INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
34100 TRIESTE (ITALY) - P.O.B. 586 - MIRAMARE - STRADA COSTIERA 11 - TELEPHONE: 2240-1
CABLE: CENTRATOM - TELEX 460392 - I

SMR/389 - 2

**WORKING PARTY ON
MODELLING THERMOMECHANICAL BEHAVIOUR OF MATERIALS
(29 May - 16 June 1989)**

"REVIEW OF THERMOELASTICITY"

C. TOME'
Universidad Nacional de Rosario
Instituto de Fisica
Bv. 27 de Febrero 210 (Bis)
2000 Rosario
Argentina

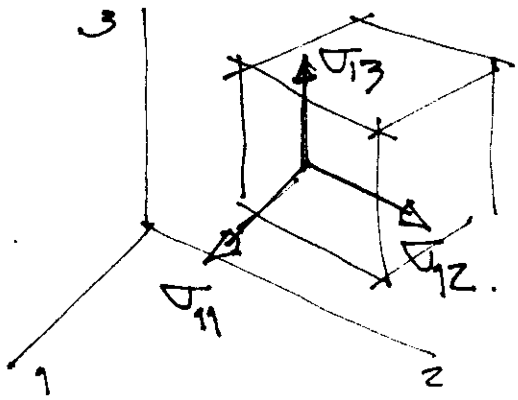
WORKING PARTY on

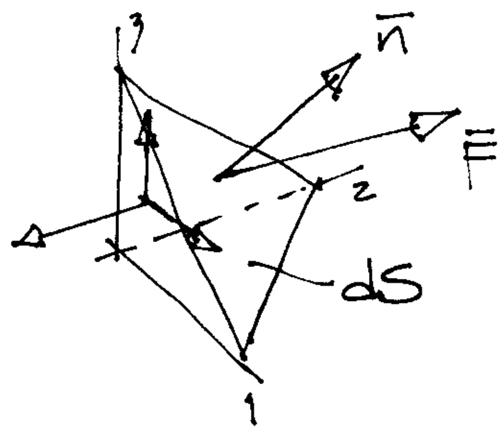
"MODELLING of THERMOMECHANICAL PPTIES"

REVIEW of THERMOELASTICITY (*) (C. TOMÉ)

1- STRESS.

Define σ_{ij} : forces per u. area acting on the 3-coordinate directions on planes parallel to the coordinate planes.





The force $\bar{F}$ acting on a surface element dS with normal $\bar{n}$ is given by

$$F_i = n_j \sigma_{ji} = \sum_{j=1}^{3} n_j \sigma_{ji}$$

(sum over repeated indices !!).

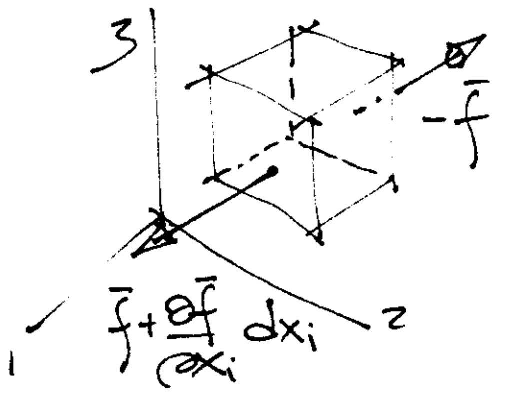
(*) see Y.C. FUNG, "FOUNDATIONS OF SOLID MECHANICS" PRENTICE-HALL (1965).

STRESS DEVIATOR.

$$\sigma'_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\cdot\delta_{ij} \quad (\text{with } \sigma_{kk} = \operatorname{tr}\sigma).$$

EQUILIBRIUM EQUATION. (STATIC).

