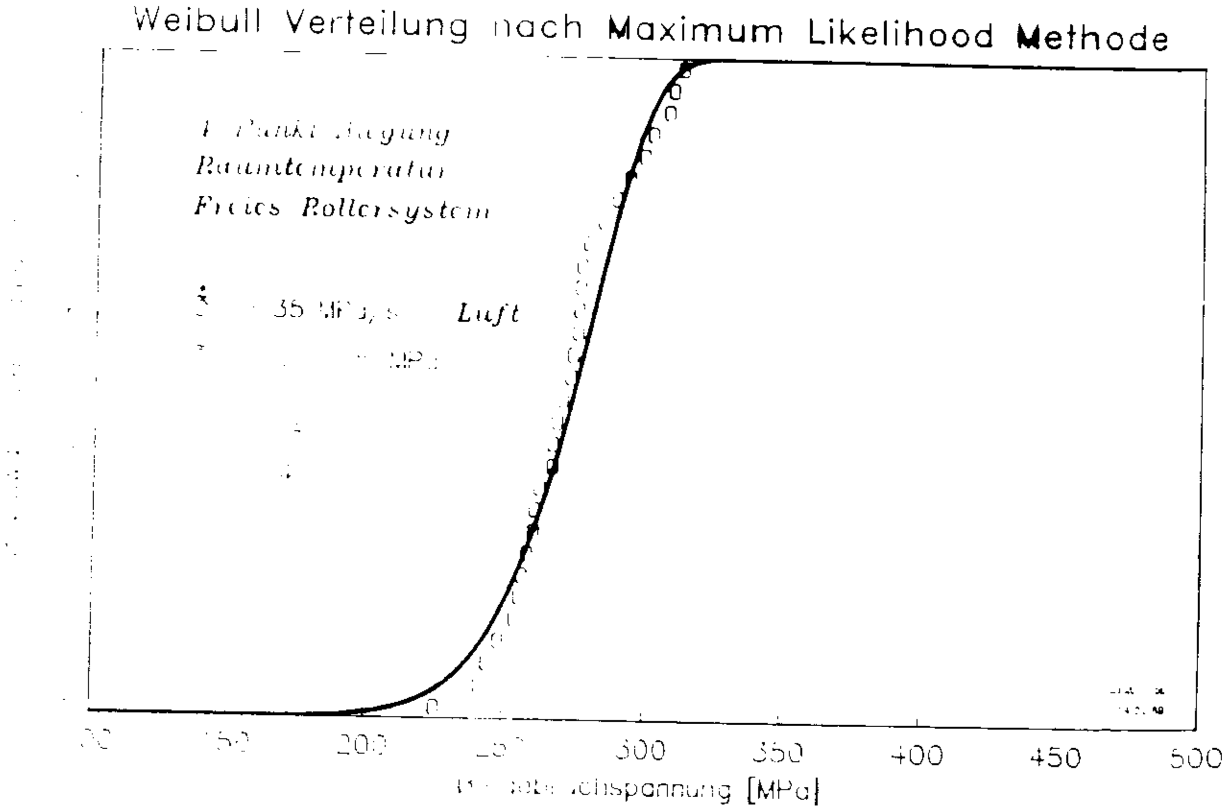
fixed roller system "Volksprüfer Jedermann"
(Bornhauser, 1984)

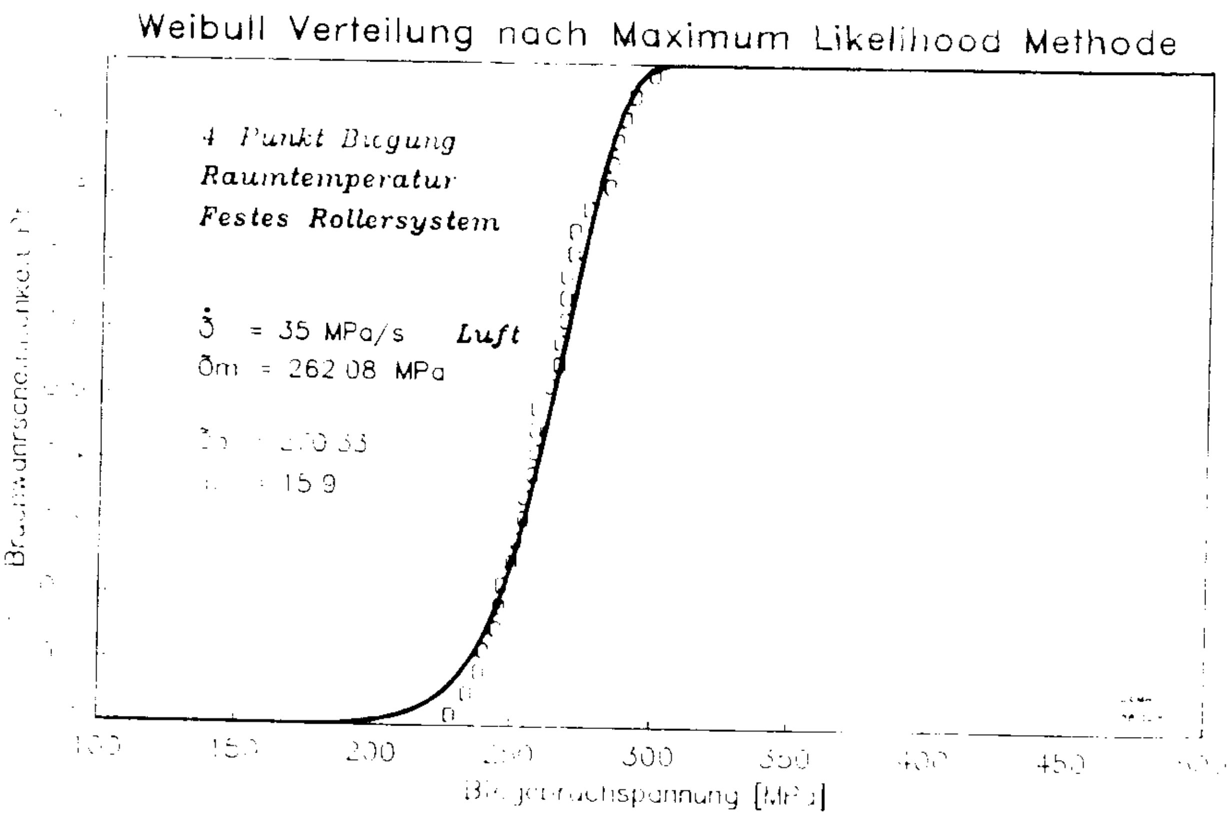
results:

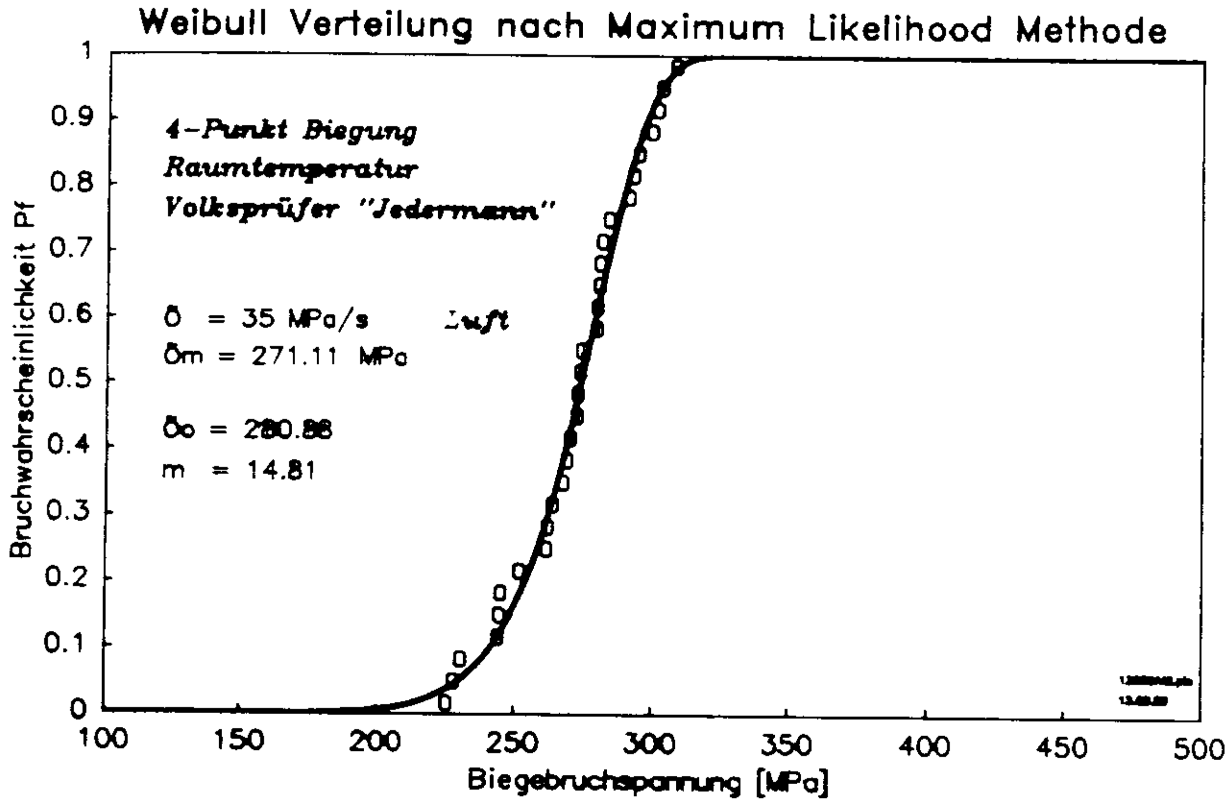
a loading rate of 50 MPa/s was applied, to take account for DIN and the practical requirements in industry;
the systems exhibit nearly identical results, only the mean value with the fixed roller system from MPI Stuttgart is located at a ~3,5% lower value – this results from a deviation in parallelism of the lower and upper surfaces of the specimen of ~0.8% – the other two systems have the possibility to pivot – the upper loading tube of Bornhauser's system has a camber and thus can pivot to some degree (diagrams on pages 40, 41, 42);

→ no influence of friction from the fixed roller system; but the systems should be able to pivot around the axis (Rief and Kromp, 1989; unpublished)

