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CONTINUUM & LATTICE MODELS OF COMPOSITE MATERIALS
(Background Material)

V.K. TEWARY
National Institute of Standards and Technology
Fracture & Deformation Division
325 Broadway
Colorado
Boulder 80303-3328
U.S.A.

Elastic Green's function for a composite solid with a planar interface

V. K. Tewary,[a] R. H. Wagoner, and J. P. Hirth[b]
Department of Materials Science and Engineering, The Ohio State University, 116 West 19th Avenue, Columbus, Ohio 43210

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The elastic plane-strain Green's function is calculated for a general anisotropic composite solid with a plane interface and a line load parallel to the composite interface. The interface may be between two different solids or between different orientations of the same solid such as a grain boundary. The equations of elastic equilibrium are solved by the Fourier transform method. Analytical expressions are obtained for the Green's function in real as well as Fourier space. These expressions should be useful for calculations of elastic properties of a composite solid containing defects. Two sum rules are also derived for matrices which constitute the Green's function and the stress tensor. These sum rules can serve as numerical checks in detailed computer simulation calculations.

I. INTRODUCTION

Many elastic properties of a solid with defects can be calculated from the elastic Green's function of the solid and/or its derivatives and integrals. The main advantage of using the Green's function is that it gives the displacement field and the stress distribution in a solid subject to any arbitrary loading which satisfies all the required compatibility and boundary conditions. The Green's function for a solid can be obtained by solving the equations of elastic equilibrium for a unit force at a point or a line subject to appropriate boundary conditions (for mathematical aspects of Green's functions, see Ref. 1). There is growing interest in the calculation of the elastic fields of cracks and dislocations near interfaces. These are important in establishing a basis for the treatment of a number of cracking problems for composites, including near interface crack propagation, crack blunting, and dislocation emission, as well as those for near interface dislocations, including pile-ups. All of these can be solved by means of Green's function techniques.

The anisotropic elastic Green's function for infinite, uniform (or homogeneous) solids containing point or line defects has been extensively studied.[2–6] The Green's function for a solid containing a crack has been calculated by Sinclair and Hirth.[7] The elastic fields of line defects residing in the interface of a composite have been recently presented.[8] However, the Green's function for the bulk phases in a composite solid containing an interface has not been reported in the literature.

In this paper we have calculated the anisotropic elastic Green's function for a line force in a composite solid containing a planar interface. The interface may be between two phases or orientations of the same material such as a phase or a grain boundary or two different materials such as in a fiber composite. The Green's function can be used to calculate various elastic properties of the solid associated with the interface such as the interaction of a dislocation or any other defect with the interface.

Planar interfaces in solids have been studied by several authors.[9–12] These studies are based upon the method developed earlier by Eshelby *et al.*[13] and Stroh.[14,15] In this method, the elliptic differential equations of elastic equilibrium are solved in terms of complex variables, a procedure that requires the solution of a complex sextic determinental equation.

In the present paper we solve the elastic equations by taking their Fourier transforms. We thus obtain an integral representation of the Green's function over the Fourier space. One advantage of this method is that it avoids the need for solving a complex eigenvalue problem which is convenient only in certain practical applications.[16] Fourier transforms have been employed by Barnett[4] as the basis for the so-called integral method of Barnett and Lothe.[10] We also calculate the Green's function in the alternative analytical form which would require solving a complex eigenvalue problem as in the Stroh method. The method is illustrated by an application to a $\Sigma 5$ grain boundary in a cubic lattice. In a later paper we shall apply these Green's functions to a composite solid containing an interfacial crack.

We find that the Green's function at a point is continuous as the point of application moves across the interface. As a by-product of this proof, we also prove two sum rules [Eqs. (B.1) and (B.2), Appendix B] which are general and should be useful as numerical checks in computer simulation calculations.

[a] Current address: National Institute of Standards and Technology, Fracture and Deformation Division, Boulder, Colorado 80303.
[b] Current address: Washington State University, Department of Mechanical and Materials Engineering, Pullman, Washington 99164-2920.

II. GREEN'S FUNCTION AND ITS FOURIER TRANSFORM

We denote the space variables by **x**, **x**′, etc., which give the position vectors of various points in the solid with respect to a suitably chosen frame of reference. The Cartesian components of **x** are denoted by x_1, x_2, and x_3 along the axes 1, 2, and 3, respectively. We follow the summation convention over repeated Roman indices i, j, k, l, etc.

The equations of elastic equilibrium for a solid can be written as follows:

$$Z_{ij}(\mathbf{x})u_j(\mathbf{x}) = -f_i(\mathbf{x}) \tag{1}$$

where $u(x)$ and $f(x)$ denote, respectively, the displacement field and the force at **x**, **Z** is a tensor operator defined by

$$Z_{ij}(\mathbf{x}) = c_{ikjl}\frac{\partial^2}{\partial x_k \partial x_l} \tag{2}$$

and c_{ikjl} are the elastic constants of the solid.

The solution of Eq. (1) can be expressed in terms of the Green's function $\mathbf{G}(\mathbf{x}, \mathbf{x}')$, which is defined as a solution of the following equation:

$$Z_{ij}(\mathbf{x})G_{jk}(\mathbf{x}, \mathbf{x}') = -\delta_{ik}\delta(\mathbf{x} - \mathbf{x}') \tag{3}$$

where δ_{ik} is the Kronecker delta function, which is 0 for $i \neq k$ and l for $i = k$, and $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function which has the following properties:

$$\delta(\mathbf{x}) = 0 \quad \text{for } x_i \neq 0$$

and

$$\int_{-\infty}^{\infty} F(\mathbf{x} - \mathbf{x}')\delta(\mathbf{x}')\,d\mathbf{x}' = F(\mathbf{x})$$

for any arbitrary $F(\mathbf{x})$.

The Green's function $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ must satisfy all the prescribed boundary conditions for $\mathbf{u}(\mathbf{x})$ over the variable **x**. Then it follows that the solution of Eq. (1) is as given below

$$u_i(\mathbf{x}) = \int_{-\infty}^{\infty} G_{ij}(\mathbf{x}, \mathbf{x}')f_j(\mathbf{x}')\,d\mathbf{x}' \tag{4}$$

We now introduce the Fourier transforms of various functions as follows:

$$G_{ij}(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^6}\iint_{-\infty}^{\infty} G_{ij}(\mathbf{q}, \mathbf{q}')e^{i\mathbf{q}\cdot\mathbf{x} - i\mathbf{q}'\cdot\mathbf{x}'}\,d\mathbf{q}\,d\mathbf{q}' \tag{5}$$

$$u_i(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} u_i(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\,d\mathbf{q} \tag{6}$$

and

$$f_i(\mathbf{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} f_i(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\,d\mathbf{q} \tag{7}$$

where **q** is the wave vector with the Cartesian components q_1, q_2, and q_3 and $\mathbf{q}\cdot\mathbf{x} = q_i x_i$.

