INTERNATIONAL ATOMIC ENERGY AGENCY
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WORKING PARTY ON
MODELLING THERMOMECHANICAL BEHAVIOUR OF MATERIALS
(29 May - 16 June 1989)

INTRODUCTION TO FINITE ELEMENT METHODS

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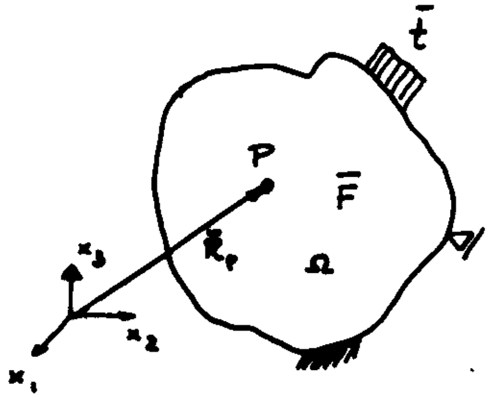
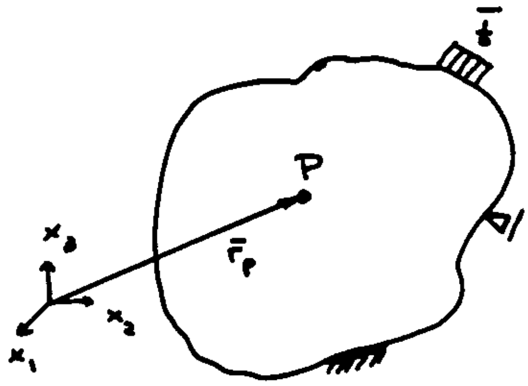
LINEAR ELASTIC THERMOELASTICITY





region Ω frontier Γ_Ω

$\Gamma_{\bar{t}}$ = frontier with surface load conditions

$\Gamma_{\bar{u}}$ = frontier with displacement conditions

$$\Gamma_{\bar{t}} \cup \Gamma_{\bar{u}} = \Gamma_\Omega$$

$$\Gamma_{\bar{t}} \cap \Gamma_{\bar{u}} = \emptyset$$

displacement $\bar{u}_p = \bar{r}_p - \bar{R}_p$

independent variable $\bar{R} \Rightarrow$ material or Lagrangian formulation

independent variable $\bar{r} \Rightarrow$ spatial or Eulerian formulation

If $|u_{i,j}| \ll 1 \Rightarrow$ small deformations

and $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ strain tensor

$\therefore$ 6 kinematic equations

Lineal Constitutive equations

$$\sigma_{ij} = C_{ijkl} \left(\varepsilon_{kl} - \beta_{ij} (T - T_0) \right)$$

thermal coeficient of linear expansion

stres free temperature

6 equations

in general C_{ijkl} has 21 independent constants

orthotropic material 9 independent constants
cubic material 3 independent constants
isotropic material 2 independent constants

Equilibrium equations

$$\sigma_{ij,j} + F_i = 0$$

3 equations

boundary conditions

$$t_i = \sigma_{ij} n_j \quad \text{in } \Gamma_t$$

Fourier law $\quad h_i = -k_{ij} T_{,j}$

heat conservation
stationary $\frac{\partial T}{\partial t} = 0$ $\quad h_{i,i} + r = 0$

$$\left.\begin{matrix} h_i = -k_{ij} T_{,j} \\ h_{i,i} + r = 0 \end{matrix}\right\} \quad -(k_{ij} T_{,j})_{,i} + r = 0$$