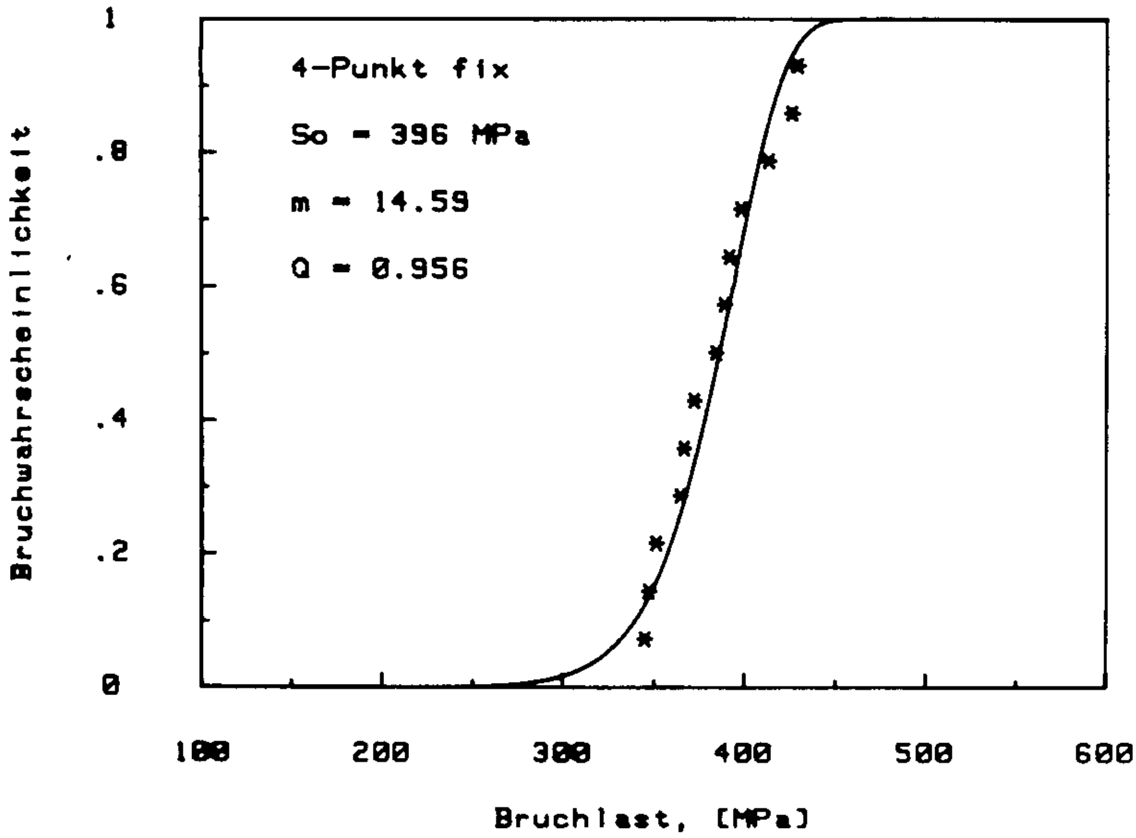
the strong influence of the loaded volume on the results is demonstrated by the diagrams on pages 43, 44:
a sintered SiC (SSiC) is loaded in 4-point and in 3-point bending in the fixed roller system, the difference in the mean values is around 15%
(Rief, Danzer and Kromp, 1988; unpublished)


Weibull Verteilung nach Maximum Likelihood Methode
Raumtemperatur
Freies Rollersystem
35 MPa/s
Luft
200
300
350
400
450
500
[MPa]

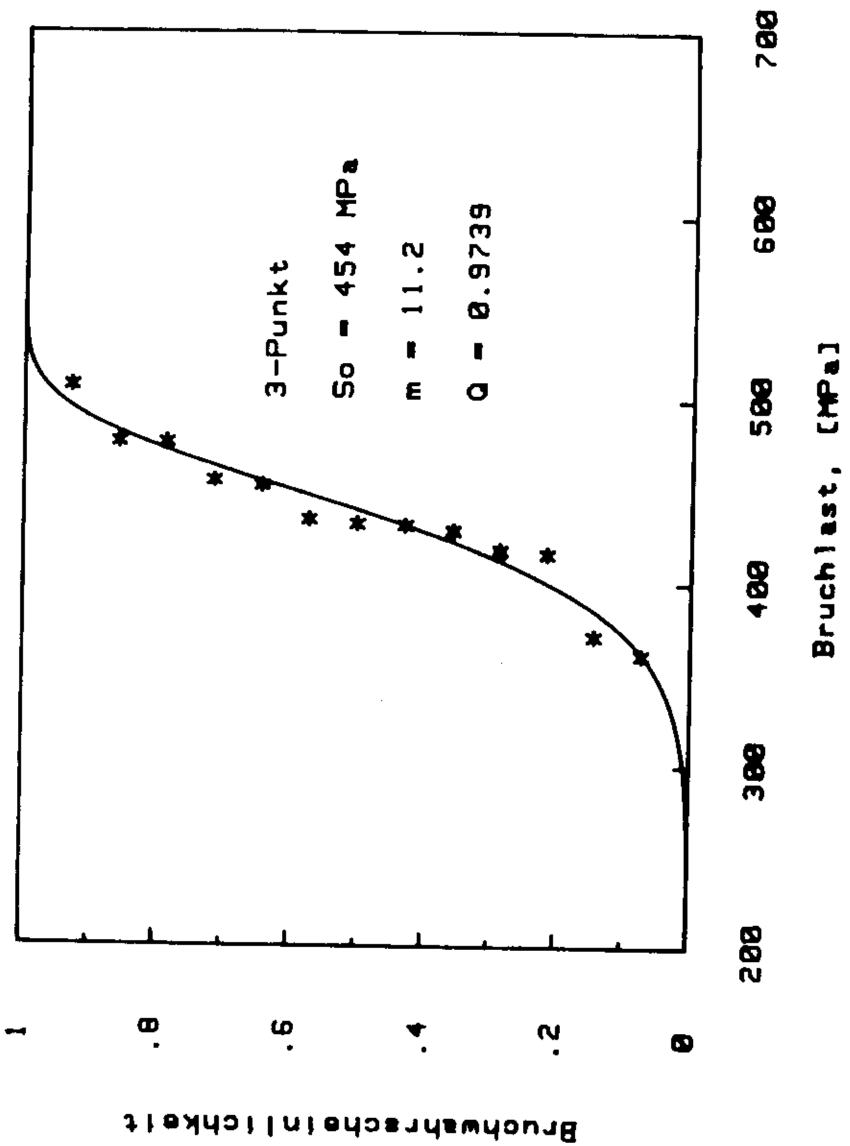

Weibull Verteilung nach Maximum Likelihood Methode
4 Punkt Biegung
Raumtemperatur
Festes Rollersystem
σ̇ = 35 MPa/s
Luft
σm = 262.08 MPa
15.9
100
150
200
300
450



Weibull Verteilung nach Maximum Likelihood Methode
4-Punkt Biegung
Raumtemperatur
Volksprüfer "Jedermann"
σ̇ = 35 MPa/s
Luft
σm = 271.11 MPa
σo = 280.88
m = 14.81
Bruchwahrscheinlichkeit Pf
Biegebruchspannung [MPa]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
100
150
200
250
300
350
400
450
500



4-Punkt fix
So = 396 MPa
m = 14.59
Q = 0.956
Bruchwahrscheinlichkeit
Bruchlast, [MPa]
0
.2
.4
.6
.8
1
100
200
300
400
500
600



- The dependence on the loading rate

there is always a joined influence of loading rate and temperature:

- at room temperature the influence of loading rate results from corrosion (subcritical crack growth)
- at high temperature the influence results from the weakened second phase

room temperature:

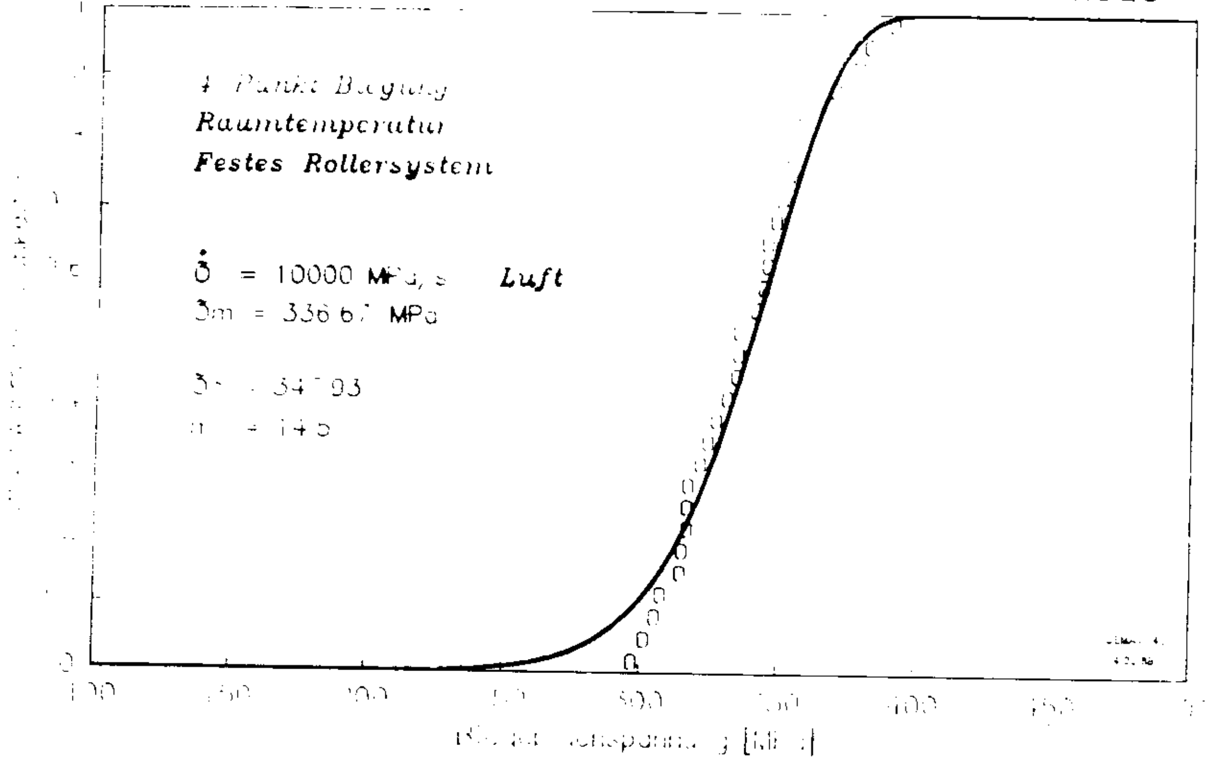
at a very high loading rate (10 000 MPa/s) the influence of corrosion is ruled out – the distributions in air and in vacuum are identical (pages 46, 47); the mean value of strength is 22% higher than at the low rate of 35 MPa/s! In the v-K curve these measurements are located probably beyond region III (see page 5);

for life-time prediction it is important to achieve an "inert strength" value, that is a value in the region of K_{Ic} (page 5), when no corrosive influence takes place; this value is measured at the low loading rate (35 MPa/s) in vacuum (page 48) – the strength level is mean between that of the low loading rate in air and that at the very high loading rate and shows clearly the influence of corrosion and thus of subcritical crack growth (compare page 48 to pages 41 and 46!)