The inverse transforms can be easily obtained by means of the following orthogonality relation:

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\mathbf{q}\cdot\mathbf{x}}\,d\mathbf{q} = \delta(\mathbf{x}) \tag{8}$$

In terms of the Fourier transforms, Eqs. (1), (3), and (4) reduce, respectively, to the following:

$$\Lambda_{ij}(\mathbf{q})u_j(\mathbf{q}) = f_i(\mathbf{q}) \tag{9}$$

$$\Lambda_{ij}(\mathbf{q})G_{jk}(\mathbf{q}, \mathbf{q}') = (2\pi)^3\delta_{ik}\,\delta(\mathbf{q} - \mathbf{q}') \tag{10}$$

and

$$u_i(\mathbf{q}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} G_{ij}(\mathbf{q}, \mathbf{q}')f_j(\mathbf{q}')\,d\mathbf{q}' \tag{11}$$

where the 3 × 3 matrix Λ (**q**) is defined by

$$\Lambda_{ij}(\mathbf{q}) = c_{ikjl}q_k q_l \tag{12}$$

To calculate the Green's function, we determine the displacement field for a unit force at a point **x**′. In order to facilitate the calculation, we define a partial Fourier transform of the Green's function as follows:

$$\mathbf{G}(\mathbf{q}, \mathbf{x}') = \int_{-\infty}^{\infty} \mathbf{G}(\mathbf{x}, \mathbf{x}')e^{-i\mathbf{q}\cdot\mathbf{x}'}\,d\mathbf{x} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \mathbf{G}(\mathbf{q}, \mathbf{q}')e^{-i\mathbf{q}\cdot\mathbf{x}'}\,d\mathbf{q}' \tag{13}$$

Its inverse transform is given by

$$\mathbf{G}(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \mathbf{G}(\mathbf{q}, \mathbf{x}')e^{i\mathbf{q}\cdot\mathbf{x}}\,d\mathbf{q} \tag{14}$$

$$\mathbf{G}(\mathbf{q}, \mathbf{q}') = \int_{-\infty}^{\infty} \mathbf{G}(\mathbf{q}, \mathbf{x}')e^{i\mathbf{q}\cdot\mathbf{x}'}\,d\mathbf{x}' \tag{15}$$

In a uniform infinite solid, since the choice of the origin of coordinates is immaterial, $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ would depend upon **x** and **x**′ only through their difference. It would be therefore a function of the single variable $\mathbf{x} - \mathbf{x}'$. Similarly, its Fourier transform $\mathbf{G}(\mathbf{q}, \mathbf{q}')$ would also depend upon the single variable $\mathbf{q} - \mathbf{q}'$. In this case, as is apparent from Eq. (10), we have

$$\mathbf{G}(\mathbf{q}) = \{\Lambda(\mathbf{q})\}^{-1} \tag{16}$$

where, in the light of the preceding remark, we have expressed the Fourier transform of the Green's function as a function of the single variable **q**. Equation (16) indicates that the 3 × 3 matrix $\mathbf{G}(\mathbf{q})$ is inverse of the matrix $\Lambda(\mathbf{q})$ where $\Lambda(\mathbf{q})$ is defined by Eq. (12).

Now we consider a composite solid with a planar interface. In this case $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ depends upon **x** as well as **x**′. Similarly, $\mathbf{G}(\mathbf{q}, \mathbf{q}')$ depends upon both **q** and **q**′ and not necessarily on their difference. Our object in this paper is to determine this dependence.

We take the interface along the plane $x_2 = 0$ (see Fig. 1). We label the material parameters of the solid in the upper half (UHP) of the x_1x_2 plane by superscript A and those of the solid in the lower half plane (LHP) by the superscript B. We assume the solids A and B to be individually uniform and to extend to infinity. Thus, the only dis-

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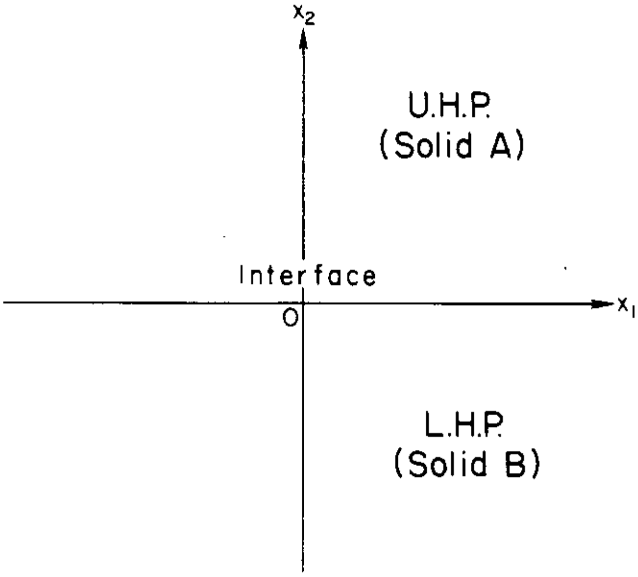

FIG. 1. The coordinate axes. The x_3 axis is normal to the plane of the paper. The interface is along the plane $x_2 = 0$.

continuity in our model solid is on the interface at the plane $x_2 = 0$.

We apply a line force $\boldsymbol{\phi}^A$ at the point R_1^A, R_2^A ($R_2^A > 0$) in the UHP and another line force $\boldsymbol{\phi}^B$ at the point R_1^B, $-R_2^B$ ($R_2^B > 0$) in the LHP. The line forces are taken to be parallel to the plane of the interface. We assume that there is no variation in the x_3 direction. Thus, the force functions and their Fourier transforms are given by

$$f_i^A(\mathbf{x}) = \phi_i^A\,\delta(x_1 - R_1^A)\,\delta(x_2 - R_2^A) \tag{17}$$

$$f_i^B(\mathbf{x}) = \phi_i^B\,\delta(x_1 - R_1^B)\,\delta(x_2 + R_2^B) \tag{18}$$

$$f_i^A(\mathbf{q}) = (2\pi)^3\phi_i^A e^{-iq_1R_1^A - iq_2R_2^A}\,\delta(q_3) \tag{19}$$

$$f_i^B(\mathbf{q}) = (2\pi)^3\phi_i^B e^{-iq_1R_1^B + iq_2R_2^B}\,\delta(q_3) \tag{20}$$

The elastic equations in the two regions, UHP and LHP, are given by

$$x_2 > 0 \qquad Z_{ij}^A(\mathbf{x})u_j^A(\mathbf{x}) = -\phi_i^A\,\delta(x_1 - R_1^A)\,\delta(x_2 - R_2^A) \tag{21}$$

$$x_2 < 0 \qquad Z_{ij}^B(\mathbf{x})u_j^B(\mathbf{x}) = -\phi_i^B\,\delta(x_1 - R_1^B)\,\delta(x_2 + R_2^B) \tag{22}$$

We assume that the two solids A and B are perfectly welded at the interfacial plane $x_2 = 0$. Thus, the prescribed boundary conditions at the interface are

$$u_i^A(x_1, x_2 = 0^+) = u_i^B(x_1, x_2 = 0^-) \tag{23}$$

and

$$\mathrm{T}_{i2}^A(x_1, x_2 = 0^-) = \mathrm{T}_{i2}^B(x_1, x_2 = 0^-) \tag{24}$$

where $\mathbf{T}$ denotes the stress tensor. In addition, we require the usual condition of zero strain at infinity on the x_1x_2 plane.

The term $\delta\,(q_3)$ in Eqs. (19) and (20) arises from the fact that the applied line forces are independent of x_3. This is the only case that we consider in this paper. Therefore, for the sake of brevity, we ignore the variable x_3 and take $q_3 = 0$ so that henceforth, unless otherwise indicated, $\mathbf{x}$ and $\mathbf{q}$ refer to two-dimensional vectors with Cartesian components x_1, x_2 and q_1, q_2, respectively.

III. CALCULATION OF DISPLACEMENT FIELD AND GREEN'S FUNCTION

In this section we solve Eqs. (21) and (22) subject to the boundary conditions given at the end of the preceding section and thus obtain the required Green's function.

We adopt the following procedure for obtaining the required solution. First we assume that $\phi^B = 0$. We write the solution of Eq. (21) as a sum of its particular integral: a solution of the homogeneous part of the equation, i.e., Eq. (21), with its RHS taken as zero. The homogeneous part of the solution would contain an integration constant which is determined by the boundary conditions.

Similarly, we write the solution of Eq. (22). This equation is already homogeneous for $\phi^B = 0$. Its particular integral therefore is zero. The two integration constants in the solution of Eqs. (21) and (22) are then determined so that the two boundary conditions as given by Eqs. (23) and (24) are satisfied.

We obtain the particular integral of Eq. (21) from Eq. (13) where for $\mathbf{G}(\mathbf{x}, \mathbf{x}')$ we take the Green's function for the uniform solid in UHP, i.e., solid A without any interface. This Green's function for a uniform solid can be written in terms of the single variable $\mathbf{x} - \mathbf{x}'$. Its Fourier transform can also be written in terms of the single variable $\mathbf{q}$.

