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WORKING PARTY ON "FRACTURE PHYSICS"
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ELASTICITY

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These are preliminary lecture notes, intended only for distribution to participants.

ELASTICITY

Gen. References

N. Muskhelishvili: "Some Basic Problems in Mathematical Theory of Elasticity" Noordhoff Leiden (1977) (translation)

Timoshenko + Goodier "Theory of Elasticity" McGraw Hill (1970)

Specialised:

Kanninen + Popelar "Advanced Fracture Mechanics" Oxford. (1985)

de Vedia, "Mecanica de Fractura" Monografia Technologica #1 Programa Regional de Desarrollo Cientifico y Tecnologico - OEA (1986)

Thomson, Solid State Physics Vol 39 (1986)

Anisotropic

Bacon, Barnett + Scattergood Progress in Matls. Sc. Vol 23 Pergamon, (1979)

Strain

Displacement: $\underline{u}(\underline{x})$ $\quad u_i(x_j)$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{Strain}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \qquad \text{Rotation}$$

First Order Theory

$$u_i(x_i + \delta x_i) = u_i(x_i) + (\epsilon_{ij} + \omega_{ij})\delta x_j + O(\delta x^2)$$

Higher order elastic terms.

Engineering Strain

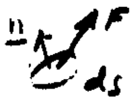
$$\gamma_{ij} = 2\epsilon_{ij} \quad i \neq j.$$

ress

Stress is a force exerted across a surface inside or on body of a body.

Arises because of the stiffness of a body. (solid or liquid)

Assume the force is proportional to the surface area.

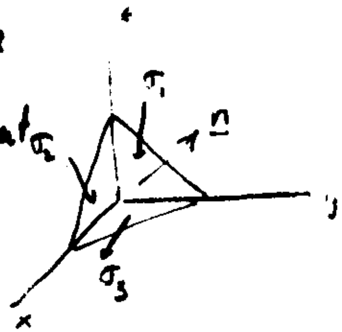


$$\text{force} = F\,dS$$
$$= \text{"Traction"}$$

ine

raction: Force acting across a surface - Positive Exerts on Negative

alance of Forces on a Volume



fine a function, $\sigma(x_i)$, such that the traction on OZY is σ_1, etc. is

$$T_{OZY} = \underline{\sigma_1}\, dA_{OZY} = \underline{\sigma_1 n_x}\, ds$$

d all tractions - must add to zero.

Thus, traction on XYZ is

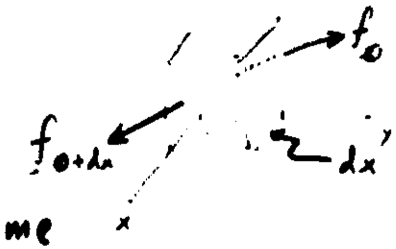
$$\boxed{f_i = \sigma_{ij} n_j}$$

Def. of stress tensor fn. f_n is traction on ds!

Equation of Equilibrium

(Elastic Field Eqn.)

$$\underline{f}_x(x+dx) = \underline{f}_{0x} + \frac{\partial f_x}{\partial x}\, dx$$



Thus total force on volume $dx\,dy\,dz$ is

$$\frac{\partial \sigma_{ix}}{\partial x} + \frac{\partial \sigma_{iy}}{\partial y} + \frac{\partial \sigma_{iz}}{\partial z} = \text{net stress force on } dV.$$

If a "body force" exists, (weight, electrical forces, etc.) Then

$$\boxed{\sigma_{ij,j} = f_{body}}$$

Transformation of Coordinates

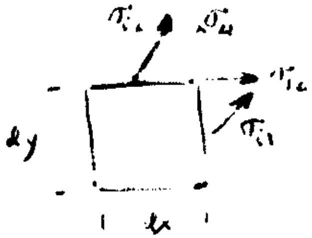
If Coordinates are transformed, (rotated

$$dx_i' = S_{ij}\, dx_j$$
$$f_i' = S_{ij} f_j$$
$$\sigma_{ij}' = S_{ik} S_{jl} \sigma_{kl}$$
$$\epsilon_{ij}' = S_{ik} S_{jl} \sigma_{kl}$$

(Note: no contra-variant - covariant distinction for rotations)

Symmetry of σ_{ij} ; Boundary Condition

If for an arbitrary volume, torque is zero, then



$$(\sigma_{21}\, dx)\, dy = (\sigma_{12}\, dy)\, dx$$

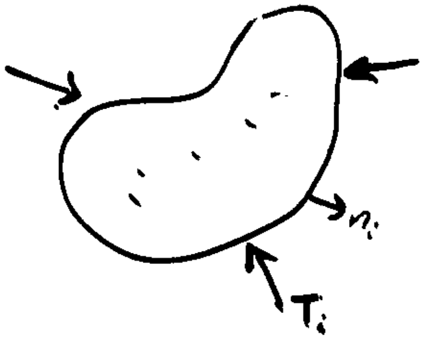
$$\sigma_{21} = \sigma_{12}$$

etc.