Equating to zero. the total force acting on an elementary cube we get:



$$\frac{\partial \sigma_{ji}}{\partial x_j} + F_i = 0.$$

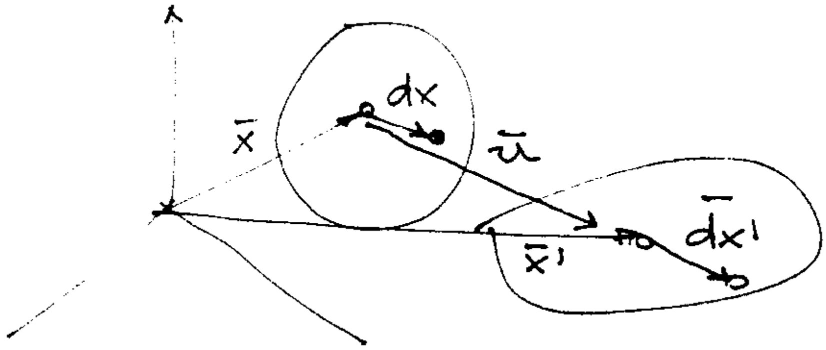
or

$$\boxed{\sigma_{ji,j} + F_i = 0.}$$

F_i = force per unit volume.

Equating to zero the moment of the forces (and assuming null moment per unit volumen) we get

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$



$$dS^2 = \sum_i dx_i^2$$

$$dS'^2 = \sum_i dx_i'^2$$

$x_i' = x_i'(x_1 x_2 x_3)$ univocal & continuous.

$$dx_i' = \frac{\partial x_i'}{\partial x_j} dx_j$$

and defining: $\bar{x}' = \bar{x} + \bar{u}$ (displacement)

$$dx_i' = \left(\delta_{ij} + \frac{\partial u_i}{\partial x_j}\right) dx_j$$

& $$dS'^2 - dS^2 = \sum_{jk} 2\,\varepsilon_{jk}\, dx_j\, dx_k + O(u_{i,j}^2).$$

with $$\boxed{\varepsilon_{jk} = \frac{1}{2}\left(u_{j,k} + u_{k,j}\right)}$$ (SYMMETRIC)

being the infinitesimal strain tensor, related to changes in length of the differential segment $\overline{dS}$

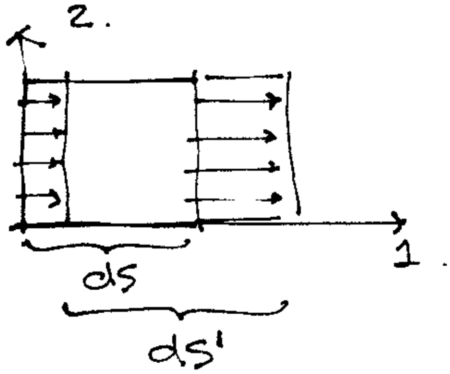
Small deformations $\longrightarrow$ $\frac{\partial u_i}{\partial x_j} \ll 1$!!

INTERPRETATION of STRAIN COMPONENTS

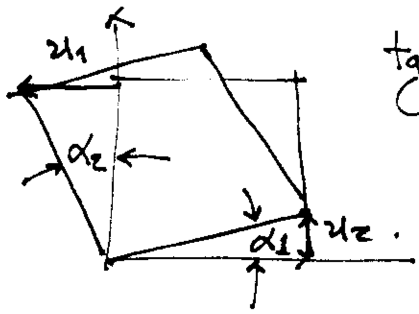
choosing $\overline{ds} = (dx_1, 0, 0)$

$$\Rightarrow \varepsilon_{11} = \frac{ds' - ds}{ds} = \frac{\Delta x_1}{dx_1}$$

elongation.



