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WORKING PARTY ON "FRACTURE PHYSICS"
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ELASTIC THEORY OF DYNAMIC CRACKS

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Dynamic Elasticity (F L J Mat Res 3, 250 (83))

The equations of dynamic elasticity (ρ is indep of t

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \sigma_{ij} = 0 \qquad (1)$$

can be obtained as extrema $\delta S = 0$ of the action integral

$$S = \frac{1}{2} \int dt\, d^3x \left[\rho \left(\frac{\partial u}{\partial t}\right)^2 - \sigma_{ij} \frac{\partial u_i}{\partial x_j} \right]$$

General invariance principles (Noether's theorem), or direct computation from (1) insure that conservation laws hold:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = 0$$

$$\frac{\partial T^{j0}}{\partial t} + \frac{\partial T^{ji}}{\partial x^i} = 0$$

where

$$T^{00} = \frac{\rho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{2}\sigma_{ij}\frac{\partial u^i}{\partial x^j} \qquad \text{energy density}$$

$$T^{0i} = -\sigma_{ij}\frac{\partial u^j}{\partial t} \qquad \text{energy flux}$$

$$T^{i0} = -\rho \frac{\partial u^j}{\partial t}\frac{\partial u^j}{\partial x^i} \qquad \text{momentum density}$$

$$T^{ij} = \sigma_{mj}\frac{\partial u^m}{\partial x^i} + \frac{1}{2}\left[\rho\left(\frac{\partial u}{\partial t}\right)^2 - \sigma_{mn}\frac{\partial u^m}{\partial x^n}\right]\delta_{ij} \qquad \text{momentum flux}$$

is the energy-momentum tensor of elasticity.

See papers by J. Rice & L.B. Freund in "Fundamentals of Deformation & Fracture" Ed. by B. Bilby et. al. Cambridge U.P. 1985

GENERAL REFS. R. Thomson in "Solid State Physics" series

i) static case

$$T^{ij}{}_{,j} = 0$$

$$J^i = \int dS_j \, T^{ij} = 0$$ for a closed surface that does not enclose a singularity

If there is a singularity, $J^i \neq 0$ in general but it is path-independent

It can be shown (see Thomson) that

$$J^i = -\nabla_i U$$ energy

the change in elastic energy per unit advance of the singularity alias "the force on the singularity"

a) Disloc. $\sigma \sim \frac{1}{r}$ but there is no self force

$$\sigma = \Sigma + \sim\frac{1}{r} \Rightarrow \text{Peach-Koehler}$$

b) Crack: $\sigma \sim \frac{1}{\sqrt{r}}$ gives the energy release rate

$$J_i = \frac{\text{mom.}}{\text{time}\ \cancel{\text{}}} = \frac{\text{force}}{\cancel{\text{}}} = \frac{\text{energy}}{\text{length}}$$

$$T^{ij} = \cancel{\delta_{ij}}\ \Sigma_{mj} \frac{\partial u^m}{\partial x^i} \qquad J_i = \underbrace{\Sigma^{mj}}_{\frac{1}{r}} \int dS_j \underbrace{\frac{\partial u^m}{\partial x^i}}_{\sim b\frac{1}{r}}$$

ii) equations :

$$\int_V d^3x \frac{\partial T^{00}}{\partial t} + \int_V d^3x \frac{\partial T^{0i}}{\partial x^i} = 0$$

$$\frac{d}{dt}\int d^3x\, T^{00} \quad \int_{\Delta V/\Delta t} d^3x\, T^{00} + \int dS^i T^{0i} = 0$$

change of energy inside volume V

$$\frac{dE}{dt} = -\int dS^i \left(T^{0i} - v^i T^{00} \right)$$

$\uparrow$

equation of motion is governed by $\frac{1}{r}$ part of T^{0i}

i) crack : coef. of $\frac{1}{\sqrt{r}}$

ii) disloc

So that no inertia ~~result of~~ of crack depends on the coeff. of $\frac{1}{\sqrt{r}}$ ~~term~~ depending on vel. and not on higher time derivatives.

$$\int d^3x \frac{\partial T^{i0}}{\partial t} + \int d^3x \frac{\partial T^{ij}}{\partial x^j} = 0$$

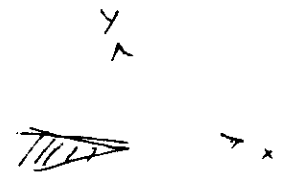
$$\frac{dP^i}{dt} = -\int dS^j \left(T^{ij} + V^j T^{i0} \right)$$

Force on a moving singular alias ~~momento~~ change in momentum per unit time

Plasticity, MIT Press 1969

Crack dynamics. A: ~~III~~ Mode III

We are interested in what goes on near the crack tip.