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

Boundary Condition on Stress.

If T_i is <u>External</u> force (traction) on body.



$$\boxed{\sigma_{ij} \cdot n_j = T_i}$$

Constitutive Relations : Hookes' Law.

<u>Assume</u> a linear force/response Relati

$$\sigma_{ij} = C_{ijkl}\, \epsilon_{kl}$$

($3^4 = 81$ C's)

But $\sigma_{ij} = \sigma_{ji}$; $\epsilon_{kl} = \epsilon_{lk}$

$$C_{ijkl} = C_{jikl} = C_{ijlk}$$ (36 C's)

Also

$$C_{ijkl} = C_{klij}$$ (From energy)

(21 C's in general)

Energy Density Fn.

$$\frac{dE}{dt} = \oint \sigma_{ij}\, \dot{u}_i\, dS_j = \int (\sigma_{ij}\, \dot{u}_i)_{,j}\, dV$$

since $\sigma_{ij,j} \to 0$

$$\frac{dE}{dt} = \int \sigma_{ij}\, \dot{\epsilon}_{ij}\, dV$$

Or per unit small volume

$$dE = \sigma_{ij}\, d\epsilon_{ij}$$

cont

to integrate from zero strain to final Strain,

$$dE = C_{ijkl}\,\epsilon_{ij}\,d\epsilon_{kl}$$

$$E = \frac{1}{2} C_{ijkl}\,\epsilon_{ij}\,\epsilon_{kl}$$

$$= \frac{1}{2}\sigma_{ij}\,\epsilon_{ij}$$

quadratic form in ϵ's or σ's.

(Note $C_{ijkl} = C_{klij}$)

Note

$$\sigma_{ij} = 2\frac{\partial E}{\partial \epsilon_{ij}}$$

which could have been taken as definition of σ.

Incompatibility

There exist 6 ϵ_{ij}, but only 3 u_i. Hence the strain components overdetermine a solution. A single valued function, $u_i(x_j)$ exists if the rotation is single valued, i.e.

$$\oint \omega_{i,n}\,dl_n = 0$$

But

$$\omega_i = \frac{1}{2}\epsilon_{ijk}\,u_{k,j}$$

ϵ_{ijk} = completely antisymmetric tensor

$\epsilon_{123} = -\epsilon_{213} = 1$, etc.

And

$$\omega_{i,n} = \epsilon_{ikl}\,\epsilon_{ln,k}$$

$$\oint \epsilon_{ijl}\,\epsilon_{ln,k} = 0.$$

Or curl [ ] = 0 :

$$\boxed{-\epsilon_{ikl}\,\epsilon_{jmn}\,\epsilon_{ln,km} = 0} \quad \boxed{= I}$$

I = incompatibility − (Burgers vector)

Reduced Notation + Isotropic Elast.

$C_{ijkl} \Rightarrow C_{ij}$

$11 \to 1$
$22 \to 2$
$33 \to 3$
$23 \to 4$
$31 \to 5$
$12 \to 6$

$(\gamma_{ij} = 2\epsilon_{ij} \quad i \neq j)$

$$\sigma_{ij} = C_{ij} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{pmatrix}$$

$\left.\begin{matrix}\gamma_4\\ \gamma_5\\ \gamma_6\end{matrix}\right\} \longrightarrow$ Note!

Cubic

$$\begin{matrix} c_{11} & c_{12} & c_{12} & & & \\ c_{12} & c_{11} & c_{12} & & & \\ c_{12} & c_{12} & c_{11} & & & \\ & & & c_{44} & & \\ & & & & c_{44} & \\ & & & & & c_{44} \end{matrix}$$

3 constants

Isotropic

$$\begin{matrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{matrix}$$

2 Constants

$$\boxed{\sigma_{ij} = \lambda \delta_{ij} u_{\ell,\ell} + \mu (u_{i,j} + u_{j,i})}$$

Isotropic Elasticity

2D Elasticity

$\left(\frac{\partial}{\partial z} = 0\right)$

Two Orthogonal Solutions

Anti Plane Strain (shear waves)

$u_3 \neq 0 \quad u_1 = u_2 = 0. \quad \left(\frac{\partial}{\partial z} = 0\right)$

$u_3(x_1, x_2)$

Introduce Complex Variable $x_1 + ix_2 = z$

Equilibrium (Field Eqns)

$\nabla^2 u = 0$

Then $u = \frac{1}{\mu} \operatorname{Im} \eta(z)$

Hooke's Law becomes

$\sigma(z) = \sigma_{32} + i\sigma_{31} = \eta'(z)$

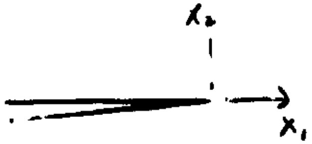
$\sigma_{r3} = \operatorname{Im}(\sigma e^{i\theta})$

$\sigma_{\theta 3} = \operatorname{Re}(\sigma e^{i\theta})$

$\eta(z)$ any complex fn. satisfying BC.

gular Solutions (2D)

ack Problem



$\eta' = z^{n/2} \quad n = \pm 1, 3, \ldots$

Energy enclosed at origin finite

$\int \sigma^2 dA = \int \sigma^2 r \, d\theta \quad ; \quad (r\sigma^2) \to 0$ at 0.