$$\Rightarrow \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$



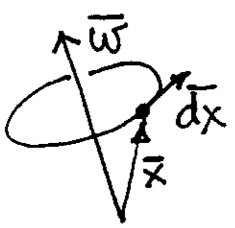
$tg\,\alpha_2 = \frac{u_1}{x_2} \simeq -|\alpha_2|$

$$= \frac{1}{2}\left(-|\alpha_2| + |\alpha_1|\right)$$

$$= \frac{1}{2} \text{ angular distortion of element.}$$

if $|\alpha_2| = |\alpha_1| \Rightarrow \varepsilon_{12} = 0$

but $\omega_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = |\alpha_1| =$



$=$ rigid rotation of element.

$$\boxed{\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = \text{rotation tensor (ANTISYMMETRIC)}}$$

$w_k = \frac{1}{2} e_{kij}\, \omega_{ij} = \frac{1}{2}(\nabla \times \overline{u})_k$ = rotation vector

$\hookrightarrow$ Levi-Civita tensor.

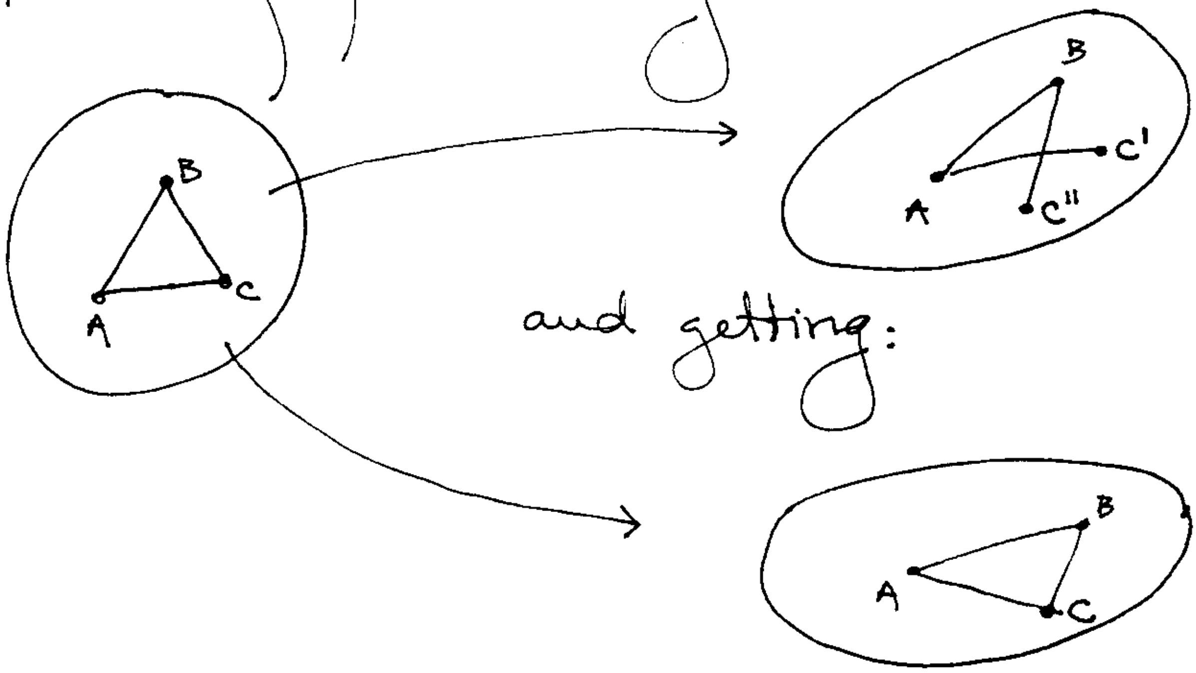
COMPATIBILITY EQUATIONS.

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

$$\Rightarrow \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} \equiv 0 .$$

a total of 6 independent equations due to redundancy in definition of strain tensor.

Necessary for avoiding:



and getting:

7 - CONSTITUTIVE EQUATION.

(Hooke + Cauchy law).

$$\boxed{\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}.}$$

with C_{ijkl} = elastic stiffness tensor.
(81 constants).