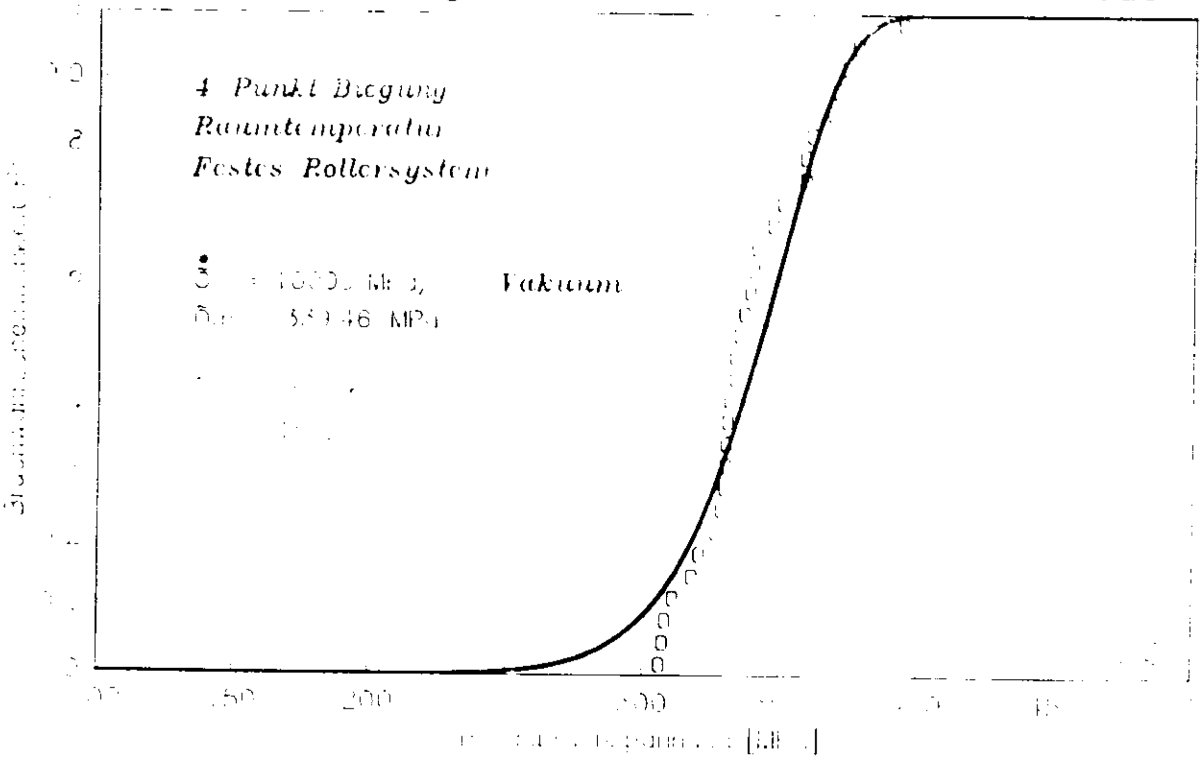
(all results: Rief and Kromp, 1989, unpublished)

(46)

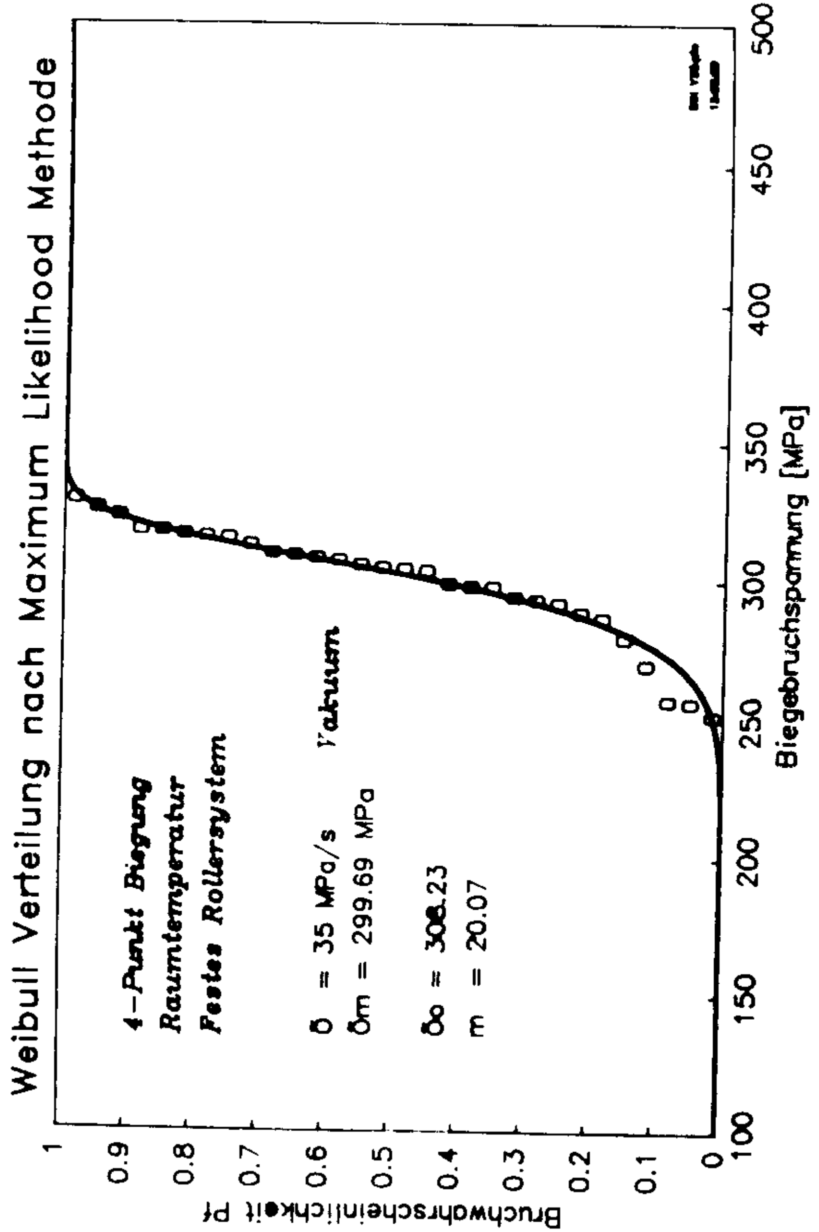
Weibull Verteilung nach Maximum Likelihood Methode



Weibull Verteilung nach Maximum Likelihood Methode



(47)

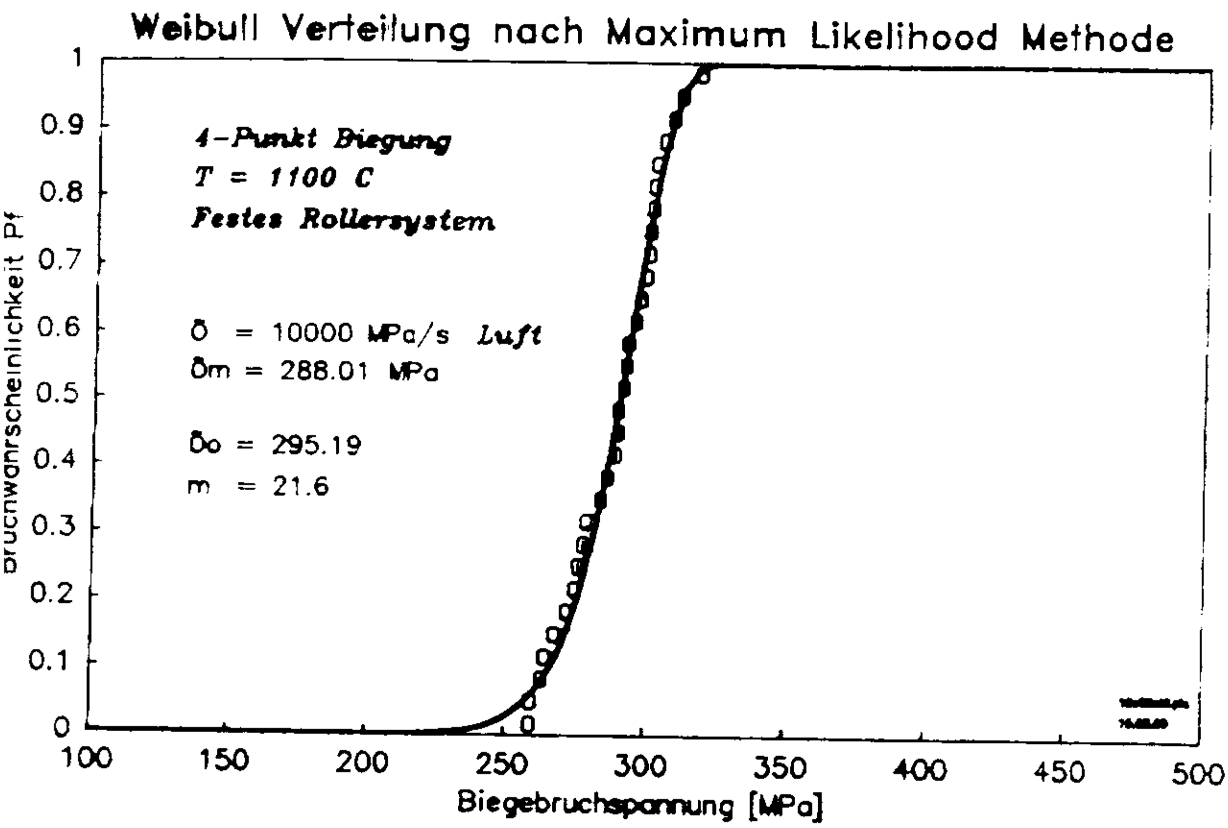


high temperature (1100°C):