k_{ij} = heat transmision tensor

h = heat source per unit volume

If k_{ij} is a scalar tensor constant in all the body

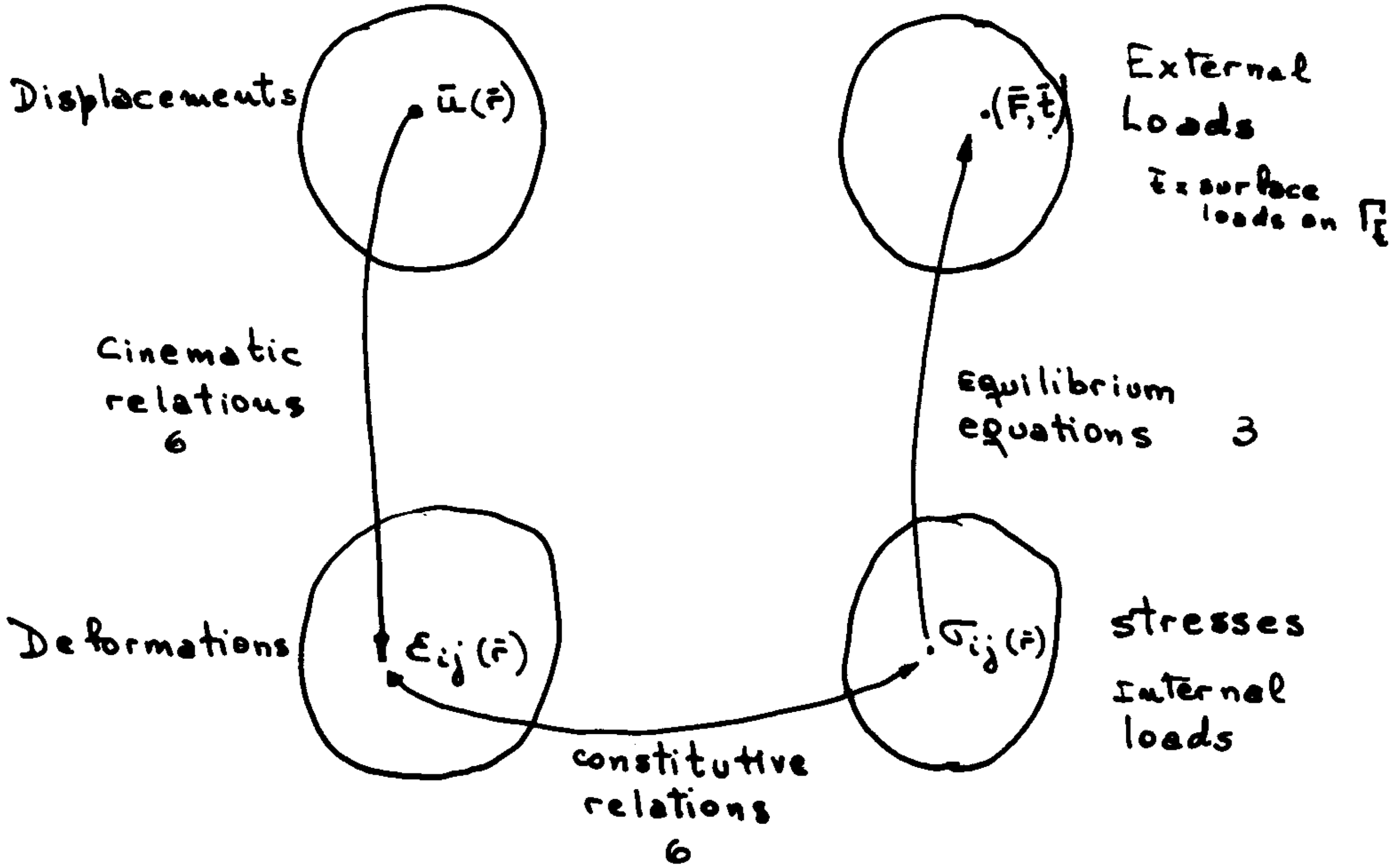
$$k_{ij} = k \delta_{ij}$$

$$-kT_{,ii} + r = 0 \quad (\text{[illegible]isson Equation})$$

unknown functions $\bar{\bar{\sigma}}(\bar{r})$ $\bar{\bar{\varepsilon}}(\bar{r})$ $\bar{u}(\bar{r})$ $T(\bar{r})$ 16