$\sigma \to 0$ at ∞

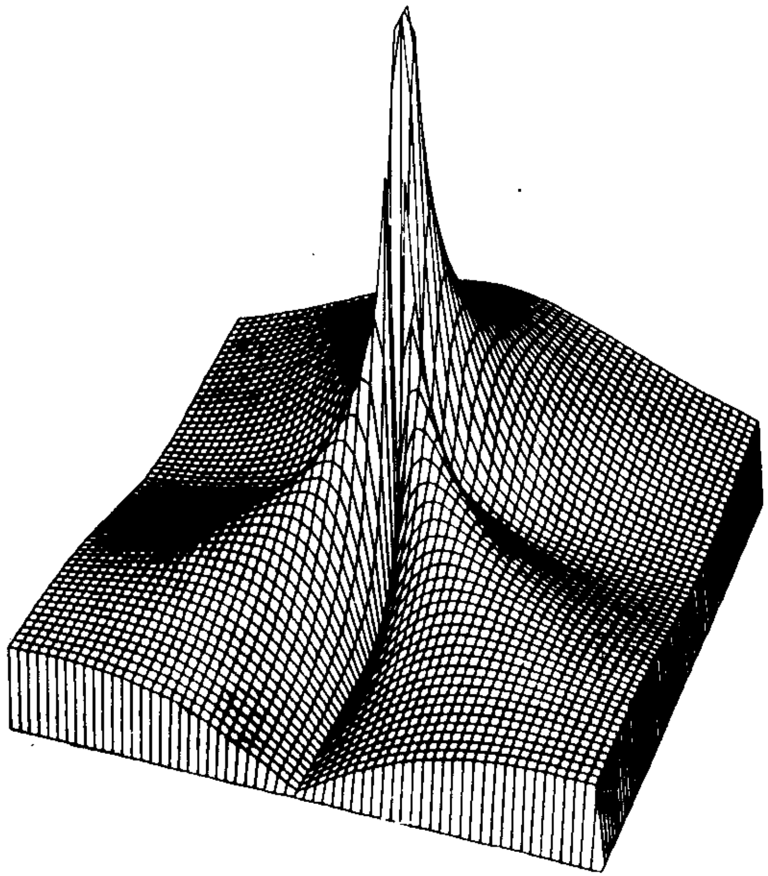
$\boxed{\sigma = \frac{K}{\sqrt{2\pi z}}}$ $\qquad \eta' = \frac{K}{2\sqrt{2\pi z}}$

K is Stress Intensity Factor

placement on $x_1 < 0$

$u_3 = \frac{2}{\mu} \operatorname{Im} \eta = \operatorname{Im} \sqrt{\frac{2}{\pi}} \, K \sqrt{z}$

On $x_2 < 0$ $\quad u_3 = \pm \sqrt{\frac{2r}{\pi}} \, K$



Dislocation

$$\oint u \, dl = b$$

(Incompatibility source)

Log functions have desired property

$$\log z = \log|z| + i\theta$$

$$\boxed{\eta(z) = \frac{\mu b}{4\pi} \ln z}$$

$\ln \ldots$

$$\boxed{\sigma(z) = \frac{\mu b}{2\pi z}}$$

Point Force

1. External Force F
2. Surrounding material exerts force $\int \sigma_{r3} \, r \, d\theta$ on material enclosed. (note sign of $\underline{n}$)
3. Total force on enclosed matter = 0.

$$-\int \sigma_{r3} \, r \, d\theta = F$$

If $\sigma = -iF/2\pi z = -iFe^{-i\theta}/2\pi r$; $\sigma_{r3} = -\frac{F}{2\pi r}$

Then integration is satisfied.

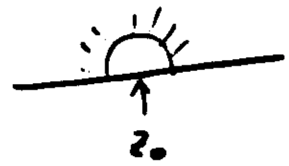
$$|F = F_z|$$

Green's Function for Crack

A crack stress field must be generated by an external stress field. Otherwise nothing opens the crack. These forces may be applied to open surfaces of the crack, or to actual external surfaces of the specimen. (In the latter case, the solution will depend on shape of specimen.)

For an ∞ surface

This is ½ previous problem of Point force.



$$\eta'_{\text{(half plane)}} = \frac{F}{2\pi(z - z_0)i}$$

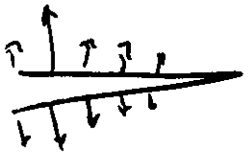
To satisfy BC for cracks, must have $1/\sqrt{z}$ at crack tip.