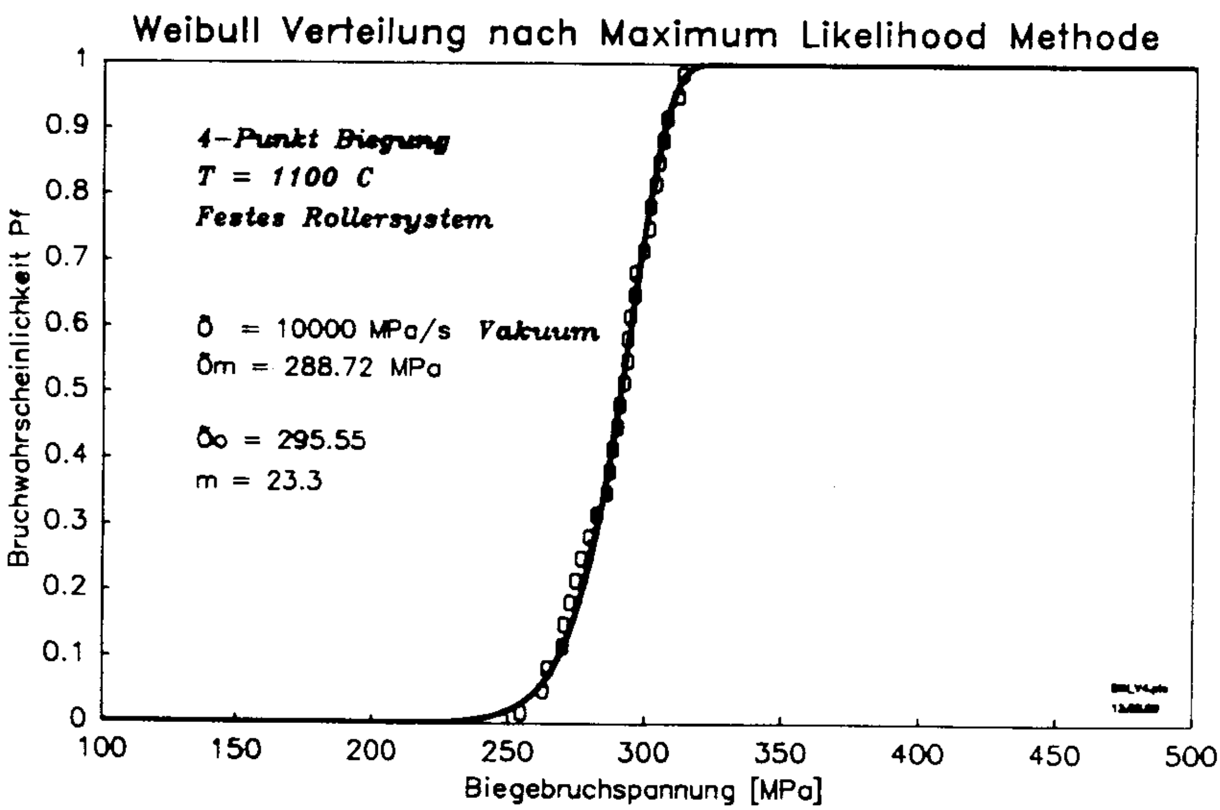
as it had been at room temperature, at the very high loading rate of 10 000 MPa/s the influence of the second phase on the results is ruled, the distributions in air and in vacuum are identical (pages 50, 51),

at the low loading rate of 35 MPa/s the results turn round with respect to room temperature: the result for vacuum, the "inert strength", is at a 19% lower level compared to that in air, pages 52, 53; the same fact was observed for the K_{IC} measurements at this temperature, see page 54; it is not cleared up to now what happens, but it is to suppose that in vacuum the second phase evaporates from the crack front, thus not assisting a blunting in microstructural ranges;

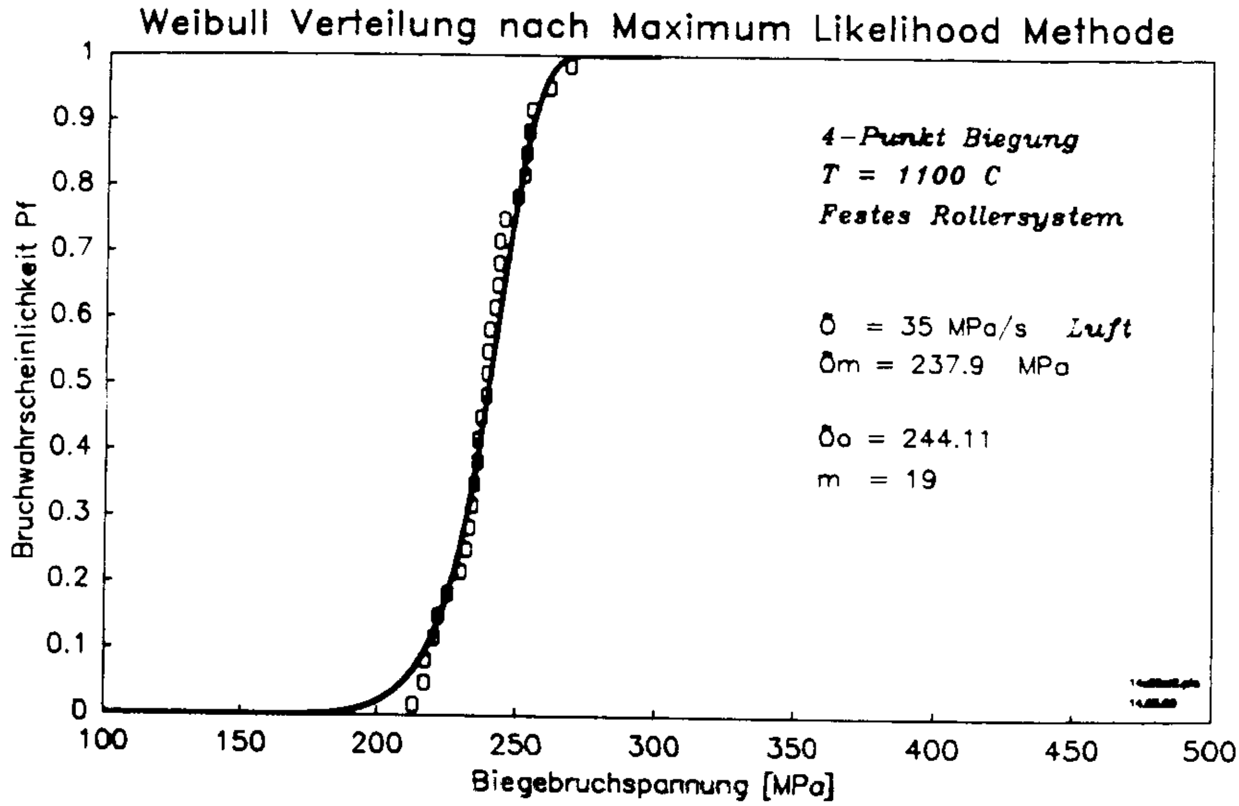
(Rief and Kromp, 1989; unpublished)

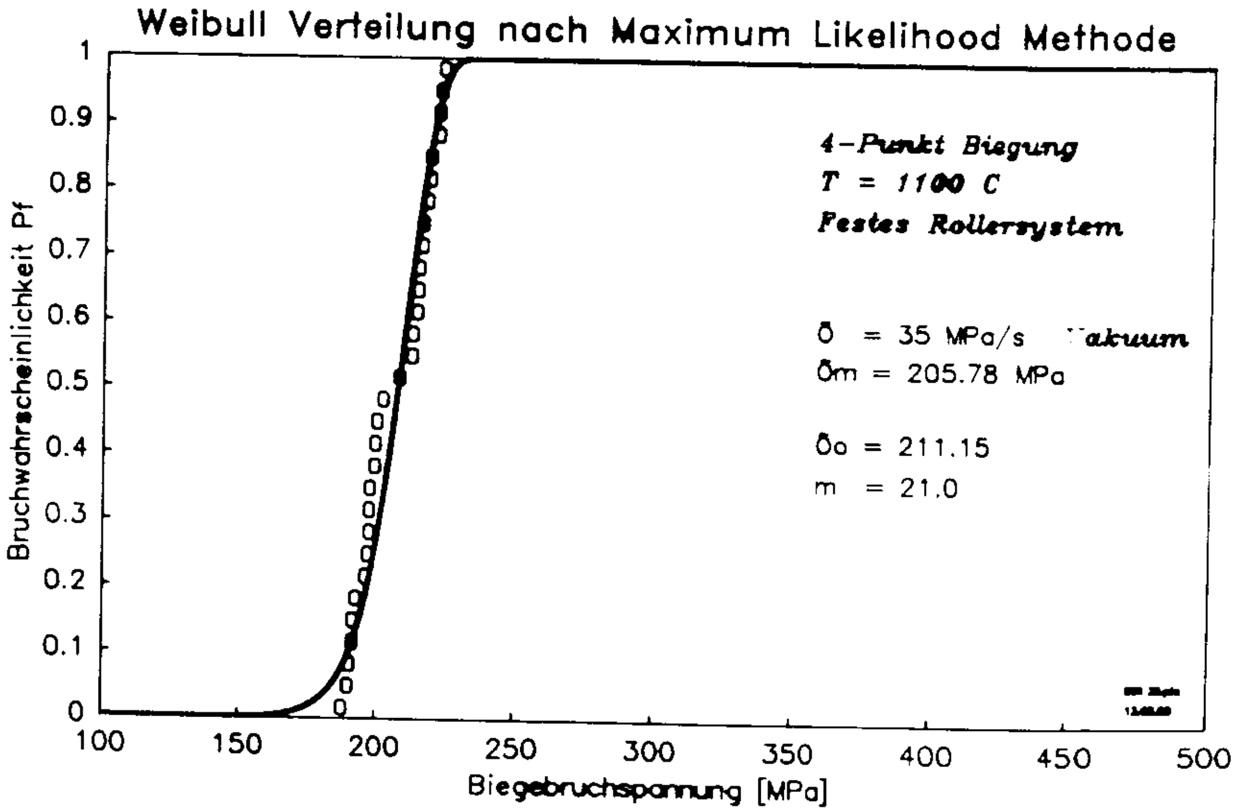

Weibull Verteilung nach Maximum Likelihood Methode
4-Punkt Biegung
T = 1100 C
Festes Rollersystem
ở = 10000 MPa/s Luft
ōm = 288.01 MPa
ōo = 295.19
m = 21.6
Bruchwahrscheinlichkeit Pf
Biegebruchspannung [MPa]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
100
150
200
250
300
350
400
450
500

50


Weibull Verteilung nach Maximum Likelihood Methode
4-Punkt Biegung
T = 1100 C
Festes Rollersystem
ở = 10000 MPa/s Vakuum
ōm = 288.72 MPa
ōo = 295.55
m = 23.3
Bruchwahrscheinlichkeit Pf
Biegebruchspannung [MPa]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
100
150
200
250
300
350
400
450
500

51


Weibull Verteilung nach Maximum Likelihood Methode
4-Punkt Biegung
T = 1100 C
Festes Rollersystem
σ̇ = 35 MPa/s Luft
σm = 237.9 MPa
σo = 244.11
m = 19
Bruchwahrscheinlichkeit Pf
Biegebruchspannung [MPa]


Weibull Verteilung nach Maximum Likelihood Methode
4-Punkt Biegung
T = 1100 C
Festes Rollersystem
σ̇ = 35 MPa/s Vakuum
σm = 205.78 MPa
σo = 211.15
m = 21.0
Bruchwahrscheinlichkeit Pf
Biegebruchspannung [MPa]

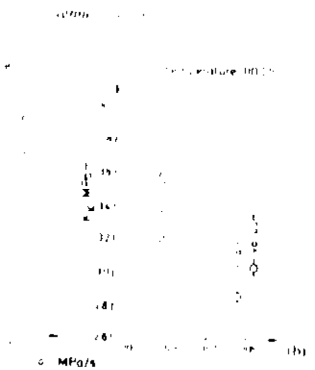


Fig. 11. Influence of loading rate on K_{Ic} results (stress rates of 35 and 10000 MPa s^{-1}) in air and in vacuum (mean values ± mean deviation): (a) at room temperature, (b) at 1100 C

while the value in vacuum was at about the same level as the high stress rate result (Fig. 11). Only subcritical crack growth prior to fast fracture could be the cause for this deviation in air. From the fracture surfaces, this fact could hardly be detected. In general, the fracture surfaces for the low stress rate in an air environment appear more intergranular than those of the specimens broken at the high stress rate (compare Figs 12(a) and (b)).