One can also write the homogeneous solution by using Eq. (13) with the same Green's function and a force function which is applied outside the interface, i.e., outside the region in which the solution is obtained. This force function must be zero everywhere inside the region of the solution. The solution obtained with this force function inserted in Eq. (13) is the solution of the homogeneous part of the equation in view of Eq. (3). The force function plays the role of the integration constant and, as mentioned before, has to be determined from the boundary conditions.

The Green's function for the composite solid is finally obtained by taking $\phi^A = 1$, corresponding to the case when the unit force is applied in the UHP. The Green's function for the composite solid for the case when the unit force is applied in the LHP can be obtained following the same procedure by taking $\phi^A = 0$ and $\phi^B = 1$.

Henceforth, the particular integral and the homogeneous solutions are denoted by the superscripts P and H, respectively. The Green's functions for the solid A in UHP and B in LHP are denoted by the superscripts A and B, respectively. The superscript C is used to denote the Green's function for the composite solid. Finally, the complex conjugate of a quantity is denoted by a star as a superscript.

First we consider Eq. (21). By taking its Fourier transform as defined by Eqs. (6) and (7) and using Eqs. (9) and

(13), we obtain the following particular solution:

$$u_0^{PA}(\mathbf{q}) = G_{ij}^A(\mathbf{q})e^{-\iota q_1 R_1^A - \iota q_2 R_2^A}\phi_j^A \tag{25}$$

where

$$G_{ij}^A(\mathbf{q}) = \{\mathbf{\Lambda}^A(\mathbf{q})\}_{ij}^{-1} \tag{26}$$

As mentioned at the end of Sec. II, the vector **q** is a two-dimensional vector with components q_1 and q_2 along the x_1 and the x_2 axes, respectively. Its magnitude is given by

$$q^2 = q_1^2 + q_2^2 \tag{27}$$

In order to satisfy the boundary conditions at the interface, we add to the integral in Eq. (26) the solution of the homogeneous part of Eq. (21), i.e., Eq. (22) with RHS = 0. This solution can be written

$$u_i^{HA}(\mathbf{q}) = G_{ij}^A(\mathbf{q})F_j^A(q_1)e^{\iota q_2 \delta} \tag{28}$$

where $\mathbf{F}^A(q_1)$ is an arbitrary vector function of q_1 and $\delta > 0$ is a constant which can be made arbitrarily small. In the end we shall take the limit $\delta = 0$. The function $\mathbf{F}^A(\mathbf{q})$ is determined so that the total displacement field as given below satisfies the prescribed boundary conditions at the interface.

$$u_i^A(\mathbf{x}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} u_i^A(\mathbf{q})e^{\iota q_1 x_1 + \iota q_2 x_2}\,dq_1\,dq_2 \tag{29}$$

where

$$u_i^A(\mathbf{q}) = u_i^{PA}(\mathbf{q}) + u_i^{HA}(\mathbf{q}) \tag{30}$$

At $x_2 = 0$, the displacement field is given by

$$u_i^A(x_1, 0) = \frac{1}{(2\pi)}\int_{-\infty}^{\infty} V_i^A(q_1)e^{\iota q_1 x_1}\,dq_1 \tag{31}$$

where

$$V_i^A(q_1) = g_{ij}^A(q_1, -R_2^A)\phi_j^A e^{-\iota q_1 R_1^A} + g_{ij}^A(q_1, \delta)F_j^A(q_1) \tag{32}$$

with

$$g_{ij}^A(q_1, \xi) = \frac{1}{(2\pi)}\int_{-\infty}^{\infty} G_{ij}^A(q_1, q_2)e^{\iota q_2 \xi}\,dq_2 \tag{33}$$

The evaluation of this integral is given in Appendix A.

We now calculate the stress components T_{i2}, which can be written as follows:

$$T_{i2}(\mathbf{x}) = c_{i2j1}\frac{\partial u_j(\mathbf{x})}{\partial x_1} + c_{i2j2}\frac{\partial u_j(\mathbf{x})}{\partial x_2} \tag{34}$$

Using Eq. (29), we obtain for T_{i2}^A from Eq. (34) at $x_2 = 0$,

$$T_{i2}(x_1, 0) = \frac{\iota}{2\pi}\int_{-\infty}^{\infty} e^{\iota q_1 x_1}S_i^A(q_1)Sgn(q_1)\,dq_1 \tag{35}$$

where

$$S_i^A(q_1) = \eta_{ij}^A(q_1, -R_2^A)\phi_j^A e^{-\iota q_1 R_1^A} + \eta_{ij}^A(q_1, \delta)F_j^A(q_1) \tag{36}$$

$$\eta_{ij}^A(q_1, \xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} L_{ik}^A(\mathbf{q})G_{kj}^A(\mathbf{q})e^{\iota q_2 \xi}\,dq_2 \tag{37}$$

and

$$L_{ik}^A(\mathbf{q}) = c_{i2k1}^A q_1 + c_{i2k2}^A q_2 \tag{38}$$

As shown in Appendix A, the integral on the RHS of Eq. (37) is independent of q_1 for $\xi = 0$.

Proceeding in a similar manner, we obtain the following expressions for material B in the LHP:

$$u_i^B(\mathbf{x}) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} u_i^B(\mathbf{q})e^{\iota q_1 x_1 + \iota q_2 x_2}\,d\mathbf{q}_1\,dq_2 \tag{39}$$

$$u_i^B(x_1, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_i^B(q_1)e^{\iota q_1 x_1}\,dq_1 \tag{40}$$

$$T_{i2}^B(x_1, 0) = \frac{i}{2\pi}\int_{-\infty}^{\infty} S_i^B(q_1)e^{\iota q_1 x_1}\,dq_1 \tag{41}$$

where

$$u_i^B(\mathbf{q}) = G_{ij}^B(\mathbf{q})e^{-\iota q_1 R_1^B + \iota q_2 R_2^B}\phi_j^B + G_{ij}^B(\mathbf{q})e^{-\iota q_2 \delta'}F_j^B(q_1) \tag{42}$$

$$V_i^B(q_1) = g_{ij}^B(q_1, R_2^B)\phi_j^B e^{-\iota q_1 R_1^B} + g_{ij}^B(q_1, -\delta')F_j^B(q_1) \tag{43}$$

$$S_i^B(q_1) = \eta_{ij}^B(q_1, R_2^B)\phi_j^B e^{-\iota q_1 R_1^B} + \eta_{ij}^B(q_1, -\delta')F_j^B(q_1) \tag{44}$$

with $G_{ij}^B(\mathbf{q})$, $g_{ij}^B(q_1, \xi)$ and $\eta_{ij}^B(q_1, \xi)$ defined, respectively, by Eqs. (26), (33), and (37) with the superscript A replaced by B. The constant δ' in Eqs. (43) and (44) is positive and in the end, we shall take the limit $\delta' = 0$.