Thus, multiply by $\sqrt{z_0/z}$. Then near z_0, have point force, near crack, get necessary $1/\sqrt{z}$

$$\boxed{\eta' = \frac{1}{\pi(z - z_0)i}\sqrt{\frac{z_0}{z}}}$$

UNIT FORCE.

-eral Solution for loaded semi ∞ crack



$$\sigma(z) = \frac{1}{\pi\sqrt{z}} \int p(\cdot)\, dt$$

singular for $(z) = -t$

$p(t)$ is stress distribution on crack surface.
(σ_{22})

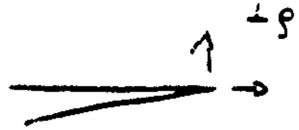
Note $P(t)$ is equal + opposite on the two surfaces.

(forces which are in same sense on the two surfaces correspond to surface stress — not discussed)

Because of crack loading, a K-field is generated.

$$K = \ldots = \frac{2}{\ldots}\int \frac{\ldots}{\sqrt{t}}\, dt$$

Dislocation + Crack



Method of Solution

1. Note crack surfaces must be traction free.
   i.e. $\sigma_{i2} = 0 \quad x_1 < 0$.
2. But Dislocation in free space generates a stress on $x_1 < 0$.
3. Add stresses which cancel the stresses on $x_1 < 0$.

$$\sigma = \sigma_{\cdot} + \ldots$$

$$\sigma_{\cdot} = \frac{K}{\ldots} + \frac{\mu b}{2\pi(z-\ldots)}$$

$$\sigma_{\cdot}(z) = \frac{1}{\pi}\int \ldots \sqrt{t}$$

$$-R_e\,\sigma_{\perp} = \frac{\mu b}{4\pi}\,\frac{t + \frac{\xi+\bar{\xi}}{2}}{(t+\xi)\ldots}$$

Final

$$\sigma = \frac{K}{\ldots} + \frac{\ldots}{4\pi\sqrt{z}}\left(\frac{\ldots}{\sqrt{z}-\sqrt{\xi}} - \frac{b}{\sqrt{z}-\sqrt{\bar{\xi}}}\right)$$

$$= \frac{K}{\ldots} + \ldots\left[\frac{1}{z-\xi}\left[\sqrt{\frac{\xi}{\ldots}}+1\right] \ldots \frac{\ldots}{z\,\bar{\xi}}\left[\sqrt{\frac{\bar{\xi}}{z}}-1\right]\right]$$

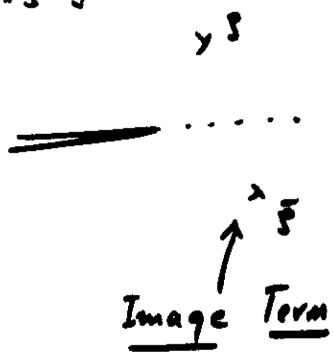
The form of solution shows an extra "$\frac{1}{\sqrt{\xi}}$" contribution from Dislocation.

This is termed SHIELDING of crack by Dislocation —

Thus

$$\underset{\text{"out side"}}{\underset{\uparrow}{K}} = \underset{\text{local crack}}{\underset{\uparrow}{k}} + \underset{\text{Dislocation term}}{\underset{\uparrow}{k^D}}$$

$$k^D = \frac{\mu b}{2}\left[\frac{1}{\sqrt{2\pi\xi}} + \frac{1}{\sqrt{2\pi\bar{\xi}}}\right]$$



Shielding Depends on sign of b.

Shielding	$b>0$
Anti shielding	$b<0$

(see Prof Lung)

PLANE STRAIN

Mathematics is generally similar— But more complicated.

Use complex variable. Need a function which represents stress as before. But there are 3 independent stresses, $\sigma_{11}, \sigma_{22}, \sigma_{23}$. Hence need $\underline{\underline{2}}$ fns.

Plane Strain

$u_3 \equiv 0.$ ...

$\sigma_{3i} \equiv 0.$ $\sigma_{11}, \sigma_{22}, \sigma_{12}$

The two complex functions describing plane strain are from Goursat.

1. Write Equil. Eqns: $\nabla\cdot\sigma = 0.$

$$\sigma_{11,1} + \sigma_{12,2} = 0$$

$$\sigma_{22,2} + \sigma_{12,1} = 0.$$

These are same as:

$$\frac{\partial}{\partial z}\left[\sigma_{11} - \sigma_{22} + 2i\sigma_{12}\right] + \frac{\partial}{\partial \bar{z}}\left[\sigma_{11} + \sigma_{22}\right] = 0.$$

The two terms are independent.