differential equations 16

Temperature is uncoupled for stationary $\left(\frac{\partial T}{\partial t}=0\right)$ and static problems $\left(\frac{\partial \bar{u}}{\partial t}=0\right)$



$$U = \frac{1}{2}\iiint_{\Omega} \sigma_{ij}\,\varepsilon_{ij}\,dV \quad \text{strain energy}$$

$$W = \iiint_{\Omega} F_i u_i\,dV + \iint_{\Gamma_t} t_i u_i\,dS \quad \text{external work}$$

$$\Pi = U - W = \text{total potential energy}$$

From the principle of virtual works

$$\iiint_{\Omega} F_i \, \delta u_i \, dV + \iint_{\Gamma_t} t_i \, \delta u_i \, dS = \iiint_{\Omega} \sigma_{ij} \, \delta \varepsilon_{ij} \, dV$$

$$\delta W = \delta U$$

$$\delta \Pi = 0$$

From the variational point of view the field of displacements that solves the thermoelastic lineal problem is the admissible field u^* that makes $\delta \Pi = 0$; for a minimum of Π, the solution is stable

Unidimensional variational problem

$$J[u] = \int_a^b F(u, u', x) \, dx \qquad \begin{matrix} u(a) = u_1 \\ u(b) = u_2 \end{matrix}$$

Euler equation $\qquad F_u - \frac{d}{dx} F_{u'} = 0$

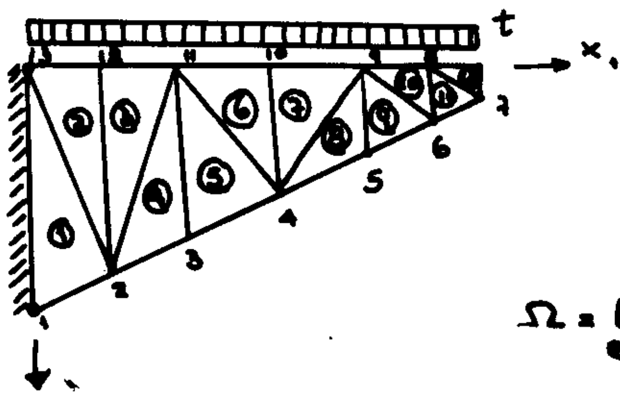
Ritz Method. $\qquad u = c_i \, \phi_i(x) + \varphi(x) \quad i = 1, N \quad \begin{cases} \varphi(a) = u_1 \\ \varphi(b) = u_2 \\ \phi_i(a) = 0 \\ \phi_i(b) = 0 \end{cases}$

$$J(c_i) = \int_a^b G(c_i, \phi_i(x), \varphi(x), x) \, dx$$

To obtain the values of c_i

$$\frac{\partial J}{\partial c_i} = 0$$ [illegible] btain a system [illegible] in N unknowns

I) Discretize the continoum



$\bar{u}(x,y) = \bar{u}(\bar{r})$

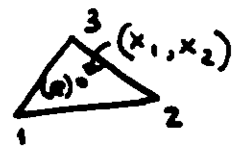
N=13 number of nodes

M=12 number of element

$\Omega = \bigcup_{e=1}^{M} \Omega^{(e)}$

$$\bar{u}(\bar{r}) = \sum_{e=1}^{M} \bar{u}^{(e)}(\bar{r}) \qquad \bar{u}^{(e)}(\bar{r}) = \begin{cases} \bar{u}(\bar{r}) & \text{if } \bar{r} \in \Omega^{(e)} \\ 0 & \text{if } \bar{r} \in \Omega' \end{cases}$$