At 1100 C, the results for the low stress rate were higher than those for the high stress rate (Fig. 11): this fact was noted previously (see Fig. 6). Subcritical crack growth, which at these temperatures can only result from thermally activated processes, does not play an important role. This high level of K_{Ic}-values at the low stress rate may result from the blunting of microcracks by the viscous flow of the glassy phase. Viscous flow is energy-absorbing and results in a higher K_{Ic}-level. This can be observed directly on the fracture surfaces; an example is given in Fig. 13. The glassy phase covers the grains and moves to triple junctions, thus blunting microcracks, which start predominantly from these positions (see especially Fig. 13(b), at the lower left of the picture).

To summarise this section:

- at low loading rates and temperatures, corrosive subcritical crack growth takes place and reduces the level of the K_{Ic}-values;
- at low loading rates and high temperatures, energy-absorbing effects caused by the low-viscosity state of the second phase occur and thus raise the K_{Ic}-values;
- at very high loading rates, these effects do not occur

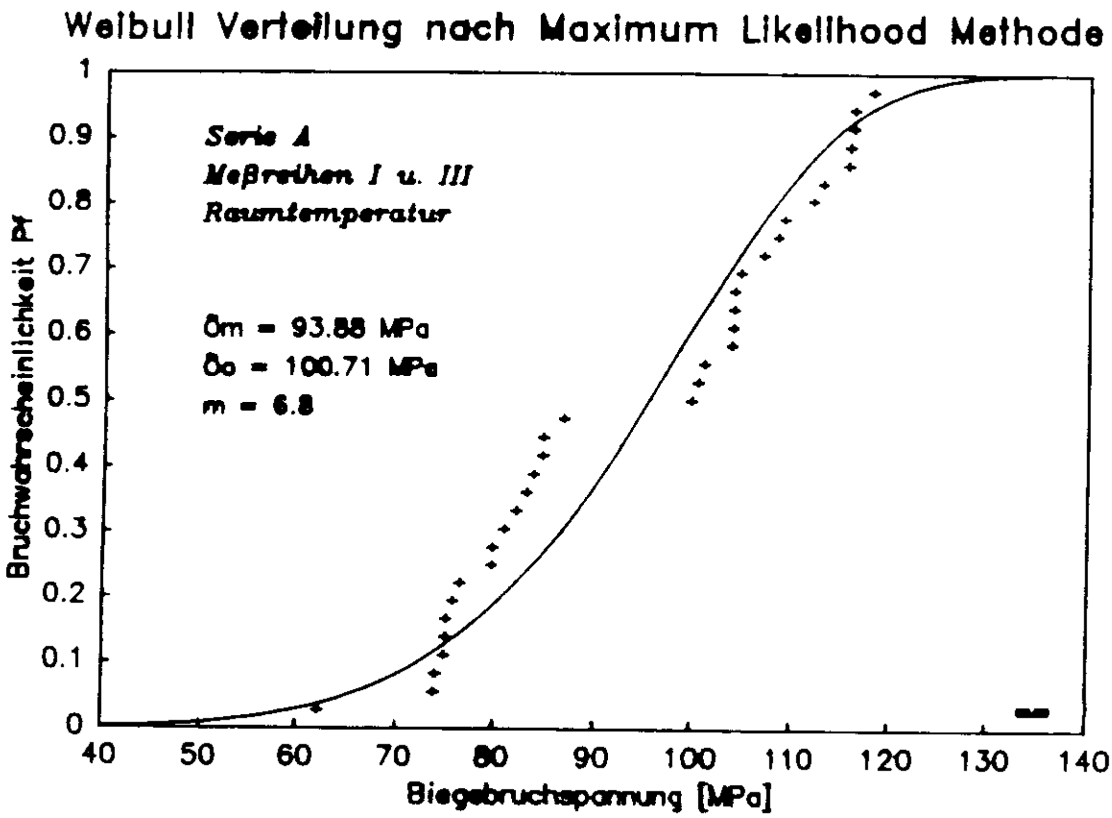
- The influence of the surface condition

The results in bending strength measurement depend to a high degree on the state of the machining of the specimen surface; this problem is difficult to standardize, because it depends on the special material investigated; in the DIN standard the end of the surface should be achieved by diamond grinding, size D15; the plan-parallelism of the surfaces should be better than 0.02 mm and the edges of the specimens should be rounded;

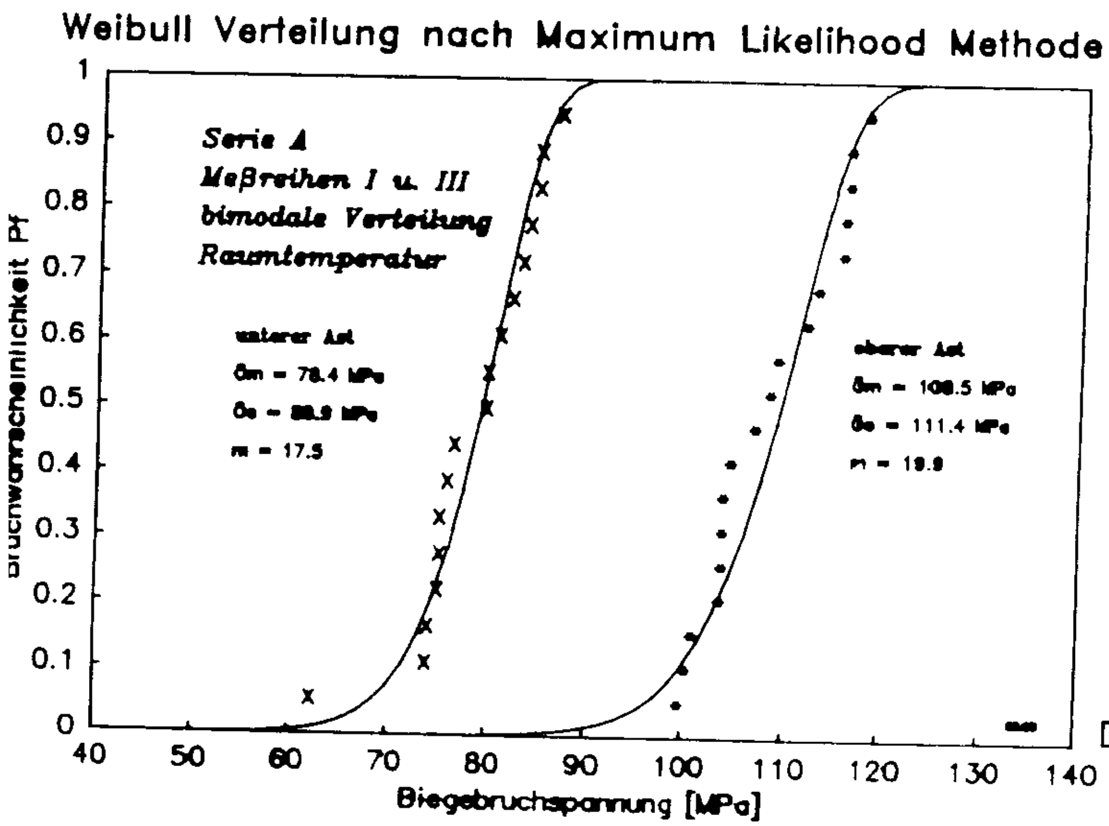
- as it could be seen from pages 40, 41 a deviation from plan-parallelism of 0.8% had some influence, when a fixed roller system is used that cannot pivot around the axis;
- for special materials the influence of surface machining can be strong; this is demonstrated by distributions measured with a recrystallized SiC-material (RSiC):

with this material at room temperature a bimodal Weibull distribution was found (page 56); the branches could be easily separated (page 57); for the branch with the lower fracture strengths the investigation of the fracture surfaces showed that the fracture origin for all of the samples had been from the edges on the tension surface, therefore for the following measurements the edges were rounded and the bimodality of the distribution vanished;

(Khalili and Kromp, 1989; unpublished)


Weibull Verteilung nach Maximum Likelihood Methode
Serie A
Meßreihen I u. III
Raumtemperatur
δm = 93.88 MPa
δo = 100.71 MPa
m = 6.8
Bruchwahrscheinlichkeit Pf
Biegebruchspannung [MPa]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
40
50
60
70
80
90
100
110
120
130
140

D. 1


Weibull Verteilung nach Maximum Likelihood Methode
Serie A
Meßreihen I u. III
bimodale Verteilung
Raumtemperatur
unterer Ast
δm = 78.4 MPa
m = 17.5
oberer Ast
δo = 111.4 MPa
Bruchwahrscheinlichkeit Pf
Biegebruchspannung [MPa]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
40
50
60
70
80
90
100
110
120
130
140

D. 3

Concluding remarks concerning the problem of bending strength measurement (observe personal opinion of the author!):

- The conditions given in the DIN standard should be observed, concerning the span 40/20 mm; the cross section: 3x4 mm²; the loading rate: 5-10 s up to fracture load; the machining of the specimens: surface grinded down to D15, rounded edges on the tension side;
- if "inert strength"-values are to be achieved, the measurements should be performed at room temperature in vacuum or Argon and at high temperatures in Argon,
- at high temperatures the measurement by fixed roller systems is to recommend for the reason of easy performance; in this case it is important to observe that the specimens are machined planparallel;