$\sigma_{ij} = \sigma_{ji} \Rightarrow C_{ijkl} = C_{jikl}$

$\varepsilon_{kl} = \varepsilon_{lk} \Rightarrow C_{ijkl} = C_{ijlk}$.

$\dfrac{\partial W}{\partial \varepsilon_{nm}} = \sigma_{nm} \Rightarrow C_{ijkl} = C_{klij}$

(21 indep. constants).

CONTRACTED VOIGT NOTATION.

11 → 1
22 → 2
33 → 3
23 → 4
13 → 5
12 → 6

$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$ $(i,j,k,l = 1,3)$

$\sigma_n = C_{nm}\, \varepsilon_m$ $(n,m = 1,6)$.

but pseudo-tensors !!

Do not transform as vectors !!

$\sigma_{ij}^T = a_{is}\, a_{jt}\, \sigma_{st}$ but $\sigma_n^T \neq a_{nm}\sigma_m$!!

CUBIC SYMMETRY.

$$
\mathbb{C} = \left[\begin{array}{ccc|ccc}
C_{11} & C_{12} & C_{12} & & \emptyset & \\
 & C_{11} & C_{12} & & & \\
 & & C_{11} & & & \\
\hline
 & & & C_{44} & & \\
 & \emptyset & & & C_{44} & \\
 & & & & & C_{44}
\end{array}\right]
$$

3 indep. constants

HEXAGONAL & TRIGONAL SYMMETRY.

$$
\mathbb{C} = \left[\begin{array}{ccc|ccc}
C_{11} & C_{12} & C_{13} & & \emptyset & \\
 & C_{11} & C_{13} & & & \\
 & & C_{33} & & & \\
\hline
 & & & C_{44} & & \\
 & \emptyset & & & C_{44} & \\
 & & & & & \frac{C_{11}-C_{12}}{2}
\end{array}\right]
$$

5 indep. cts.

ISOTROPIC ELASTICITY.

$$
\mathbb{C} = \left[\begin{array}{ccc|ccc}
\lambda+2\mu & 2\mu & 2\mu & & & \\
 & \lambda+2\mu & 2\mu & & & \\
 & & \lambda+2\mu & & & \\
\hline
 & & & 2\mu & & \\
 & & & & 2\mu & \\
 & & & & & 2\mu .
\end{array}\right]
$$

2 indep. cte

(λ , μ : Lamé's coefficients).

4 - ISOTROPIC ELASTICITY EQUATIONS (STATIC).

a) $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$ (HOOKE'S).

b) $\sigma_{ij,j} + F_i = 0$ (EQUILIBRIUM).

c) $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$a+b+c \Rightarrow \boxed{2\mu\, u_{i,jj} + (\lambda+\mu) u_{j,ji} + F_i = 0}$$

NAVIER EQ.

d) BOUNDARY CONDITIONS

$f_i^s = \sigma_{ij} n_j = \left(\lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i})\right) n_j$

on the surface. <u>or</u>

u_i^s imposed on the surface.

Observe that compatibility eqs. are not used.

(≠ STATIONARY)

a) $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} - \beta_{ij}\,(T - T_0)$

where $\beta_{ij} = \beta_{ji}$ → thermal moduli

and T_0 ↔ reference state (stress free).

and $T = T(x_1, x_2, x_3)$.

b) $h_i = -k_{ij}\,\dfrac{\partial T}{\partial x_j}$ = heat flux.

where k_{ij} = heat conduction coefficients.

c) $\nabla \cdot \bar{h} = h_{i,i} = 0$ (conservation).

ISOTROPIC THERMOELASTICITY.

a) $\sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} - \beta\,(T - T_0)\,\delta_{ij}$

with $\beta = (3\lambda + 2\mu)\,\alpha$ (linear coeff thermal exp.)

b) $h_i = -k\,\dfrac{\partial T}{\partial x_j}$ (or $\bar{h} = -k\,\nabla T$)

(uncoupled elastic and thermal behavior)