II) Select interpolation functions



$$\underset{(2\times 1)}{\bar{u}_e^{(e)}(x_1,x_2)} = \underset{(2\times 6)}{\bar{N}^{(e)}(x_1,x_2)} \cdot \underset{(6\times 1)}{\tilde{u}^{(e)}}$$

$$\tilde{u}^{(e)} = \begin{bmatrix} u_1(1) \\ u_2(1) \\ u_1(2) \\ u_2(2) \\ u_1(3) \\ u_2(3) \end{bmatrix}$$

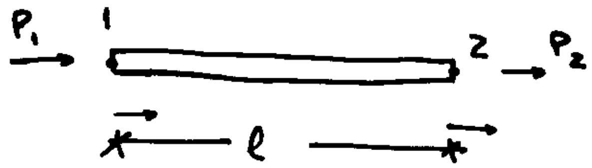
→ displacement in x_1 direction of node

→ displacement in x_2 direction of node

Conditions for monotonic convergence

Compatibility: At element interface the displacement must be continous

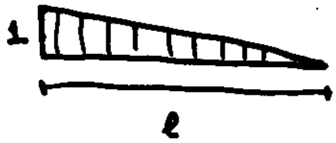
Completeness: All uniform states of displacements and partial derivatives of first order should have representation when the element shrinks to 0



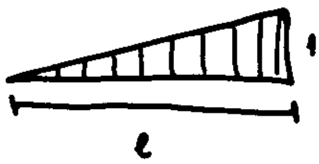
$\sigma = E\varepsilon$

A = section of the ba

$$u(x) = \phi_1(x)\, u_1 + \phi_2(x)\, u_2$$



$$\phi_1(x) = 1 - \frac{x}{\ell}$$



$$\phi_2(x) = \frac{x}{\ell}$$

$$u = \left(1 - \frac{x}{\ell}\right) u_1 + \frac{x}{\ell}\, u_2$$

$$\varepsilon = \frac{du}{dx} = -\frac{u_1}{\ell} + \frac{u_2}{\ell}$$

$$\Pi(u) = \frac{1}{2} \iiint_V \sigma_{(u)} \varepsilon_{(u)}\, dV - P_1 u_1 - P_2 u_2$$

$$\Pi(u) = \frac{1}{2} \int_0^\ell E\, \varepsilon^2_{(u)} \left[\iint_A dA \right] dx - P_1 u_1 - P_2 u_2$$

$$\Pi(u) = \frac{1}{2} EA \int_0^\ell \varepsilon^2_{(u)}\, dx - P_1 u_1 - P_2 u_2$$

$$\Pi(u_1, u_2) = \frac{1}{2} \frac{EA}{\ell} (u_2 - u_1)^2 - P_1 u_1 - P_2 u_2$$

minimum

$$\frac{\partial \Pi}{\partial u_1} = 0 = -\frac{EA}{\ell}(u_2 - u_1) - P_1$$

$$\frac{\partial \Pi}{\partial u_2} = 0 = \frac{EA}{\ell}(u_2 - u_1) - P_2$$

$$P_1 = \frac{EA}{\ell} u_1 - \frac{EA}{\ell} u_2$$

$$P_2 = -\frac{EA}{\ell} u_1 + \frac{EA}{\ell} u_2$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

stifness
matrix for
a bar.

For the triangular beam we can write

$$\underset{(6\times 1)}{\overline{P}^{(e)}} = \underset{(6\times 6)}{[K]_L^{(e)}} \underset{(6\times 1)}{\tilde{u}^{(e)}}$$

III) Assemble the element properties to obtain the system equations. Transform local nodes to global nodes, and local coordinates to global coordinat

For example for element ④

Local	Global
1	4
2	9
3	5

It results

$$\underset{(26\times 1)}{\overline{P}} = \underset{(26\times 26)}{[K]_G} \underset{(26\times 1)}{\tilde{u}}$$

Global stiffness

5) Boundary conditions

$$u_1(1) = 0$$
$$u_2(1) = 0$$
$$u_1(13) = 0$$
$$u_2(13) = 0$$

6) Solve the linear system of equations to get the solution

SAP IV : Structural Analysis Program for static and dynamic response of linear systems.
BERKELEY - CALIFORNIA - USA.