The boundary conditions given by Eqs. (23) and (24) are satisfied if $\mathbf{F}^A(q_1)$ and $\mathbf{F}^B(q_1)$ are determined from the following equations, which are obtained from Eqs. (31), (40), (35), and (41):

$$V_i^A(q_1) = V_i^B(q_1) \tag{45}$$

and

$$S_i^A(q_1) = S_i^B(q_1) \tag{46}$$

After some algebraic manipulation, we obtain the following expressions for $\mathbf{F}^A(q_1)$ and $\mathbf{F}^B(q_1)$ in terms of various 3 × 3 matrices:

$$\mathbf{F}^A(q_1) = \mathbf{M}[\mathbf{P}(q_1)\boldsymbol{\phi}^A e^{-\iota q_1 R_1^A} - \mathbf{Q}(q_1)\boldsymbol{\phi}^B e^{-\iota q_1 R_1^B} \tag{47}$$

and

$$\mathbf{F}^B(q_1) = \mathbf{N}[\mathbf{T}(q_1)\boldsymbol{\phi}^A e^{-\iota q_1 R_1^A} - \mathbf{W}(q_1)\boldsymbol{\phi}^B e^{-\iota q_1 R_1^B} \tag{48}$$

where

$$M = [\mathbf{g}^A(q_1, \delta)]^{-1}[\boldsymbol{\eta}^B(-\delta')\{g^B(q_1, -\delta')\}^{-1} - \eta^A(\delta)\{\mathbf{g}^A(q_1, \delta)\}^{-1}]^{-1} \tag{49}$$

$$N = [\mathbf{g}^B(q_1, -\delta')]^{-1}[\boldsymbol{\eta}^B(-\delta')\{g^B(q_1, -\delta')\}^{-1} - \eta^A(\delta)\{\mathbf{g}^A(q_1, \delta)\}^{-1}]^{-1} \tag{50}$$

$$\mathbf{P}(q_1) = [\boldsymbol{\eta}^A(-R_2^A)\{\mathbf{g}^A(q_1, -R_2^A)\}^{-1} - \boldsymbol{\eta}^B(-\delta')\{\mathbf{g}^B(q_1, -\delta')\}^{-1}]\mathbf{g}^A(q_1, -R_2^A) \quad (51)$$

$$\mathbf{Q}(q_1) = [\boldsymbol{\eta}^B(R_2^B)\{\mathbf{g}^B(q_1, R_2^B)\}^{-1} - \boldsymbol{\eta}^B(-\delta')\{\mathbf{g}^B(q_1, -\delta')\}^{-1}]\mathbf{g}^B(q_1, R_2^B) \quad (52)$$

$$\mathbf{T}(q_1) = [\boldsymbol{\eta}^A(-R_2^A)\{\mathbf{g}^A(q_1, -R_2^A)\}^{-1} - \boldsymbol{\eta}^A(\delta)\{\mathbf{g}^A(q_1, \delta)\}^{-1}]\mathbf{g}^A(q_1, -R_2^A) \quad (53)$$

$$\mathbf{W}(q_1) = [\boldsymbol{\eta}^B(R_2^B)\{\mathbf{g}^B(q_1, R_2^B)\}^{-1} - \boldsymbol{\eta}^A(\delta)\{\mathbf{g}^A(q_1, \delta)\}^{-1}]\mathbf{g}^B(q_1, R_2^B) \quad (54)$$

The evaluation of various matrices in Eqs. (47)–(54) is given in Appendix A.

The displacement field in the UHP and LHP can now be calculated from Eqs. (29) and (39), respectively. The Green's function is just the displacement field for a unit force, i.e., for $\boldsymbol{\phi}^A$ and $\boldsymbol{\phi}^B$ equal to the unit matrix. Thus, we obtain the following expression for the partial Fourier transform of the Green's function which is defined by Eq. (13):

(i) Unit line force in the UHP at $\mathbf{x}' = \mathbf{R}^A \equiv (R_1^A, R_2^A)$

$$\mathbf{G}^{CA}(\mathbf{q}, \mathbf{R}^A) = \mathbf{G}^A(\mathbf{q})e^{-iq_1R_1^A - iq_2R_2^A} + \mathbf{G}^A(\mathbf{q})\mathbf{M}\mathbf{P}(q_1)e^{-iq_1R_1^A} \quad (55)$$

$$\mathbf{G}^{CB}(\mathbf{q}, R) = \mathbf{G}^B(\mathbf{q})\mathbf{N}\mathbf{T}(q_1)e^{-iq_1R_1^A} \quad (56)$$

(ii) Unit line force in the LHP at $\mathbf{x}' = \mathbf{R}^B \equiv (R_1^B, -R_2^B)$

$$\mathbf{G}^{CA}(\mathbf{q}, \mathbf{R}^B) = -\mathbf{G}^A(\mathbf{q})\mathbf{M}\mathbf{Q}(q_1)e^{-iq_1R_1^B} \quad (57)$$

$$\mathbf{G}^{CB}(\mathbf{q}, \mathbf{R}^B) = \mathbf{G}^B(\mathbf{q})e^{-iq_1R_1^B + iq_2R_2^B} - \mathbf{G}^B(\mathbf{q})\mathbf{N}\mathbf{W}(q_1)e^{-iq_1R_1^B} \quad (58)$$

The final result for the Green's function can be obtained from Eq. (14). Carrying out the integral over $\mathbf{q}$ in Eq. (14) as shown in Appendix A, we obtain the following expressions ($\mathbf{x}'$ stands for $\mathbf{R}^A$ or $\mathbf{R}^B$):

(i) <u>$\mathbf{x}$ and $\mathbf{x}'$ in UHP</u> ($x_2 > 0$, $x_2' > 0$)

$$G^{CA}(\mathbf{x}, \mathbf{x}') = -\frac{1}{\pi}\sum_{\alpha} \boldsymbol{\nu}^A(p_\alpha^A) \cdot \ln[(x_1 - x_1') + p_\alpha^A(x_2 - x_2')] - \frac{1}{\pi}\sum_{\alpha\beta} \boldsymbol{\nu}^A(p_\alpha^A)M \cdot [\boldsymbol{\sigma}^{*A}(p_\beta^A) - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\nu}_s^{*B^{-1}}\boldsymbol{\nu}^{*A}(p_\beta^A)] \cdot \ln[(x_1 - x_1') + x_2p_\alpha^A - x_2'p_\beta^{*A}] \quad (59)$$

(ii) <u>$\mathbf{x}$ in LHP, $\mathbf{x}'$ in UHP</u> ($x_2 < 0$, $x_2' > 0$)

$$G^{CB}(\mathbf{x}, \mathbf{x}') = -\frac{1}{\pi}\sum_{\alpha\beta} \boldsymbol{\nu}^{*B}(p_\alpha^B)N \cdot [\boldsymbol{\sigma}^{*A}(p_\beta^A) - \boldsymbol{\sigma}_s^A\boldsymbol{\nu}_s^{A^{-1}}\boldsymbol{\nu}^{*A}(p_\beta^A)] \cdot \ln[(x_1 - x_1') - |x_2|p_\alpha^{*B} - x_2'p_\beta^{*A}] \quad (60)$$

(iii) <u>$\mathbf{x}$ in UHP, $\mathbf{x}'$ in LHP</u> ($x_2 > 0$, $x_2' < 0$)

$$G^{CA}(\mathbf{x}, \mathbf{x}') = \frac{1}{\pi}\sum_{\alpha\beta} \boldsymbol{\nu}^A(p_\alpha^B)\mathbf{M} \cdot [\boldsymbol{\sigma}^B(p_\beta^B) - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\nu}_s^{*B^{-1}}\boldsymbol{\nu}^B(p_\beta^B)] \cdot \ln[(x_1 - x_1') + x_2p_\alpha^A + |x_2'|p_\beta^B] \quad (61)$$

(iv) <u>$\mathbf{x}$ and $\mathbf{x}'$ in LHP</u> ($x_2 < 0$, $x_2' < 0$)

$$G^{CB}(\mathbf{x}, \mathbf{x}') = -\frac{1}{\pi}\sum_{\alpha} \boldsymbol{\nu}^{*B}(p_\alpha^B) \cdot \ln[(x_1 - x_1') - p_\alpha^{*B}(|x_2| - |x_2'|)] + \frac{1}{\pi}\sum_{\alpha\beta} \boldsymbol{\nu}^{*B}(p_\alpha^B)\mathbf{N} \cdot [\boldsymbol{\sigma}^B(p_\beta^B) - \boldsymbol{\sigma}_s^A\boldsymbol{\nu}_s^{A^{-1}}\boldsymbol{\nu}^B(p_\beta^B)] \cdot \ln[(x_1 - x_1') - |x_2|p_\alpha^{*B} + |x_2'|p_\beta^B] \quad (62)$$

where the matrices $\boldsymbol{\nu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\nu}_s$, and $\boldsymbol{\sigma}_s$ are given by Eqs. (A.9), (A.17), (A.22), and (A.23), respectively, in Appendix A, and the matrices **M** and **N** as defined by Eqs. (49) and (50) are expressed in terms of $\boldsymbol{\nu}_s$ and $\boldsymbol{\sigma}_s$ in Eqs. (A.20) and (A.21) in Appendix A. It is understood that the displacement field is given by the real part of the Green's function in Eqs. (49)–(62).