In the static case (Notation of Eshelby in Argon book 1969)

$$\sigma_{zx} = \mu \frac{\partial w}{\partial x} = \frac{-K}{\sqrt{2\pi}} \frac{\sin \theta/2}{\sqrt{r}}$$

$K = K_{III}$

$$\sigma_{zy} = \mu \frac{\partial w}{\partial y} = \frac{K}{\sqrt{2\pi}} \frac{\cos \theta/2}{\sqrt{r}}$$

Note: i) $\frac{1}{\sqrt{r}}$ singularity. Corrections are of order $\left(\frac{r}{l}\right)$
↳ length of crack, only length scale around

ii) $\theta = \pm\pi \Rightarrow \sigma_{zy} = 0$
→ zero traction across crack surfaces

iii) $w = \sqrt{\frac{2}{\pi\mu^2}}\, K \sqrt{r} \sin\frac{\theta}{2}$

$\Delta w = 2\sqrt{\frac{2}{\pi\mu^2}}\, K \sqrt{x}$ discontinuity in displacement behind crack tip.

Dynamics: eqn is now

$$\frac{\partial^2 w}{\partial t^2} - c^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad c \text{ - shear wave speed}$$

boundary cond: $\sigma_{zy}(x,0) = 0 \quad x < \xi(t)$

uniform motion

$$w(x,y,t) = w(\overbrace{x-vt}^{x^*}, y)$$

$$\overbrace{\left(1-\frac{v^2}{c^2}\right)}^{\gamma^2}\frac{\partial^2 w}{\partial x^{*2}} + \frac{\partial^2 w}{\partial y^2} = 0$$

$$\tilde{x} = \frac{x-vt}{\gamma}$$

$$\frac{\partial^2 w}{\partial \hat{x}^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

for a screw dislocation boundary cond is

$$\Delta w\Big|_{\tilde{x}<0} = b \quad \text{(Burgers}$$

which is the same as static case

$$\Rightarrow w = \frac{b}{2\pi}\tan^{-1}\frac{y}{\tilde{x}}$$

for a mode III crack

note $\sigma_{zx} = \mu\dfrac{\partial w}{\partial x}$ still $\quad \sigma_{zy} = \mu\dfrac{\partial w}{\partial y}$

$$= \frac{\mu}{\gamma}\frac{\partial w}{\partial \tilde{x}}$$

and b.c. is

$$\sigma_{zy}(x,0) = 0 \quad \tilde{x} < 0$$

in the static case we had

$$r^2 = x^2 + y^2 \qquad \theta = tg^{-1}\frac{y}{x}$$

A moving crack is obtained by a "Lorentz" transfo

$$\tilde{x} = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\tilde{y} = y$$

However, as in the static case in which the stress intensity factor K is not determined if the finite length of the crack is not considered, here there is also an undetermined velocity dependent constant of proportionality. This constant can be fixed in terms of K if we consider a crack that is at rest and starts moving with velocity v at $t=0$ (Eshelby) :

$$K \to K\left(\frac{1-v/c}{1+v/c}\right)^{1/4}$$

to see this :

$$w_0 = B\sqrt{r}\,\sin\frac{\theta}{2}$$

$$w_v = A(v)\sqrt{\tilde{r}}\,\sin\frac{\tilde{\theta}}{2} \qquad \gamma = \sqrt{1-v^2/c^2}$$

$$\tilde{r} = \sqrt{\tilde{x}^2+\tilde{y}^2}$$

$$\tilde{\theta} = \operatorname{Arctg}\frac{\tilde{y}}{\tilde{x}}$$

Continuity condition

$$w_v\big|_{r=ct} = w_0\big|_{r=ct}$$

$$w_v \Big|_{r=ct} = A(v) \left(\frac{1}{\gamma^2} (ct\cos\theta - vt)^2 + c^2t^2 \sin^2\theta \right)^{1/4} \cdot$$

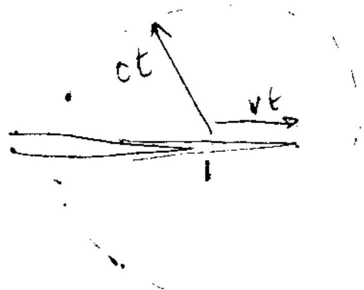
$$\cdot \sin \left(\frac{1}{2} \operatorname{Arctg} \left(\frac{\gamma ct \sin\theta}{ct\cos\theta - vt} \right) \right)$$

let $\theta \approx 0$

$$= A(v) \left(\frac{1}{\gamma^2} (ct - vt)^2 \right)^{1/4} \frac{\theta}{2} \frac{\gamma ct}{ct - vt}$$