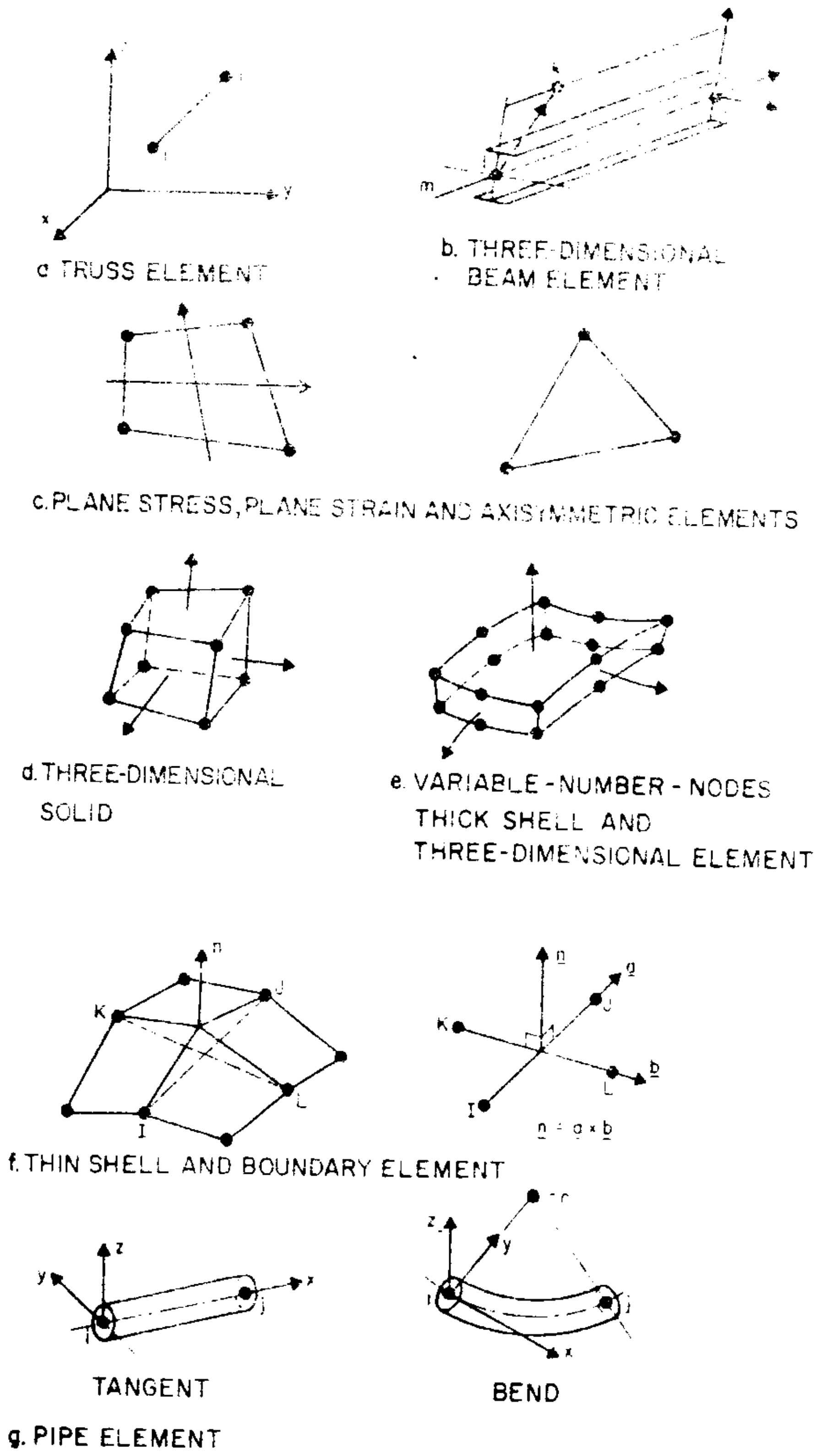


FIGURE 7: ELEMENT LIBRARY OF SAP IV

SAP DATA INPUT

I) Heading Card

II) Master control card

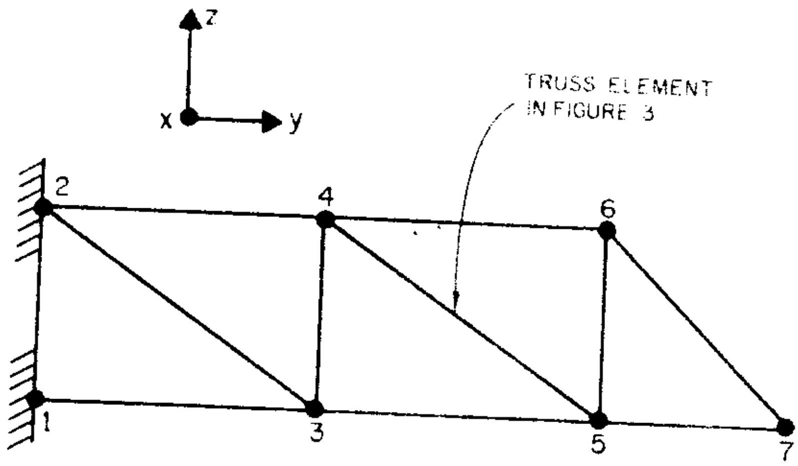
III) Nodal point data

IV) Element data

V) Concentrated load, mass data

VI) Element load multipliers

VII) Dynamic Analysis



NODAL POINT LAYOUT OF TRUSS

$$
ID = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

DEGREES OF FREEDOM

NODAL POINT NUMBERS

FIGURE 1: NODAL POINT LAYOUT OF TRUSS-EXAMPLE AND ID-ARRAY AS READ AND/OR GENERATED

TRUSS FILE INPUT

```
TRUSS ELEMENTS INPUT FOR SAP
     7     1     1     0     0     0     0     0     0
     1     1     1     1     1     1     1     0.0        0.0        0.0
     2     1     1     1     1     1     1     0.0        0.0        1.0
     3     1     0     0     1     1     1     0.0        1.0        0.0
     4     1     0     0     1     1     1     0.0        1.0        1.0
     5     1     0     0     1     1     1     0.0        2.0        0.0
     6     1     0     0     1     1     1     0.0        2.0        1.0
     7     1     0     0     1     1     1     0.0        3.0        0.0