Limits of fracture statistics

- dependence on defect distribution

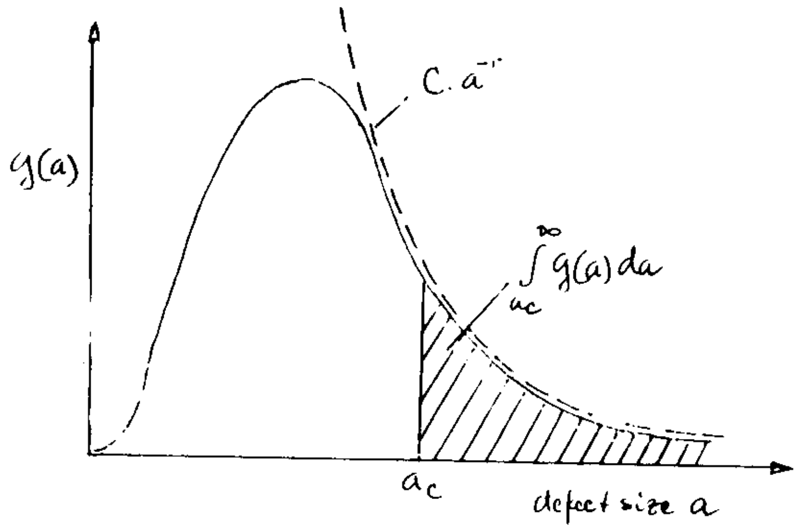
the Weibull distribution bases on an exponential defect distribution $g(a) = C \cdot a^{-r}$

with the Weibull model $m = 2(r-1)$

$$P_f = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] = 1 - \exp\left[-V\int_{a_c}^{\infty} g(a)\,da\right]$$

$g(a)$	number of defects per defect length and volume
C, r	material parameters
a_c	critical defect size

it can be shown that the Weibull-formula follows directly by basing such a distribution; Jayatilaka, 1977; Danzer 1989,



the Weibull modul m depends on this defect-distribution $m = 2(r-1)$ → the Weibull function is a special case of a more general distribution function for a certain defect distribution (→ physical basis for the Weibull distribution function);

- dependence on loaded volume and surface — effective volume, effective surface

in general different volume elements are differently stressed - only tension stresses are considered, the integration is therefore to perform only over volume elements under tension ; $V \to \int dV$:

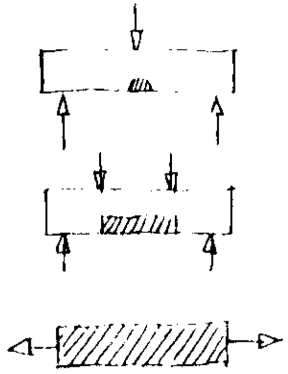
$$P_f = 1 - \exp\left[-\frac{1}{V_0}\int\left(\frac{\sigma}{\sigma_0}\right)^m dV\right]$$

def. effective volume is the volume of the tension specimen, which has the same fracture probability under the stress σ_{max}, as the component with the nonuniform stress distribution with a maximum stress σ_{max}:

$$P_f = 1 - \exp\left[-\frac{1}{V_0}\int_V\left(\frac{\sigma}{\sigma_0}\right)^m dV\right] = 1 - \exp\left[-\frac{V_{eff}}{V_0}\left(\frac{\sigma_{max}}{\sigma_0}\right)^m\right]$$

$$V_{eff} = \left(\frac{\sigma_0}{\sigma_{max}}\right)^m \int_V \left(\frac{\sigma}{\sigma_0}\right)^m dV$$

example::



3-point bending $\qquad V_{eff} = V\frac{1}{2(m+1)^2}$

4-point bending $\qquad V_{eff} = V\frac{m+2}{4(m+1)^2}$

tension $\qquad V_{eff} = V$

σ_{max} = maximum stress below central loading rod in 3-point bending or between central loading rods in 4-point bending;

bending specimen: $3 \times 4 \times 45$ mm³ (DIN)
→ $V = V_{tension} = 540$ mm³

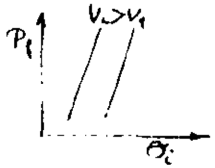
m	3-point	4-point	tension
20	0.6	7	540
10	2	13	540
5	8	26	540

m = 5 bad material, m = 10 medium, m = 20 excellent

it is obvious that in bending only a small, unrepresentative part of the volume contributes to the result;

for the following considerations it is presumed that the defect population, basing the Weibull function, in the different batches (volumes) of specimen is the same, m is the same (same material);
batches of equal sample size with different volumes, fracture stresses for the same fracture probability are calculated by (page 54):

$$P_f(\sigma_1, V_1) = P_f(\sigma_2, V_2) \qquad \sigma_2 = \sigma_1 \left(\frac{V_1}{V_2}\right)^{1/m}$$



σ_2/σ_1 (%)

V_2/V_1	1	10^2	10^4	10^6
m = 20	1	79	63	50
m = 10	1	63	40	25
m = 5	1	40	16	6

(Danzer, 1989)

it is evident that at high volume ratios and with low Weibull moduli the applicable strength drops very fast;
under the same assumptions as above, the size ratios of the largest defects causing fracture are calculated:

a_{c2}/a_{c1}

V_2/V_1	1	10^2	10^4	10^6
m = 20	1	1.6	2.5	4
m = 10	1	2.5	6.3	16
m = 5	1	6.3	39.0	250

(Danzer, 1989)

the size of the largest defects rises rapidely with the volume → this fact should be compared to the range, over which informations on the defect sizes are available!

Obviously the defect-distribution is known only in a very narrow range – extrapolations of 10^4 to 10^6 exceed the range of experience and should be avoided!

On the other hand: V_{eff} for 4-point bending is about $10\ mm^3$, constructive parts reach volumina of $10^6\ mm^3$ – V_{eff} of these may be 10% – there remains a volume extrapolation of 10^4!

→ A reduction of this extrapolation could be achieved by rising the specimen size, the sample size or with another testing technique e.g. testing in tension!

Fracture surface investigations show that often the failure starts from surface flaws and defects, mostly introduced by machining;
an argumentation, completely analogous to volume size influence, can be followed for surface defects:

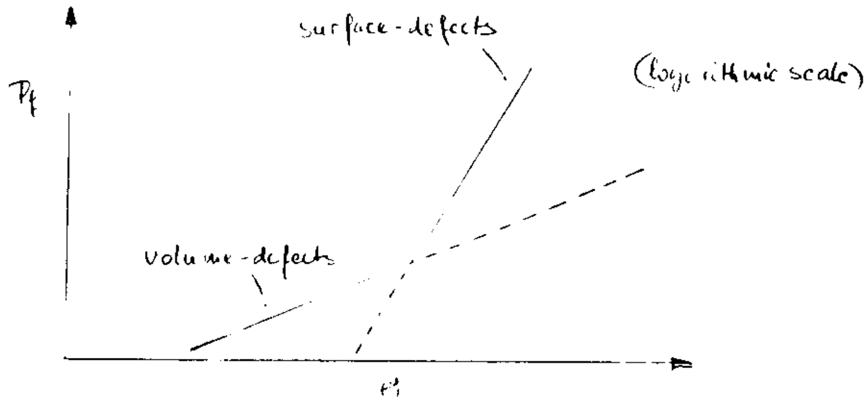
$$P_f = 1 - \exp\left[-\frac{A}{A_0}\left(\frac{\sigma}{\sigma_0^*}\right)^{m^*}\right] \quad ; \quad \frac{\sigma_2}{\sigma_1} = \left(\frac{A_1}{A_2}\right)^{1/m^*}$$

it is to observe that with rising specimen size the ratio between surface and volume goes down and the influence of volume defects gets more and more decisive!

in general both defect populations should be taken into account:

$$P_f = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^{m} - \frac{A}{A_0}\left(\frac{\sigma}{\sigma_0'}\right)^{m'}\right]$$

then the Weibull distributions get bimodal, as schematically shown below:



What cannot be realized by statistical methods are "very unusual defects" – these can arise from imperfections during processing of the materials and have a chance to be detected in bending tests by ~1 : 1000, these defects could be avoided only by processing in extremely clean surrounding.

Consequences

- dimensions of specimens and testing conditions should be chosen that V_{eff}, A_{eff} do not differ too much from the real constructive parts (1 : 20 ?);
- fractographic investigation of the fracture surfaces, especially of those broken at the lowest strength levels;
- sample size should be chosen so that the sum of V_{eff} of all samples exceeds the V_{eff} of the constructive part (detection of "very unusual defects"!);
- in many cases a "proof test" of the constructive part before application is inevitable!

the Weibull-distribution

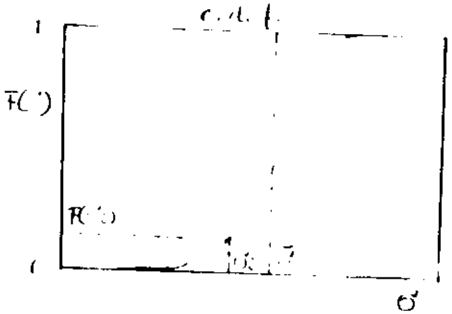
appearance, evaluation procedures and confidence

(all simulations and experimental results: Khalili and Kromp, 1989; unpublished)

for simplification: two parametric, empirical function; σ

$$P_f = F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right]$$

"cumulative density function" (c.d.f.)

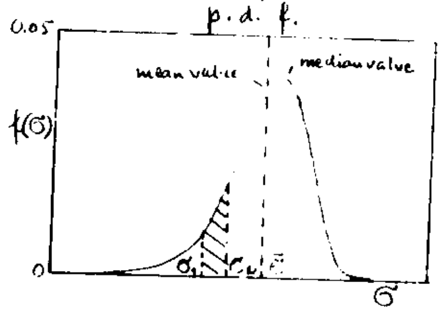


$P_{fi} = F(\sigma_i)$ probability to fail at a stress $\sigma \leq \sigma_i$

the derivative

$$f(\sigma) = \frac{dF(\sigma)}{d\sigma} = \frac{dP_f}{d\sigma} = \frac{m}{\sigma_0}\left(\frac{\sigma}{\sigma_0}\right)^{m-1} \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right]$$

"probability density function" (p.d.f.)
(relative frequency distribution)

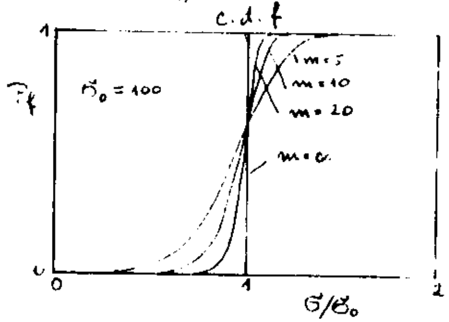


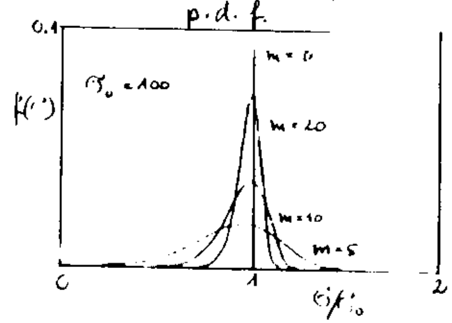
total area: unity,
area between σ_1, σ_2 probability that σ_i between σ_1, σ_2

→ tendency, asymmetry

m: Weibull modulus

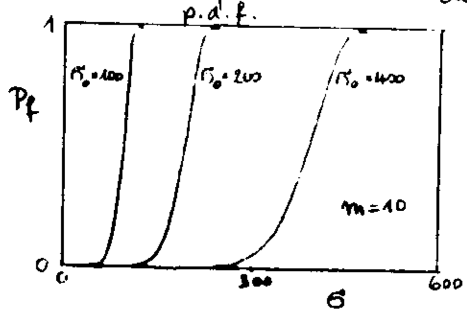
→ width; skewness, for $m > 3$ negative skewness
i.e. the more values are below $\bar{\sigma}$;
the higher m, the smaller the scatter (width), only for σ_0 = const.!