$$= A(v) \sqrt{ct} \, \frac{\theta}{2} \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right)^{1/4}$$

$$\Rightarrow \qquad A(v) = B \left(\frac{1 - v/c}{1 + v/c} \right)^{1/2}$$



Δt ~ typical time between ~~acc~~ changes in velocity

$r < ct \sim c \frac{v}{A} \sim \lambda$

cannot have singularities at a distance $< \lambda$

If, at time t_1 $v \longrightarrow v_1$, solution is again given by

$$w_{v_1} = A(v_1, v) \sqrt{\tilde{r}} \sin \frac{\tilde{\theta}}{2}$$

$$\tilde{x} = \frac{(x - vt_1) - v_1(t - t_1)}{\sqrt{1 - v_1^2/c^2}}$$

$$\frac{1-\dfrac{v_1+v_2}{c\left(1+\frac{v_1v_2}{c^2}\right)}}{1+\dfrac{v_1+v_2}{c\left(1+\frac{v_1v_2}{c^2}\right)}} = \frac{c+\frac{v_1v_2}{c}-v_1-v_2}{c+\frac{v_1v_2}{c}+v_1+v_2}$$

$$w_v\Big|_{r=c(t-t_1)} = A(v)\left[\frac{1}{\gamma^2}\left(c(t-t_1)-vt\right)^2\right]^{1/4} \cdot$$

$\theta\sim 0$

$$\cdot\ \frac{\theta}{2}\ \frac{\gamma c(t-t_1)}{c(t-t_1)-vt}$$

$$= A(v)\left[\frac{(c(t-t_1)-vt)^{1/2}}{\gamma^{1/2}}\ \frac{\gamma c(t-t_1)}{c(t-t_1)-vt}\right]\ \frac{\theta}{2}$$

$$= A(v)\ \frac{\gamma^{1/2}\ c(t-t_1)}{(c(t-t_1)-vt)^{1/2}}\ \frac{\theta}{2} = B\left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)^{1/4}\frac{\left(1-\frac{v^2}{c^2}\right)^{1/4}c(t-t_1)}{(c(t-t_1)-vt)}$$

$$= B\ (1-v/c)^{1/2}\ c(t-t_1)/(c(t-t_1)-vt)$$

$$w_{v_1} = D\sqrt{\tilde{\tilde{r}}}\ \sin\frac{\tilde{\tilde{\theta}}}{2}$$

$$\tilde{\tilde{r}}^2 = \left(\frac{r\cos\theta-vt_1-v_1(t-t_1)}{\gamma_1}\right)^2 + r^2\sin^2$$

$$\tilde{\tilde{\theta}} = \operatorname{Arctg}\frac{\gamma r\sin\theta}{r\cos\theta-vt_1-v_1(t-t_1)}$$

$$\tilde{\tilde{r}}\underset{\theta\sim 0}{\sim}\frac{\left(c(t-t_1)-vt_1-v_1(t-t_1)\right)}{\gamma_1}$$

$r=c(t-t_1)$

$$\tilde{\tilde{\theta}} = \theta\ \frac{\gamma c(t-t_1)}{c(t-t_1)-vt_1-v_1(t-t_1)}$$

$$\cancel{w_{v_1} \approx C\frac{\theta}{2}\sqrt{\frac{ct - vt_1 - v_1(t-t_1)}{1 - v_1^2/c^2}}}$$

$$w_{v_1} = D\frac{\theta}{2}\left(\frac{c(t-t_1) - vt_1 - v_1(t-t_1)}{\gamma_1}\right)^{1/2} \frac{\gamma_1 c(t-t_1)}{c(t-t_1) - vt_1 - v_1(t-t_1)}$$

$$= D\frac{\theta}{2}\frac{\gamma_1^{1/2}\, c(t-t_1)}{\left(c(t-t_1) - vt_1 - v_1(t-t_1)\right)^{1/2}}$$

$$D\frac{\gamma_1^{1/2}\, c(t-t_1)}{\left(c(t-t_1) - vt_1 - v_1(t-t_1)\right)^{1/2}} = B\frac{\left(1-\frac{v}{c}\right)^{1/2} c(t-t_1)}{\left(c(t-t_1) - vt\right)^{1/2}}$$

Freund

Mode I

* J. Mech. Phys. Solids 20 129-140, 141-152 (1972)

* in "Mech. Today Vol 3" ed. by S. Nemat-Nasser Pergamon 17(

Energy Flux $\sim$ const $\frac{K_I^2(v)}{D} \cdot v^2 \gamma_\ell$

$$\cancel{D = 4 \gamma^3 - (1+}$$

$$D = 4\gamma_s \gamma_\ell - (1+\gamma_s^2)^2$$

$$\gamma_s = \sqrt{1 - v^2/c_s^2}$$

$K_I(v) \xrightarrow[v \to v_R]{}$

$D = 0$ vel of Rayleigh waves

talk about R waves if time permit

We want to study crack dynamics when a defect is close (dist < λ) to crack tip.