IV. AN ILLUSTRATIVE EXAMPLE

In this section we illustrate the method given above by applying it to the simple case of a planar interface between two cubic solids. The interface is taken to be parallel to the crystallographic z-axis. The two solids on the opposite sides of the interface may be different materials as in a composite solid or different crystallographic orientations of the same material as in a grain boundary.

We consider the case where the x and y (x_1 and x_2) axes of our frame of reference, as given in Fig. 1, coincide with the crystallographic axes. The results for a different orientation of the x and y axes can then be obtained by an orthogonal transformation.

For the cubic case the elements of the Λ matrix as defined by Eq. (12) reduce to the following:

$$\Lambda_{ij}(\mathbf{q}) = c_{44}[(q^2 + \delta_0 q_k^2)\delta_{ki}\delta_{kj} + \beta_0 q_i q_j] \quad (63)$$

where

$$\delta_0 = (c_{11} - c_{12} - 2c_{44})/c_{44} \quad (64)$$

$$\beta_0 = (c_{12} + c_{44})/c_{44}$$

and

$$q^2 = q_1^2 + q_2^2 + q_3^2 \quad (65)$$

Our interest is only in the case $q_3 = 0$, as remarked at the end of Sec. II. In this case the matrix defined by Eq. (63) reduces to a 2×2 matrix, which is referred to as Λ, and a 1×1 matrix (i.e., a pure number) given by

$$\Lambda_{33}(\mathbf{q}) = c_{44}q^2 \tag{66}$$

The elements of the 2 × 2 matrix $\mathbf{\Lambda}$ are formally defined by Eq. (63) with i and j assuming values of only 1 and 2 and with $q_3 = 0$. The matrix $\mathbf{\Lambda}$ is relevant for the plane strain problems for which $f_3 = 0$; whereas $\Lambda_{33}(\mathbf{q})$ is required for the antiplane strain problems where $f_1 = f_2 = 0$ and $f_3 \neq 0$. Here, we consider the plane strain problem so that $f_3 = 0$.

The roots of $D(\mathbf{q})$, the determinant of $\mathbf{\Lambda}$, as defined by Eq. (A.5) are given as

$$p_1^2 = -K^2 + (K^4 - 1)^{1/2} \tag{67}$$

and

$$p_2^2 = -K^2 - (K^4 - 1)^{1/2} \tag{68}$$

where

$$K^2 = 1 + \frac{\delta_0}{\zeta}(\beta_0 + \delta_0/2) \tag{69}$$

and

$$\zeta = c_{11}/c_{44} \tag{70}$$

with a, as defined in Eq. (A.5), equal to c_{11}.

For $\delta_0 > 0$, we can write

$$p_1 = \iota\mu_1 \quad \text{and} \quad p_2 = \iota\mu_2$$

where the μ_i are real and are given by

$$\mu_1^2 = K^2 - (K^4 - 1)^{1/2} \tag{71}$$

and

$$\mu_2^2 = K^2 + (K^4 - 1)^{1/2} \tag{72}$$

In this case after some algebraic manipulations, we obtain the following expressions for the matrix elements of $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}$ ($\alpha, \beta = 1, 2$; $\beta \neq \alpha$):

$$\nu_{11}(p_\alpha) = -\frac{1}{2a}\frac{1 - \zeta\mu_\alpha^2}{\mu_\alpha(\mu_\alpha^2 - \mu_\beta^2)} \tag{73}$$

$$\nu_{12}(p_\alpha) = \nu_{21}(p_\alpha) = \frac{\iota}{2a}\frac{\beta_0}{\mu_\alpha^2 - \mu_\beta^2} \tag{74}$$

$$\nu_{22}(p_\alpha) = -\frac{1}{2a}\frac{\zeta - \mu_\alpha^2}{\mu_\alpha(\mu_\alpha^2 - \mu_\beta^2)} \tag{75}$$

$$\sigma_{11}(p_\alpha) = -\frac{\iota}{2c_{11}(\mu_\alpha^2 - \mu_\beta^2)}[c_{44}(1 - \zeta\mu_\alpha^2) - \beta_0 c_{44}] \tag{76}$$

$$\sigma_{12}(p_\alpha) = \frac{-c_{44}}{2c_{11}(\mu_\alpha^2 - \mu_\beta^2)}[\beta_0\mu_\alpha + (\zeta - \mu_\alpha^2)/\mu_\alpha] \tag{77}$$

$$\sigma_{21}(p_\alpha) = \frac{-1}{2c_{11}(\mu_\alpha^2 - \mu_\beta^2)} \cdot [c_{11}\beta_0\mu_\alpha + c_{12}(1 - \zeta\mu_\alpha^2)/\mu_\alpha] \tag{78}$$

$$\sigma_{22}(p_\alpha) = \frac{-\iota}{2c_{11}(\mu_\alpha^2 - \mu_\beta^2)}[c_{11}(\zeta - \mu_\alpha^2) - \beta_0 c_{12}] \tag{79}$$

From the above equations we obtain the following matrix sums:

$$\nu_{sij} = \frac{1 + \zeta}{2c_{11}(\mu_1 + \mu_2)}\delta_{ij} \tag{80}$$

$$\sigma_{s11} = \sigma_{s22} = \iota/2 \tag{81}$$

$$\sigma_{s12} = \frac{c_{44}}{2c_{11}(\mu_1 + \mu_2)}[1 - \beta_0 + \zeta] \tag{82}$$

and

$$\sigma_{s21} = \frac{1}{2c_{11}(\mu_1 + \mu_2)}[c_{12} + \zeta c_{12} - \beta_0 c_{11}] \tag{83}$$

We note that, as demonstrated by Eqs. (B.1) and (B.2) in Appendix B, $\boldsymbol{\nu}_s$ is real and

$$\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_s^* = \iota\mathbf{I} \tag{84}$$

where $\mathbf{I}$ is the unit matrix.

In the case $\delta_0 < 0$, the roots p_α obtained from Eq. (67) and Eq. (68) are given below:

$$p_1 = R + \iota\mu \tag{85}$$

and

$$p_2 = -R + \iota\mu \tag{86}$$

where

$$R = [(1 - K^2)/2]^{1/2} \tag{87}$$

and

$$\mu = [(1 + K^2)/2]^{1/2} \tag{88}$$

We also have the relation that

$$p_2^2 = p_1^{*2} \tag{89}$$

Proceeding as before, we find that

$$\nu_{11}(p_\alpha) = Z_\alpha(1 + \zeta p_\alpha^2) \tag{90}$$

$$\nu_{12}(p_\alpha) = \nu_{21}(p_\alpha) = -Z_\alpha\beta_0 p_\alpha \tag{91}$$

$$\nu_{22}(p_\alpha) = Z_\alpha(\zeta + p_\alpha^2) \tag{92}$$

$$\sigma_{11}(p_\alpha) = Z_\alpha p_\alpha(1 - \beta_0 + \zeta p_\alpha^2) \tag{93}$$

$$\sigma_{12}(p_\alpha) = Z_\alpha[(1 - \beta_0)p_\alpha^2 + \zeta] \tag{94}$$

$$\sigma_{21}(p_\alpha) = Z_\alpha[-1 + \beta_0 - \zeta p_\alpha^2] \tag{95}$$

$$\sigma_{22} = c_{44}Z_\alpha p_\alpha(\zeta^2 - \beta_0^2 + \beta_0 + \zeta p_\alpha^2) \tag{96}$$

where

$$Z_\alpha = -\frac{(-1)^\alpha}{8c_{11}\mu R p_\alpha} \tag{97}$$

The matrix sums can now be determined but are not given here. However, with the help of Eqs. (85), (86), and (89), we can verify that

$$\nu(p_1) = \nu^*(p_2) \tag{98}$$

$$\sigma_{ij}(p_1) = \sigma_{ij}^*(p_2)\,(i \neq j) \tag{99}$$

and

$$\sigma_{s11} = \sigma_{s22} = \frac{c_{44}\zeta(p_1^2 - p_2^2)}{8c_{11}\mu R} = \frac{\iota}{2} \tag{100}$$

From Eqs. (98)–(100) we find that in this case also Eqs. (B.1) and (B.2) are obeyed.