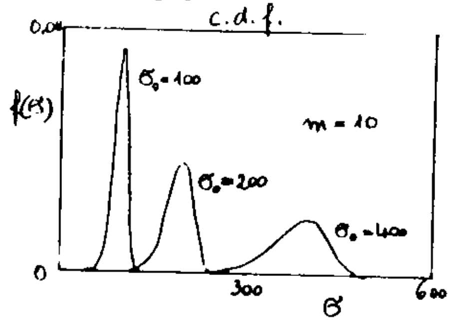




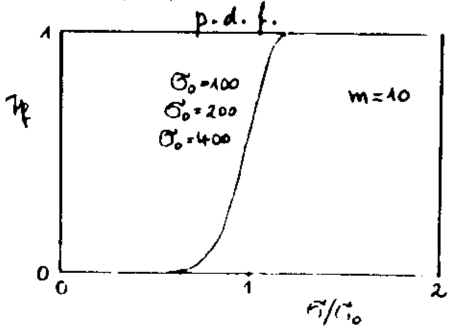
σ_0: scaling factor

→ influences scatter and mean value



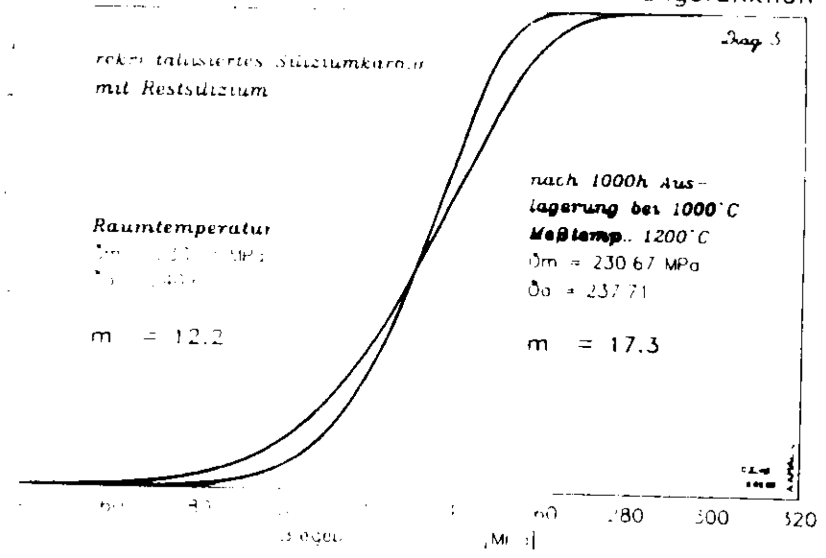


if σ is not scaled absolutely but relatively, σ/σ_0,
all three materials from the diagrams above show
the same scatter:



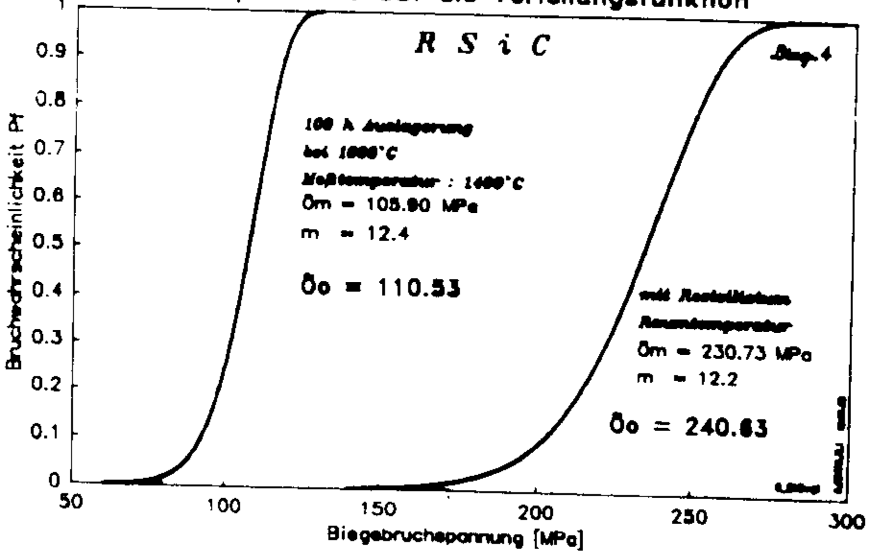
example: RSiC σ_0 = const, m varies

Einfluß des Weibullmoduls m auf die Verteilungsfunktion



example: RSiC m = const, σ_0 varies

Einfluß von σo auf die Verteilungsfunktion



evaluation procedures.

definition of fracture probability P_f and ranking:

$\sigma_i = 0 \rightarrow P_f = 0$, pressure loading ($\sigma_i < 0$) is not included

$\sigma_i \gg 0 \rightarrow P_f = 1$

n : rank of data, N : sample size

N specimens broken $\rightarrow$ ranking of fracture stresses

$$\sigma_1 < \sigma_2 < \ldots < \sigma_i < \ldots < \sigma_{N-1} < \sigma_N$$

analogous to the fracture probabilities

$$P_{f1} < P_{f2} < \ldots < P_{fi} < \ldots < P_{fN-1} < P_{fN}$$

different definitions for P_f:

(1) $P_f = \dfrac{n - 0.5}{N}$

(2) $P_f = \dfrac{n}{N+1}$ ("mean ranking")

(3) $P_f = \dfrac{n - 0.3}{N + 0.4}$ ("median ranking")

(4) $P_f = \dfrac{n - 0.5}{N + 0.25}$

equation (2) is the most applied, but not optimal (see later);

evaluation procedures:

to find the exact Weibull distribution an infinite number of tests would be necessary – the N tested specimens are only a random sample out of an „infinite mother's distribution" – the task will be to achieve the best estimate of an approximation!

linear regression.

mean least square method,

$y = a + bx$ $\qquad \ln \ln \frac{1}{1-P_i} = m \ln \sigma_i - m \ln \sigma_0$

slope : $b = m$ → Weibull modulus m

axial intercept : $a = -m \ln \sigma_0$ → scaling factor σ_0

moments method:

batch of data, which piles up around a central point, can be described by its "moments";

i^{th} - moment : sum of the data points to the power i

example: mean value $\bar{\sigma}$: first moment

$$\bar{\sigma} = \frac{1}{N} \sum_{i=1}^{N} (\sigma_i)^1$$

variance, square root = standard deviation : second moment

$$Var(\sigma_i) = S_\sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\sigma_i - \bar{\sigma})^2$$

$$\sqrt{var(\sigma_i)} = S_\sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (\sigma_i - \bar{\sigma})^2}$$

procedure:

mean value and variance of the theoretical mother distribution are compared to mean value and variance of the measured random sample $\{\sigma_i\}$;

from the p.d.f. function →:

$$\bar{\sigma} = \int_0^\infty f(\sigma)\sigma\, d\sigma \quad , \quad S_\sigma^2 = \int_0^\infty (\sigma - \bar{\sigma})^2 f(\sigma)\, d\sigma$$

result:

$$\bar{\sigma} = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right) \quad , \quad S_\sigma^2 = \sigma_0^2 \left\{\Gamma\left(1 + \frac{2}{m}\right) - \left[\Gamma\left(1 + \frac{1}{m}\right)\right]^2\right\}$$

def. : coefficient of variation $:= \frac{\text{standard deviation}}{\text{mean value}}$

$$\frac{S_\sigma}{\bar{\sigma}} = \frac{\sigma_0 \left\{\Gamma\left(1+\frac{2}{m}\right) - \left[\Gamma\left(1+\frac{1}{m}\right)\right]^2\right\}^{1/2}}{\sigma_0 \Gamma\left(1+\frac{1}{m}\right)}$$

S_σ and $\bar{\sigma}$ from measured results → equation depends only on m and can be solved iteratively!

→ m calculated, σ_0 from $\bar{\sigma} = \sigma_0 \Gamma\left(1 + \frac{1}{m}\right)$

(Γ function tabulated or in software, calculator for iterative solution)

maximum - likelihood - method:

from the title one could assume that m, σ_0 are chosen to fit a sample of measured data – actually it is vice versa: for the actual m, σ_0 the probability is estimated for the sample of N specimens to result exactly in the measured data $\{\sigma_i\}$.

the combined probability f_N that the N specimens fail at the stresses $\{\sigma_1, \dots, \sigma_N\}$ is the product:

$$f_N = f(\sigma_1) \cdot \ldots \cdot f(\sigma_i) \cdot \ldots \cdot f(\sigma_N)$$

$$f_N \equiv L = \prod_i f(\sigma_i)$$

the maximizing of L – with estimated m, σ_0 – corresponds to the highest probability that the N specimens result in the measured $\{\sigma_i\}$'s;

procedure:

The highest probability – "maximum likelihood" – is achieved by inserting $f(\sigma)$ in L, taking the logarithm of the product, derivating by m and σ_0 and equalizing zero;

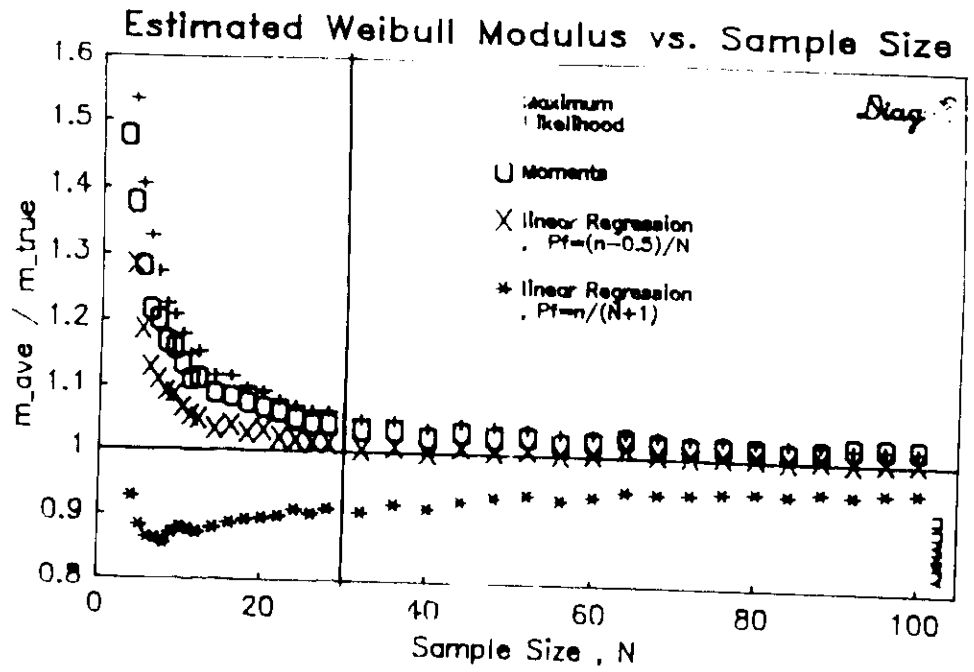
the result is, similar to moments method, an equation containing σ_i and m:

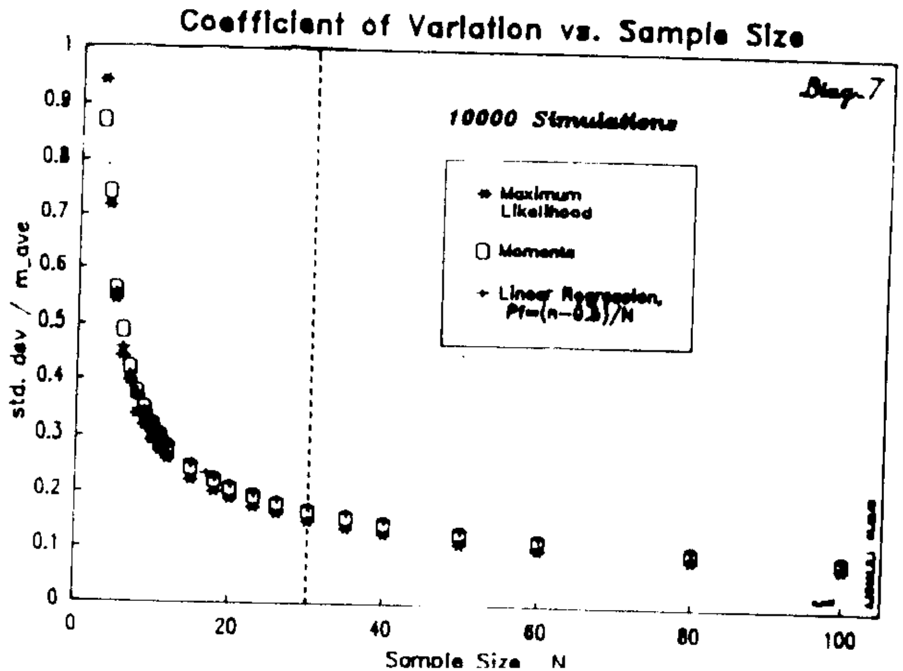
$$g(m) = \frac{N}{m} + \sum \ln \sigma_i - N \frac{\sum \sigma_i^m \ln \sigma_i}{\sum \sigma_i^m} = 0$$

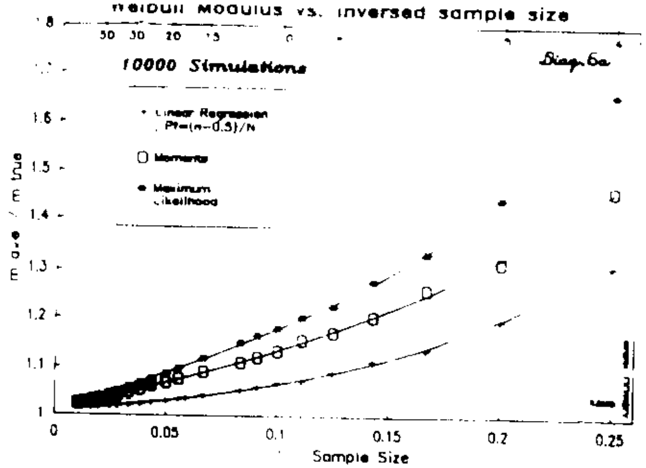
the solution is by iteration (calculator); if m has been calculated → $\sigma_o = \left(\frac{\sum \sigma_i^m}{N}\right)^{1/m}$

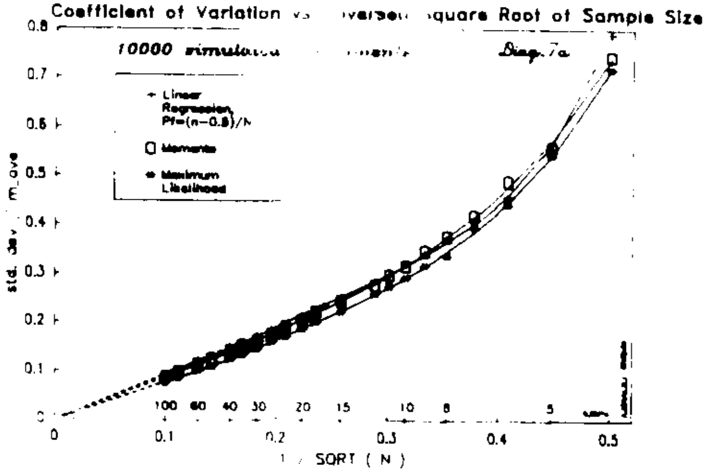
Comparison of the different evaluation methods:

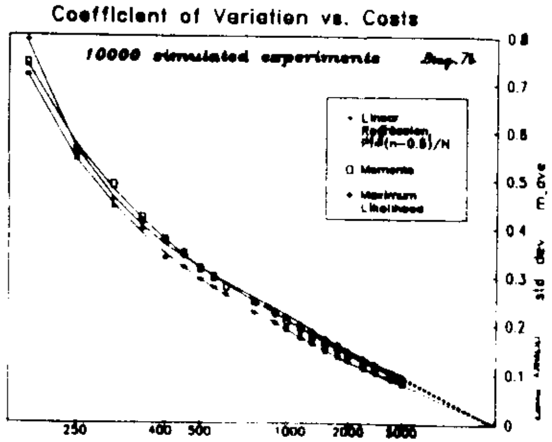
up to now there exists no analytical method, to decide, which of the evaluation procedures is the most reliable; computer simulations offer a solution at least qualitatively; basis is the infinite mother distribution with m_{true}, σ_{true}, from which by a random generator simple sizes N were taken out to estimate m and σ_o; the sample size was varied from N = 4 to N = 100 in steps, 10000 simulations for each step;



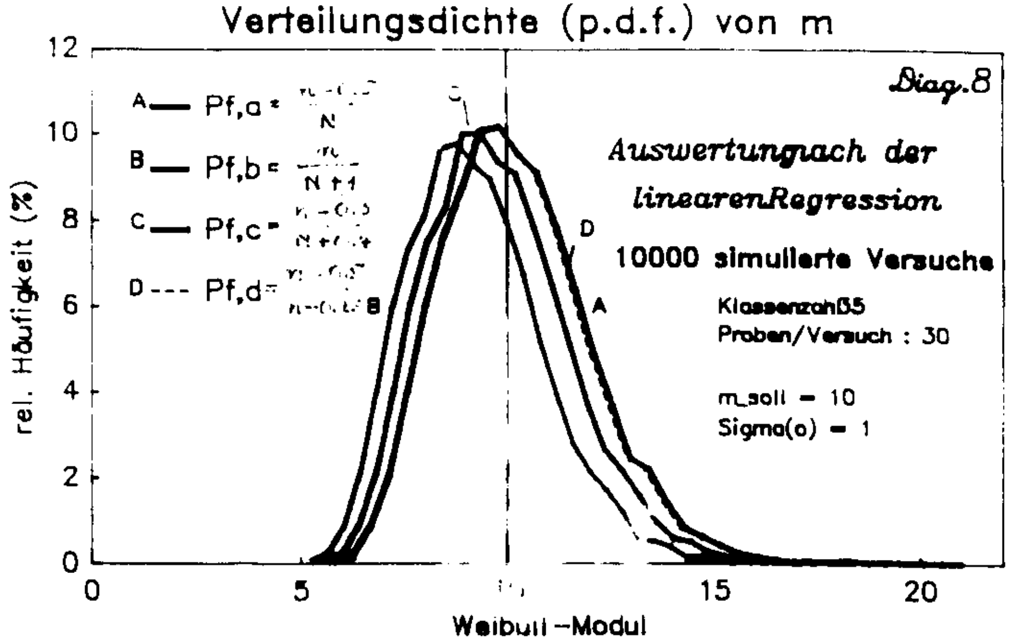


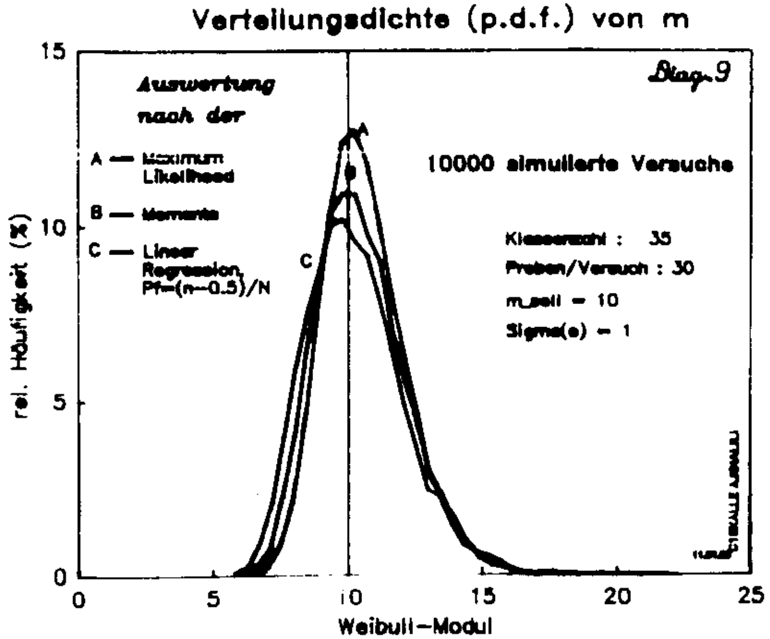






sample size: 30, 10000 simulations in 35 classes





Concluding remarks:

- the maximum likelihood method exhibits the lowest coefficient of variation for all sample sizes N (other authors: ML only for large sample sizes N!); for sample sizes $N \geq 50$, which should be applied, this method should be preferred;
- the p.d.f (page 75) show that the distributions are asymmetric – for small N the mean value can overestimate the true value, the median will be a more reliable quantity;
- the most reliable definition for P_f is not the usual $P_f = \frac{n}{N+1}$ but $P_f = \frac{n-0.5}{N}$; all the four investigated definitions exhibit the same coefficient of variation, but the median value of this latter definition moves nearest to m_{true}!
- the evaluation of σ_0 shows nearly the same result for all the three evaluation methods; the coefficient of variation for σ_0 is one order of magnitude less than that for m (reason: see the formulas for the evaluation of σ_0);