Second is real. + in terms of a $\phi(z)$,

$$\sigma_{11} + \sigma_{22} = 2\left(\phi'(z) + \overline{\phi'(z)} \right)$$

by integration of first [ ], See (Appendix A)

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left[\bar{z}\, \phi''(z) + \psi'(z) \right]$$

ϕ + ψ are now two independent complex fns. from which stresses (and displacements) can be obtained. Use a different pair, convenient for crack problem.

$$\left[\begin{array}{l} \sigma_{11} + \sigma_{22} = 2\left(\phi(z) + \overline{\phi'(z)}\right) \\ \sigma_{22} - i\sigma_{12} = \varphi' + \overline{\omega'} + (z - \bar{z})\,\overline{\varphi''} \\ 2\mu(u_1 + i u_2) = 2\mu u = \kappa\varphi - (z - \bar{z})\overline{\varphi'} - \overline{\omega} \end{array}\right]$$

φ + ω are elastic potentials + are analytic complex fns.

—

Solution of Plane Strain Crack Problem.

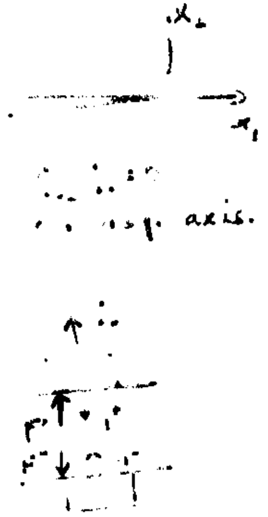
$$(\sigma_{22} + i\sigma_{12})^{+}(\ldots) + F^{+} = 0$$

$$(\sigma_{22} - i\sigma_{12})^{+} = \left[\varphi' + \overline{\omega'}\right] = -\overline{F}^{+}$$

Likewise, on bottom

$$(\sigma_{22} - i\sigma_{12})^{-} = \left[\varphi' + \overline{\omega'}\right]^{-} = -\overline{F}^{-}$$

Note, $F = \underset{\text{mode I}}{F_{I}} + i \underset{\text{mode II}}{F_{II}}$



Load analysis - F is a Force distribution on the two cleavage surfaces.

$\varphi + \overline{\omega}$ is not an analytic fn. of z because of $\overline{\omega}$.

Define

$f(z)$ analytic

Then $f^{*}(z) = \overline{f(\bar{z})}$

If $f = az + bz^2$,

$f^{*} = \bar{a}z + \bar{b}z^2 = f^{*}(z)$

Now on x_1 axis,

$$\overline{\omega(z)}\Big]_{x\,\text{axis}} = \omega^{*}(z)\Big]_{x\,\text{axis}}$$

Thus BC on crack:

$$[\varphi'(z) + \omega'^*(z)]^+ = -\bar{F}^+$$

$$[\varphi'(z) + \omega'^*(z)]^- = -\bar{F}^-$$

Hence $\varphi'(z) + \omega'^*(z)$ has discontinuity on crack plane

$$[\varphi'(z) + \omega'^*(z)]^+_- = -2\bar{F} \qquad (\bar{F}^+ = -\bar{F}^-)$$

But $\varphi'(z) - \omega'^*(z)$ is analytic every where.

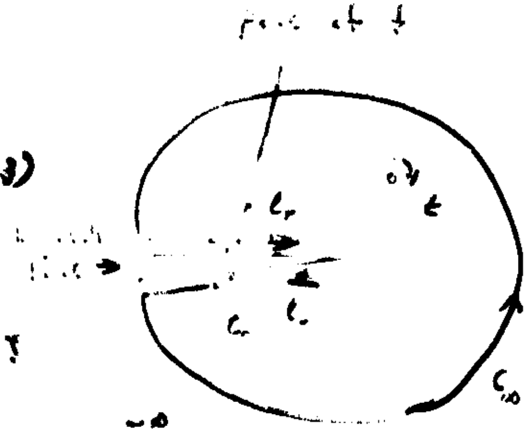
$$[\varphi'(z) - \omega'^*(z)]^+_- = 0.$$

Thus we require a function $\varphi + \omega$ which satisfy these b.c..

This is a Hilbert problem.

Hilbert Problem

Cauchy Integral for $f(z)$



$$f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{f(t)}{t-z} \, dt + \frac{1}{2\pi i} \int_{0}^{-\infty} \frac{f(t)}{t-z} \, dt$$

$$+ \frac{1}{2\pi i} \int_{C_\infty} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

If f is regular at ∞ the $\int_{C_\infty} = 0$.

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{0} \frac{g(t)}{t-z} \, dt$$

$$g(t) = f^+ - f^- \qquad \left\{ \begin{array}{l} g \text{ is discontinuity} \\ \text{at branch line.} \end{array} \right.$$

$$\boxed{\text{If } g(t) \text{ is known on branch then } f(z) \text{ is known every where}}$$

For crack, we have a discontinuity of form

$$\varphi'(z) + \omega'^*(z) = \chi(z)$$

where $\chi^+(z) + \chi^-(z) = h(t)$ on branch.

skhelishvili Approach:

Find an "integration factor" which makes $h(t)$ have property of $g(t)$ on branch

s

If $\left[\chi(z)\right]^+ + \left[\chi(t)\right]^- = h(t)$ (known)

Find $\Xi(z)$

$$\left[\frac{\chi(t)}{\Xi(t)}\right]^+ - \left[\frac{\chi(t)}{\Xi(t)}\right]^- = g(t) \quad \text{(known)}$$

is not surprising to find $\sqrt{z}$ is an interesting function :

$$\frac{\left[\sqrt{z}\right]^+}{\left[\sqrt{z}\right]^-} = -1$$

e can therefore write

$$\sqrt{z}\left(\varphi' + \omega'^{*}\right) = \frac{1}{2\pi i}\int_{-\infty}^{0} \frac{2\,\overline{F}(t)\sqrt{t}}{(t-z)}\,dt$$

and we have derived a Green's function for the stress distribution on the cleavage plane.