In deriving the above formulae, we have assumed that the x and y axes of our frame of reference coincide with the crystallographic axes. The corresponding matrices for a different orientation can be found with the help of the orthogonal transformation of rotation S, viz.:

$$S_{11}(\theta) = S_{22}(\theta) = \cos\theta \tag{101}$$

$$S_{12}(\theta) = -S_{21}(\theta) = \sin\theta \tag{102}$$

where θ is the angle between the crystallographic x axis and the x_1 axis. The transformed matrices are simply given by

$$\mathbf{\Lambda}_\theta(\mathbf{q}) = \mathbf{S}(\theta)\mathbf{\Lambda}(\mathbf{q})\mathbf{S}^{-1}(\theta) \tag{103}$$

$$\boldsymbol{\nu}_\theta(p_\alpha) = \mathbf{S}(\theta)\boldsymbol{\nu}(p_\alpha)\mathbf{S}^{-1}(\theta) \tag{104}$$

and

$$\sigma_\theta(p_\alpha) = \mathbf{S}(\theta)\boldsymbol{\sigma}(p_\alpha)^{-1}\mathbf{S}(\theta) \tag{105}$$

Since $\mathbf{S}(\theta)$ is an orthogonal real matrix, it follows from Eqs. (80), (84), (98), and (100) that Eqs. (B.1) and (B.2) are obeyed for any orientation θ.

Once the $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}$ matrices are known, the various terms in Eqs. (59)–(62) for the Green's function can be calculated. Here we give the results for a $\Sigma 5$ tilt grain boundary in stainless steel. We take the crystallographic axes in the UHP (region A) to be parallel to the chosen x_1, x_2 axes. In the LHP (region B), the crystallographic axis is taken to be oriented to the x_1 by $\theta = \tan^{-1} 3/4$. The elastic constants of stainless steel as taken from Hirth and Lothe[5] are given below in units of c_{44}:

$$c_{11} = 2.2 \qquad c_{12} = 1.3 \qquad c_{44} = 1$$

The calculated values of $G^{CA}(\mathbf{x}, \mathbf{x}')$ and $G^{CB}(\mathbf{x}, \mathbf{x}')$ as a function of x_2 ($x_1 = 0$) are depicted in Fig. 2.

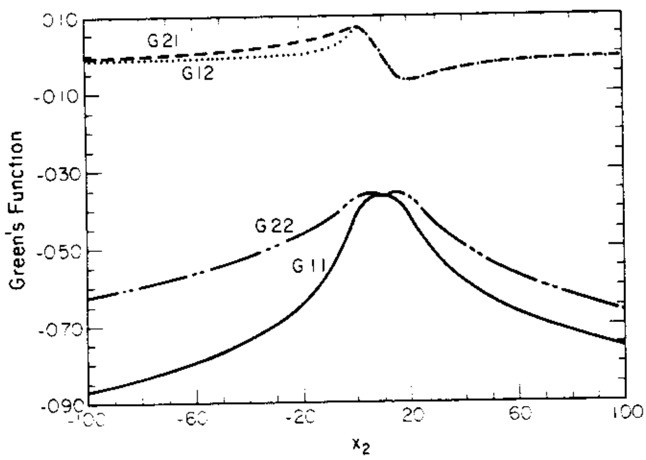

FIG. 2. Real part of the Green's function plotted as a function of x_2 ($x_1 = 0$) for a unit line force in UHP at $x_1 = 10$, $x_2 = 10$ for a $\Sigma 5$ boundary in stainless steel. In the UHP, the crystallographic axes are parallel to chosen frame of reference (Fig. 1). In the LHP, the crystallographic axes are rotated by an angle $\tan^{-1} 3/4$ relative to the frame of reference, which explains why $G_{12} \neq G_{21}$ in LHP.

The case of antiplane strain is particularly simple. In this case all the matrices are 1×1, i.e., pure numbers. The $\mathbf{\Lambda}$ matrix is given by Eq. (66). The only root of $\Lambda_{33}(q) = 0$ which contributes is $p_1 = \iota$. This yields the following expressions for ν, s, M, and N:

$$\nu(p_1) = \nu_s = 1/2c_{44} \tag{106}$$

$$\sigma(p_1) = \sigma_s = \iota/2 \tag{107}$$

$$M = 2\iota c_{44}^A/(c_{44}^A + c_{44}^B) \tag{108}$$

$$N = 2\iota c_{44}^B/(c_{44}^A + c_{44}^B) \tag{109}$$

Again we see that Eqs. (106) and (107) satisfy Eqs. (B.1) and (B.2). The final results for the Green's function as obtained from Eqs. (59)–(62) are given below:

(i) $\underline{x_2 > 0,\ x_2' > 0}$

$$G^{CA}(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi}\frac{1}{c_{44}^A}\ln(\mathbf{z} - \mathbf{z}') + \frac{1}{2\pi}\frac{\zeta}{c_{44}^A}\ln(\mathbf{z} - \mathbf{z}'^*)$$

(ii) $\underline{x_2 < 0,\ x_2' > 0}$

$$G^{CB}(\mathbf{x}, \mathbf{x}') = -\frac{1}{\pi}\frac{1}{c_{44}^A + c_{44}^B}\ln(z^* - z'^*)$$

(iii) $\underline{x_2 > 0,\ x_2' < 0}$

$$G^{CA}(\mathbf{x}, \mathbf{x}') = -\frac{1}{\pi}\frac{1}{c_{44}^A + c_{44}^B}\ln(z - z')$$

(iv) $\underline{x_2 < 0,\ x_2' < 0}$

$$G^{CB}(\mathbf{x}, \mathbf{x}') = -\frac{1}{2\pi}\frac{1}{c_{44}^B}\ln(z^* - z'^*) - \frac{1}{2\pi}\frac{\zeta}{c_{44}^B}\ln(z^* - z')$$

where

$$z = x_1 + \iota x_2 \tag{110}$$

$$z' = x_1' + \iota x_2' \tag{111}$$

and

$$\zeta = (c_{44}^B - c_{44}^A)/(c_{44}^B + c_{44}^A) \tag{112}$$

V. DISCUSSION

The Fourier transform of the Green's function is given by Eqs. (55)–(58) from which the Green's function (or the displacement field) can be calculated by means of Eq. (14). This would involve a numerical calculation of an infinite integral over q_1. Alternatively, one can use the analytical expressions given by Eqs. (59)–(62), which would require solving a complex sextic equation.

We notice that the Green's function in real space has the well-known (see Ref. 5) logarithmic dependence on the space variables as found for the displacement field by Tucker.[9] One important physical test of the theory is that there should be no discontinuity in the displacement field as the point of application of the line force moves across the interface between UHP and LHP. We show in Appendix B that this discontinuity is indeed absent in the result given by Eqs. (59)–(61). In Appendix B we prove the two sum rules that $\boldsymbol{\nu}_s - \boldsymbol{\nu}_s^* = 0$, i.e., the matrix $\boldsymbol{\nu}_s$ is real, and that $\sigma_s - \sigma_s = \iota \mathbf{I}$, which is independent of the material parameters of the solid. A direct verification of this interesting result has been given for cubic lattices in Sec. IV. Because of their generality, these sum rules should be useful as numerical checks in detailed computer simulation calculations.

An application of the Green's function derived in this paper for the problem of an interfacial crack in a composite solid will be reported in a forthcoming paper.