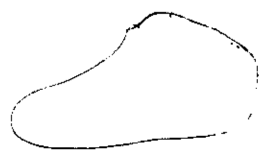
Idea: to use ~~the~~ fact that, mathematically at least, crack "is" a continuous distribution of dislocations.

So, first, study dislocation dynamics

(Mura, Philos. Mag $\underline{8}$ 843 (63))

Solve
$$\rho \frac{\partial^2 u_i}{\partial t^2} - c_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0$$

using Green function with $\Delta \vec{u} = \vec{b}$ on slip plane



get
$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= G * \\ \frac{\partial u}{\partial t} &= G * \end{aligned}\right\rangle$$ sources localized on dislocation loop

$$S = \frac{1}{2}\int \rho \frac{\partial u}{\partial t} - c_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l}$$

replace $u = u_s + U$

get
$$S = S[X, U] \qquad \frac{\delta S}{\delta X} = 0 \text{ gives equ}' \text{ of motion}$$

F Lund, J. Mater. Res $\underline{3}$, 280 (83); Phys. Rev Lett. $\underline{54}$, 14 (85)

$$\frac{dP}{dt} = 0 \qquad P = \text{total momentum inside core}$$

~~1st ...~~

need short-distance behaviour. Leading term does not contribute. Need next one

say $\frac{\partial u}{\partial t} \sim \frac{bV}{r} + \frac{bVA}{c^2} \ln\frac{L}{r} = \frac{bV}{r}\left(1 + \frac{r}{c^2/A}\ln\frac{L}{r}\right)$

$$r \ll L \ll \frac{c^2}{A} \qquad \left(\frac{r}{L}\right)\left(\frac{L}{c^2/A}\right)$$

gives meaning to "small" distance.

same for $\frac{\partial u}{\partial x}$

get Eqn of motion which turns out to be the same as for relat particle with mass (per unit length)

long distance cut-off

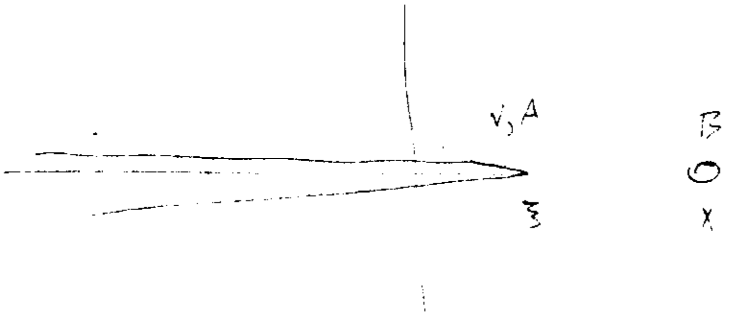
$$M = \frac{\mu b^2}{4\pi c^2} \ln\left(\frac{L}{\epsilon}\right)$$

short dist. cut-off

logarithmic dependence, not serious

Everything can be done in 3-D with loop of arb. shape

Mode III



Díaz - Lund
Philos. Mag. - to appear

Small acceleration

$$\sigma \sim \int \frac{D}{\text{dist}} + A \int D \ln \frac{L}{\text{dist}} + \frac{B}{\text{dist}}$$

zero traction on "crack" surface

Need coef of $\frac{1}{\sqrt{\text{dist}}}$ in σ near crack tip

get

$$\sigma \sim \frac{1}{\sqrt{\text{dist}}} \left(\frac{K}{\sqrt{2\pi}} \left(\frac{1 - v/c}{1 + v/c} \right)^{1/2} - \frac{\mu B}{\sqrt{X - \xi}} - 2\mu B \frac{A}{c^2} \sqrt{L} \right)$$

external stress if no dislocation

dislocation shielding

dislocation-induced inertia

Energy release rate

$$= \cancel{\frac{1}{2\mu\left(1 - \frac{v^2}{c^2}\right)} \left(K (1 - v/c)^{1/2} - \frac{2\pi}{\sqrt{X - \xi}} \right)}$$

$$= \frac{1}{2\mu\gamma} \left[K (1 - v/c)^{1/2} - \sqrt{\frac{2\pi}{X - \xi}} \frac{\mu B}{2\pi} - 2\sqrt{2\pi L} \frac{\mu B}{2\pi} \frac{A}{2c^2} \right]^2$$

$B = 0 \Rightarrow$ Eshelby $\qquad A = 0 \Rightarrow$ Lin - Thomson (1986) J. Mat. Res 1, 73