Since

$$\left[\varphi'(t) - \omega^{*\prime}(t)\right]_-^+ = 0$$

$\varphi' - \omega^{*\prime}$ have no discontinuities. Hence

$$\varphi'(z) = \omega^{*\prime}(z) \quad \text{everywhere}$$

Thus

$$\left[\begin{aligned} \varphi'(z) = \omega^{*\prime}(z) &= \int_{-\infty}^{0} \overline{F}(t)\, g(z,t)\, dt \\ \eta'(z) &= \int_{-\infty}^{0} F_{\mathrm{III}}\, g(z,t)\, dt \\ g(z,t) &= \frac{1}{2\pi i\,(t-z)}\sqrt{\frac{t}{z}} \end{aligned}\right]$$

Unified result for Mode III + Mode I, II. Thus, η', + φ' are on same footing as potentials for elastic 2D problem.

Now any problem in plane strain is solved in similar way as mode III, because Green's fn. are same. B.C. are always that $\varphi'(t)^+ + \varphi'^- = 2\overline{F}$

Crack K-fields

From the Green's Fns. the stress field of the crack can be determined (For Green's Fns for a finite crack, see SSP,29)

From the Definition of K:

$$\bar{K} = \lim_{z \to 0} z\sqrt{2\pi z}\, \varphi'(z)$$

$$K = K_I + i K_{II}$$

For a point force at t: $(t>0)$

$$\bar{K} = \sqrt{\frac{2}{\pi}}\, \frac{\bar{F}}{\sqrt{t}} \qquad F = F_I + i F_{II}$$

For finite crack - see SSP.

$$K = F\sqrt{\pi a}$$

$a = \frac{1}{2}$ crack length
F = constant force

Dislocation Stress Fields

Definition of Dislocation

Thus φ and ω are double valued function with $\underline{b}$ discontinuity at origin

$$\oint \frac{du}{ds}\, ds = \underline{b}$$

Try

$$\varphi' = \frac{A}{z} \qquad \varphi = A \ln z$$

$$\omega' = \frac{B}{z} \qquad \omega = B \ln z$$

From Eqn for $u(z)$,

$$2\mu u = \kappa A \ln z - (z - \bar{z})\, \frac{\bar{A}}{\bar{z}} - \bar{B} \ln \bar{z}$$

$$\Delta u = b = b_1 + i b_2 = \frac{\kappa}{2\mu}\, 2\pi i A + \frac{\bar{B}}{2\mu}\, 2\pi i$$

$$A = \bar{B} = \frac{\mu b}{\pi i (\kappa + 1)} \qquad \kappa = 3 - 4\nu.$$

Satisfies B.c. at origin - also gives $\sigma \to 0$ at ∞

ut such a solution gives very strange translation properties. That is,

If $\varphi' = \dfrac{A}{z-\zeta}$,

$$\sigma_{11} - i\sigma_{12} = \frac{A}{z-\zeta} + \frac{A}{\bar z - \bar\zeta} - \frac{2\zeta\bar A}{(\bar z - \bar\zeta)^2}$$

not invariant

ence, Add a term which makes third term invariant with traslation.

$$\varphi' = \frac{A}{z-\zeta} \qquad \omega' = \frac{\bar A}{z-\zeta} - \frac{(\zeta-\bar\zeta)}{(z-\zeta)^2}A$$

---

rom these solutions for dislocation, can solve crack-dislocation problem.

$$\varphi' = \varphi'_{crack} + \varphi'_{xloc} + \varphi'_1 \; ; \; \omega' = \cdots$$

where φ'_1 chosen to make

$$\varphi' \to 0 \text{ on neg. real axis}$$

$$\left[\varphi'_1 = -\varphi'_{xloc}\right]_{\text{neg. axis}}.$$

See SSP. p 34. for details

Result is again the induced K-field at crack tip.

$$K^D = K_I^D + i K_{II}^D = \frac{i\mu}{2(1-\nu)}\left[\frac{b}{\sqrt{2\pi\zeta}} + \frac{\bar b}{\sqrt{2\pi\bar\zeta}} + \frac{\pi\bar b(\zeta-\bar\zeta)}{(2\pi\bar\zeta)^{3/2}}\right]$$

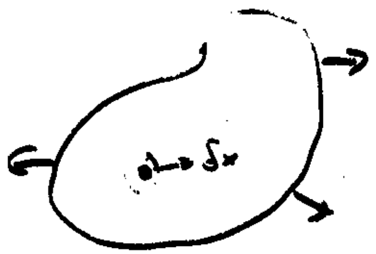
image term

"extra" complexity

Eshelby's Theorem: Force on Elastic Singularities

SSP p41

3 stages to analysis



① Assume an energy density fn. exists [single valued.]