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APPENDIX A: EVALUATION OF THE INVERSE FOURIER TRANSFORM OF THE GREEN'S FUNCTION

In this appendix we shall show how to evaluate the inverse Fourier transform of the Green's function for the composite solid by contour integration of the various expressions given in Sec. III. First we shall give the background formulation which leads to the well-known logarithmic expression for the Green's function of a uniform homogeneous solid (solid A in UHP or B in LHP in the present case). We shall then use the same technique to obtain the inverse Fourier transform of the Green's function for the composite solid and the result as quoted at the end of Sec. III.

Background formulation

We note that $\Lambda_{ij}(\mathbf{q})$ is a homogeneous function of q_2 and q_1 (for all i and j = 1, 2, and 3). The three eigenvalues of $\Lambda(\mathbf{q})$ can therefore be written in the following form for α = 1, 2, or 3:

$$\begin{aligned}\in_\alpha(\mathbf{q}) &= a_\alpha(q_2^2 + b_\alpha q_1 q_2 + c_\alpha q_1^2) \\ &= a_\alpha(q_2 - p_\alpha q_1)(q_2 - p_\alpha^* q_1)\end{aligned} \tag{A.1}$$

where a_α, b_α, and c_α are constants and p_α and p_α^* are the roots of the sextic equation $\in_\alpha(\mathbf{q}) = 0$, and * denotes complex conjugate.

For elastically stable solids the eigenvalues of Λ are real and positive[13] and are not zero for any real value of q_1 and q_2 except for $q_1 = q_2 = 0$. Each root p_α therefore must be complex. We label the roots so that

$$\operatorname{Im} p_\alpha > 0 \quad (\text{for } \alpha = 1, 2, 3) \tag{A.2}$$

The Fourier transform of the Green's function for a uniform solid as defined by Eq. (16) can then be written as

$$G_{ij}(\mathbf{q}) = \{\Lambda(\mathbf{q})\}_{ij}^{-1} = \frac{\Gamma_{ij}(\mathbf{q})}{D(\mathbf{q})} \tag{A.3}$$

where Γ_{ij} is a cofactor of Λ, viz.,

$$\Lambda_{ij}(\mathbf{q})\Gamma_{jk}(\mathbf{q}) = D(\mathbf{q})\delta_{ik} \tag{A.4}$$

and $D(\mathbf{q})$ is the determinant of Λ which is given by

$$D(q) = \prod_{\alpha=1}^{3} \in_\alpha(q) = a \prod_{\alpha=1}^{3} (q_2 - p_\alpha q_1)(q_2 - p_\alpha^* q_1) \tag{A.5}$$

and

$$a = a_1 a_2 a_3 .$$

The inverse Fourier transform of $G_{ij}(\mathbf{q})$, i.e., the Green's function for a uniform solid in real space can be written as follows in terms of Eq. (A.3):

$$G_{ij}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\iota q_1 x_1} g_{ij}(q_1, x_2)\, dq_1 \tag{A.6}$$

where

$$\begin{aligned} g_{ij}(q_1, x_2) &= \int_{-\infty}^{\infty} G_{ij}(\mathbf{q}) e^{\iota q_2 x_2}\, dq_2 \\ &= \frac{1}{2\pi a} \int_{-\infty}^{\infty} \frac{\Gamma_{ij}(\mathbf{q}) e^{\iota q_2 x_2}}{\prod_{\alpha=1}^{3} (q_2 - p_\alpha q_1)(q_2 - p_\alpha^* q_1)}\, dq_2 \end{aligned} \tag{A.7}$$

We consider the case where all of the eigenvalues are distinct. The case of degenerate eigenvalues, such as those for an isotropic solid, can be tackled by means of a standard limiting procedure. We also assume that $x_2 > 0$.

The integral on the RHS of Eq. (A.7) can be obtained by means of a semicircular contour in the upper half of the complex q plane, as shown by the solid line in Fig. 3. In view of the condition (A.2), the integral over the semicircle vanishes in the infinite limit. The only contribution comes from the poles at $q_2 = p_\alpha q_1$ in the upper half plane. Thus we obtain

$$g_{ij}(q_1, x_2) = \frac{1}{q_1} \sum_\alpha \nu_{ij}(p_\alpha) e^{\iota q_1 p_\alpha x_2} \tag{A.8}$$

where

$$\nu_{ij}(p_\alpha) = \frac{\iota}{a q_1^4} \frac{\Gamma_{ij}(q_2 = q_1 p_\alpha)}{(p_\alpha - p_\alpha^*) \prod_{\beta \neq \alpha} (p_\alpha - p_\beta)(p_\alpha - p_\beta^*)} \tag{A.9}$$

Since $\Lambda_{ij}(q)$ is a homogeneous function of second degree in q_1 and q_2, its cofactor $\Gamma_{ij}(q)$ is a homogeneous function of degree 4 in q_1 and q_2. The matrix elements of $\boldsymbol{\nu}(p_\alpha)$, as defined by Eq. (A.9), therefore depend only upon p_α and not upon q_1.

In order to evaluate $g_{ij}(q_1, -x_2)$ (for $x_2 > 0$), we choose a clockwise contour in the lower half plane, as indicated by the dotted line in Fig. 3. Moreover, g_{ij} has the property that

$$g_{ij}(q_1, -x_2) = g_{ij}^*(q_1, x_2) \tag{A.10}$$

In order to evaluate $G_{ij}(x)$ from Eq. (A.6), we notice that it has a singularity at $q_1 = 0$. We can, however, obtain its principal value by considering the following integral:

$$I(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\iota q_1 x_1}}{|q_1|} e^{\iota q_1 p_\alpha x_2} dq_1$$

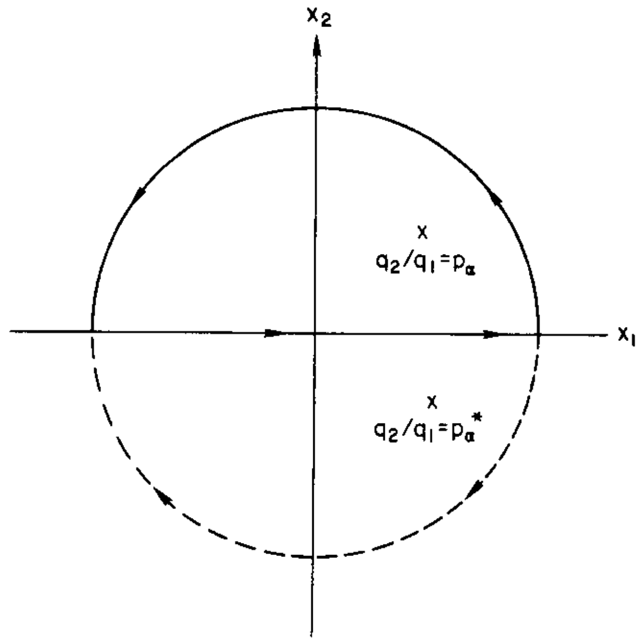

FIG. 3. Semicircular contours chosen for integration. The clockwise solid line contour in the UHP encloses only those poles for which Im p_α is positive. The anticlockwise broken line contour in the LHP will enclose poles for which Im p_α is negative. Poles are denoted by crosses.

$$= \frac{1}{\pi} Rl \int_0^\infty \frac{e^{\iota q_1 (x_1 + p_\alpha x_2)}}{q_1} dq_1 \tag{A.11}$$

or

$$\frac{dI}{dx_1} = Rl \frac{\iota}{\pi} \int_0^\infty e^{\iota q_1 (x_1 + p_\alpha x_2)} dq_1 \tag{A.12}$$

$$= -Rl \frac{1}{\pi} \frac{1}{(x_1 + p_\alpha x_2)} \tag{A.13}$$

or

$$I(x_1, x_2) = -\frac{1}{\pi} Rl \ln(x_1 + p_\alpha x_2), \tag{A.14}$$

In writing Eq. (A.13) we have used the fact that the imaginary part of p_α is positive so that the integral is zero at the upper limit $q_1 = \infty$. Thus, we obtain the following expression for $G_{ij}(x)$:

$$G_{ij}(\mathbf{x}) = -\frac{1}{\pi} Rl \sum_a \nu_{ij}(p_\alpha) \ln(x_1 + p_\alpha x_2) \tag{A.15}$$