     1    10     1
     1    10.0       0.0             0.0        1.0        0.0
1.0
1.0
1.0
1.0
     1     1     3     1
     2     2     3     1
     3     2     4     1
     4     3     5     1
     5     4     5     1
     6     4     6     1
     7     5     7     1
     8     6     7     1
     9     3     4     1
    10     5     6     1
     7     1   0.0        0.0         10.0     0.0        0.0         0.0
     0
       1.0
```

TRUSS OUTPUT

1PRUEBA DEL SAP4 ELEMENTOS DE TRUSS

C O N T R O L   I N F O R M A T I O N

```
     NUMBER OF NODAL POINTS   =     7
     NUMBER OF ELEMENT TYPES  =     1
     NUMBER OF LOAD CASES     =     1
     NUMBER OF FREQUENCIES    =     0
     ANALYSIS CODE (NDYN)     =     0
       EQ.0,  STATIC
       EQ.1,  MODAL EXTRACTION
       EQ.2,  FORCED RESPONSE
       EQ.3,  RESPONSE SPECTRUM
       EQ.4,  DIRECT INTEGRATION
     SOLUTION MODE (MODEX)    =     0
       EQ.0,  EXECUTION
       EQ.1,  DATA CHECK
     NUMBER OF SUBSPACE
     ITERATION VECTORS (NAD)  =     0
     EQUATIONS PER BLOCK      =     0
     TAPE10 SAVE FLAG (N10SV) =     0
```