Assume the soln. is rigidly displaced.

$$W(x) = W_0 - \frac{\partial W}{\partial x_i}\,\delta x_i \qquad \delta x = \text{const. displ.}$$

$$\delta U^1 = -\int \frac{\partial W}{\partial x_i}\,\delta x_i\, dV = -\oint W\, dS_k\, \delta x_k$$

② Resatisfy BC. on surface $(\sigma_{ij}\, dS_j = 0)$

Assume new stresses are added to surface to satisfy the new BC. These generate additional displ., Δu_i

$$\Delta u_i = u_i^{final} - \underbrace{\left(u_i^0 - \frac{\partial u_i^0}{\partial x_j}\,\delta x_j\right)}_{\text{rigid disp part.}}$$

Work done by these stresses is

$$\delta U^2 = \oint (\sigma_{ij}^0 + \Delta\sigma_{ij})\,\Delta u_i\, dS_j$$

$$\simeq \oint \sigma_{ij}^0\,\Delta u_i\, dS_j$$

③ Final contribution is due to change in external sources of force on body of body (i.e. a weight might fall)

$$\delta U^3 = -\int \sigma_{ij}^0\,(u_i^{final} - u_i^0)\, dS_j$$

$$\delta U = \delta U^1 + \delta U^2 + \delta U^3$$

$$= -\int (W\,\delta_{ik} - \sigma_{ij}\, u_{j,k})\, dS_k\, \delta x_i$$

$$\boxed{f_i = \oint (W\,\delta_{ik} - \sigma_{ij}\, u_{j,k})\, dS_k}$$

Note

1. W need not be linear — But must be single valued fn. — otherwise Gauss theorem invalid.
2. This is a 3-D result.
3. Independent of surface.

roof of Surface independence:

$$f^{1}-f^{2}=\oint\left(w\,\delta_{kj}-\sigma_{ij}u_{i,k}\right)ds_j$$

$$=\int_{\substack{\text{enclosed}\\ \text{Vol.}}}\left(W_{,k}-\sigma_{ij}u_{i,jk}\right)\qquad \sigma_{ij,j}=0.$$

Now

$W=\int\sigma_{ij}\,d\epsilon_{ij}$ If $\sigma_{ij}(\epsilon_{ij})$ – No explicit dep. on x.

$$W_{,k}=\epsilon_{\ell m,k}\frac{d}{d\epsilon_{\ell m}}\left[\int\sigma_{ij}\,d\epsilon_{ij}\right]=\epsilon_{\ell m,k}\,\sigma_{\ell m}$$

$$\epsilon_{\ell m,k}=\frac{1}{2}\left[u_{\ell,mk}+u_{m,\ell k}\right]$$

$$W_{,k}=\sigma_{ij}u_{j,ik}$$

QED.

Note 1) Independence depends on not having singularities between S_1 + S_2.

Again – elastic need not be linear.

But matl must be homogeneous.

2) These theorems apply to cracks because integrand is either symmetric or zero.

Force in 2D

Independence of f on contour is very reminiscent of Cauchy theorem. Suggests the force is a property only of the singularity itself.

Thus

Convert the Contour integral by Cauchy theorem to a statement about residues of stress functions:

Plane Strain

$$\bar{f}=f_1-if_2=\sum_{\text{Res}}\frac{2\pi(1-\nu)}{\mu}\left[2\,\text{Re}\left(\varphi'\omega'-\varphi'^2-\bar{z}\varphi'\omega''\right)+\text{Res}\left(\varphi^2\right)\right]$$

Anti Plane Strain

$$\bar{f}=\frac{\pi}{\mu}\sum_{\text{Res}}\text{Res}\left(\sigma^2\right)$$

These expressions are useful when dislocations + cracks interact.

Anti plane strain – especially simple

plane strain. φ, ω, already given.

<u>Simple cases</u>

<u>Dislocation</u>:

$$\sigma = \sigma_0(x) + \frac{\mu b}{2\pi z} \qquad \text{(anti plane)}$$

$$\bar{f} = b\,\sigma_0 \qquad \text{(Peach-Koehler)}$$

(no self force)!

<u>Crack</u>

$$\sigma = \frac{K_{III}}{2(\pi z)^{1/2}} \qquad \text{(pure self force)}$$

$$\bar{f} = \frac{k_{III}^2}{2\mu} \qquad \text{(anti plane strain)}$$

"known as energy release rate" – Derived by Irwin

$$\bar{f} = \frac{1-\nu}{2\mu}\left(k\bar{k} + \frac{k^2 - \bar{k}^2}{2}\right)$$

$$f_1 = \frac{1-\nu^2}{E}\left(k_I^2 + k_{II}^2\right) = \frac{1-\nu}{2\mu}(\ldots)$$

Crack force unphysical

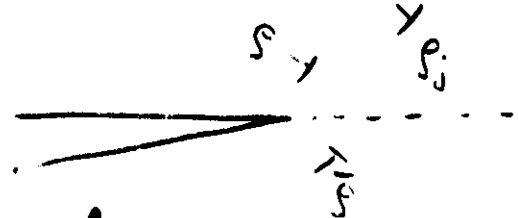
Dislocation force w/ crack.

(Mode III)

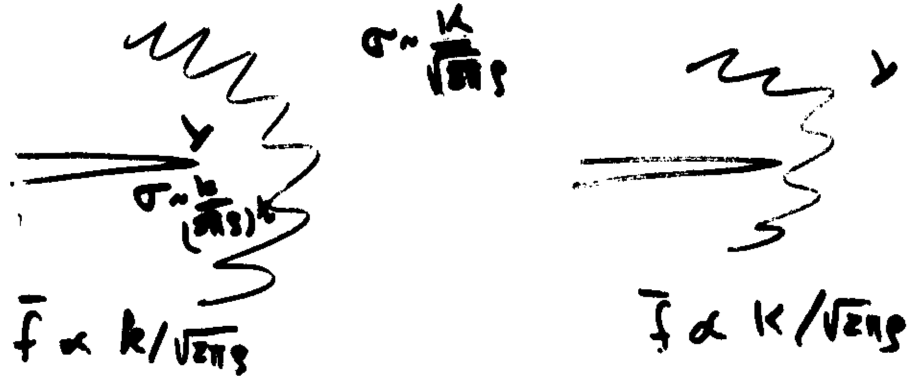
$$\bar{f} = \frac{K_{III}\, b}{\sqrt{2\pi\rho}} - \frac{\mu b^2}{4\pi}\left[\frac{1}{\ldots} \cdot \frac{1}{\rho-\bar{\rho}} - \left(\frac{\bar{\rho}}{\rho}\right)^{1/2} \frac{1}{\rho-\bar{\rho}}\right]$$

$$+ \sum_j \frac{\mu b b_j}{4\pi}\left[\frac{1}{\rho-\rho_j} - \frac{1}{\rho-\bar{\rho}_j} + \sqrt{\frac{\rho_j}{\rho}}\,\frac{1}{\rho-\bar{\rho}_j} + \sqrt{\frac{\bar{\rho}}{\rho}}\,\frac{1}{\rho-\bar{\rho}_j}\right]$$

disl – disl.



Considerable change from crack term.

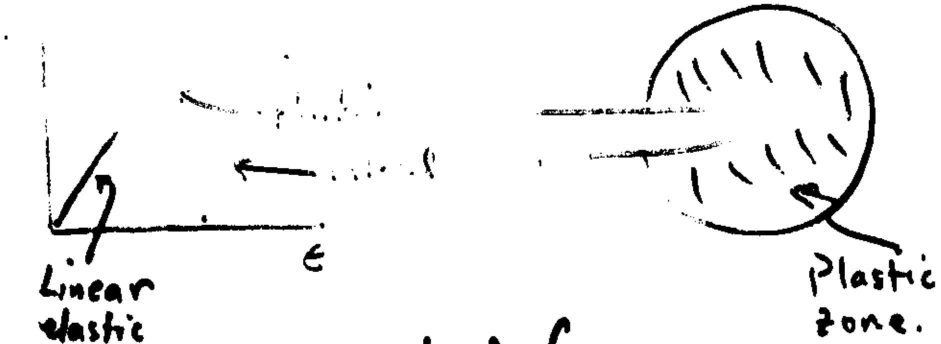


For single crack singularity,

$$f = \frac{k^2}{2\mu}$$

But Eshelby theorem valid for general nonlinear solid.

Known in this form as Rice J-integral.



$\sigma(\epsilon)$ a multiple valued fn.

So Eshelby theorem not valid for unload

(Applied anyway!)

In this way toughness for plastic matl. can be measured using J-ideas — but not for unload (fatigue)

Idea is that J can be integrated in macro region, + not need to do it thru plastic zone.

Mechanics

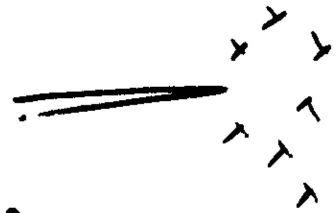
Solve elastic-plastic problem from a plastic continuum constitutive law (replaces Hooke's Law)

Highly numerical (finite element)

Cannot deal (except empirically) with zone where material physically fails.

Used with considerable success to treat "hole growth" ductile fracture.

Physical



Consider plastic zone to be pure elastic plus dislocations.

Then consider details of crack-dislocation interactions.

Can then hope to deal with close-in zone when disl. # not too large.

Must adopt continuum if large #'s of disl., + revert to mechanics.