NODAL POINT INPUT DATA

0NODE NUMBER	BOUNDARY CONDITION CODES X	Y	Z	XX	YY	ZZ	NODAL POINT COORDINATE X	Y
1	1	1	1	1	1	1	.000	.000
2	1	1	1	1	1	1	.000	.000
3	1	0	0	1	1	1	.000	1.000
4	1	0	0	1	1	1	.000	1.000
5	1	0	0	1	1	1	.000	2.000
6	1	0	0	1	1	1	.000	2.000
7	1	0	0	1	1	1	.000	3.000

1GENERATED NODAL DATA

0NODE NUMBER	BOUNDARY CONDITION CODES X	Y	Z	XX	YY	ZZ	NODAL POINT COORDINATE X	Y
1	1	1	1	1	1	1	.000	.000
2	1	1	1	1	1	1	.000	.000
3	1	0	0	1	1	1	.000	1.000
4	1	0	0	1	1	1	.000	1.000
5	1	0	0	1	1	1	.000	2.000
6	1	0	0	1	1	1	.000	2.000
7	1	0	0	1	1	1	.000	3.000

1EQUATION NUMBERS

N	X	Y	Z	XX	YY	ZZ
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	1	2	0	0	0
4	0	3	4	0	0	0
5	0	5	6	0	0	0
6	0	7	8	0	0	0
7	0	9	10	0	0	0

```
1NUMBER OF TRUSS MEMBERS=   10
 NUMBER OF DIFF. MEMBERS=    1


 TYPE              E              ALPHA                DEN                AREA
    1    .1000000E+02    .0000000E+00    .0000000E+00    .1000000E+01


 ELEMENT LOAD MULTIPLIERS
                      A                  B                  C
 X-DIR     .100000E+01     .000000E+00     .000000E+00     .000000E+0
 Y-DIR     .100000E+01     .000000E+00     .000000E+00     .000000E+0
 Z-DIR     .100000E+01     .000000E+00     .000000E+00     .000000E+0
  TEMP     .100000E+01     .000000E+00     .000000E+00     .000000E+0


1      N       I       J   TYPE        TEMP    BAND
       1       1       3      1         .00       2
       2       2       3      1         .00       2
       3       2       4      1         .00       2
       4       3       5      1         .00       6
       5       4       5      1         .00       4
       6       4       6      1         .00       6
       7       5       7      1         .00       6
       8       6       7      1         .00       4
       9       3       4      1         .00       4
      10       5       6      1         .00       4
1E Q U A T I O N   P A R A M E T E R S

  TOTAL NUMBER OF EQUATIONS        =   10
  BANDWIDTH                        =    6
  NUMBER OF EQUATIONS IN A BLOCK =   10
  NUMBER OF BLOCKS                 =    1
1N O D A L   L O A D S   (S T A T I C)   O R   M A S S E S   (D Y

   NODE    LOAD          X-AXIS           Y-AXIS            Z-AXIS
 NUMBER    CASE           FORCE            FORCE             FORCE

      7       1     .00000E+00      .00000E+00      .10000E+02     .


 STRUCTURE              ELEMENT     LOAD     MULTIPLIERS
 LOAD CASE            A            B            C            D

     1          1.000         .000         .000         .000
1N O D E   D I S P L A C E M E N T S / R O T A T I O N S

   NODE    LOAD           X-            Y-             Z-
 NUMBER    CASE     TRANSLATION   TRANSLATION   TRANSLATION       RO

0      7       1    .00000E+00    .60000E+01    .29485E+02     .000

0      6       1    .00000E+00   -.30000E+01    .17657E+02     .000

0      5       1    .00000E+00    .50000E+01    .16657E+02     .000
```