By using the same technique, we can calculate the Fourier integral required for $\eta(\theta_1, \xi)$ in the stress component T_{i2} in Eq. (37). Assuming $\xi > 0$, we take the same contour in the upper half plane, as shown in Fig. 3. The result is (for superscripts A or B)

$$\eta_{ij}(q_1, \xi) = \sum_a \sigma_{ij}(p_\alpha) e^{\iota q_1 p_\alpha \xi} \tag{A.16}$$

where

$$\sigma_{ij}(p_\alpha) = L_{ik}(p_\alpha) \nu_{kj}(p_\alpha) \tag{A.17}$$

is independent of q_1,

$$L_{ik}(p_\alpha) = c_{i2k1} + c_{i2k2} p_\alpha \tag{A.18}$$

and

$$\eta_{ij}(q_1, -\xi) = \eta_{ij}^*(q_1, \xi) \tag{A.19}$$

Derivation of the inverse Fourier transform of the Green's function for the composite solid

Using the background formulation and the contour integration technique as described in the preceding subsection, we shall now derive the inverse Fourier transform of the Green's function for the composite solid.

We note that in the limit $\xi = 0$, $\eta(q_1, \xi)$ is independent of q_1 and, from Eq. (A.8), $g_{ij}(q_1, \xi)$ is proportional to $1/q_1$. The matrices **M** and **N** as defined by Eqs. (49) and (50) are therefore independent of q_1 in the limit δ and $\delta' = 0$. Their explicit forms are given below.

$$\mathbf{M} = \boldsymbol{\nu}_s^{A^{-1}} Y^{-1} \tag{A.20}$$

$$\mathbf{N} = \boldsymbol{\nu}_s^{*B^{-1}} Y^{-1} \tag{A.21}$$

where

$$\boldsymbol{\nu}_s^{A,B} = \sum_\alpha \boldsymbol{\nu}^{A,B}(p_\alpha) \tag{A.22}$$

$$\boldsymbol{\sigma}_s^{A,B} = \sum_\alpha \boldsymbol{\sigma}^{A,B}(p_\alpha) \tag{A.23}$$

-9-

and

$$Y = \boldsymbol{\sigma}_s^{*B}\nu_s^{*B^{-1}} - \boldsymbol{\sigma}_s^A\boldsymbol{\nu}_s^{A^{-1}}$$

Similarly, we obtain the following expressions for the other matrices in Eqs. (51)–(54):

$$\mathbf{P}(q_1) = \sum_\alpha \boldsymbol{\sigma}^{*A}(p_\alpha^A)e^{-\iota q_1 p_\alpha^{*A}R_2^A} - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\nu}_s^{*B^{-1}}\sum_\alpha \boldsymbol{\nu}^{*A}(p_\alpha^A)e^{-\iota q_1 p_\alpha^{*A}R_2^A} \quad (A.24)$$

$$\mathbf{Q}(q_1) = \sum_\alpha \boldsymbol{\sigma}^B(p_\alpha^B)e^{\iota q_1 p_\alpha^B R_2^B} - \boldsymbol{\sigma}_s^{*B}\nu_s^{*B^{-1}}\sum_\alpha \boldsymbol{\nu}^B(p_\alpha^B)e^{\iota q_1 p_\alpha^B R_2^B} \quad (A.25)$$

$$\mathbf{T}(q_1) = \sum_\alpha \boldsymbol{\sigma}^{*A}(p_\alpha^A)e^{-\iota q_1 p_\alpha^{*A}R_2^A} - \boldsymbol{\sigma}_s^A\boldsymbol{\nu}_s^{A^{-1}}\sum_\alpha \boldsymbol{\nu}^{*A}(p_\alpha)e^{-\iota q_1 p_\alpha^{*A}R_2^A} \quad (A.26)$$

$$\mathbf{W}(q_1) = \sum_\alpha \boldsymbol{\sigma}^B(p_\alpha^B)e^{\iota q_1 p_\alpha^B R_2^B} - \boldsymbol{\sigma}_s^A\boldsymbol{\nu}_s^{A^{-1}}\sum_\alpha \boldsymbol{\nu}^B(p_\alpha^B)e^{\iota q_1 p_\alpha^B R_2^B} \quad (A.27)$$

Now we evaluate the inverse Fourier transforms of various terms in Eqs. (55)–(58) as defined by Eq. (14). The inverse Fourier transform of the first term on the RHS of Eq. (55) and also Eq. (58) can be obtained directly from Eq. (A.13). The other terms are given below ($x_2 > 0$, $R_2^A > 0$, $R_2^B > 0$).

(i)

$$\begin{aligned}\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} dq_1\, \mathbf{G}^A(\mathbf{q})\mathbf{M}\mathbf{P}(q_1)e^{-\iota q_1 R_1^A}e^{\iota x_1 q_1 + \iota q_2 x_2}dq_2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{g}^A(q_1, x_2)\mathbf{M}\mathbf{P}(q_1)e^{\iota q_1(x_1 - R_1^A)}dq_1 \\ &= -\frac{1}{\pi}\sum_{\alpha\beta}\boldsymbol{\nu}^A(p_\alpha^A)\mathbf{M}[\sigma^{*A}(p_\beta^A) - \sigma_s^{*B}\nu_s^{*B^{-1}}\nu^{*A}(p_\beta^A)] \\ &\quad\cdot \ln[(x_1 - R_1^A) + x_2 p_\alpha^A - R_2^A p_\beta^{*A}]\end{aligned} \quad (A.28)$$

where we have used Eqs. (A.8), (A.14), and (A.24). In a similar manner we evaluate the remaining terms as given below.

(ii)

$$\begin{aligned}\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} dq_1\, \mathbf{G}^B(\mathbf{q})\mathbf{N}\mathbf{T}(q_1)e^{-\iota q_1 R_1^A}e^{\iota q_1 x_1 - \iota q_2 x_2}dq_2 &= -\frac{1}{\pi}\sum_{\alpha\beta}\boldsymbol{\nu}^{*B}(p_\alpha^B)\mathbf{N}[\sigma^{*A}(p_\beta^A) - \boldsymbol{\sigma}_s^A\boldsymbol{\nu}_s^{A^{-1}}\boldsymbol{\nu}^{*A}(p_\beta^A)] \\ &\quad\cdot \ln[(x_1 - R_1^A) - x_2 p_\alpha^{*B} - R_2^A p_\beta^{*A}]\end{aligned} \quad (A.29)$$

(iii)

$$\begin{aligned}\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \mathbf{G}^A(\mathbf{q})\mathbf{M}\mathbf{Q}(q_1)e^{-\iota q_1 R_1^B}e^{\iota q_1 x_1 + \iota q_2 x_2}dq_1\, dq_2 &= -\frac{1}{\pi}\sum_{\alpha\beta}\boldsymbol{\nu}^A(p_\alpha^A)\mathbf{M}[\sigma^B(p_\beta^B) - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\nu}_s^{*B^{-1}}\boldsymbol{\nu}^B(p_\beta^B)] \\ &\quad\cdot \ln[(x_1 - R_1^B) + x_2 p_\alpha^A + R_2^B p_\beta^B]\end{aligned} \quad (A.30)$$

(iv)

$$\begin{aligned}\frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \mathbf{G}^B(\mathbf{q})\mathbf{N}\mathbf{W}(q_1)e^{-\iota q_1 R_1^B}e^{\iota q_1 x_1 - \iota q_2 x_2}dq_1\, dq_2 &= -\frac{1}{\pi}\sum_{\alpha\beta}\boldsymbol{\nu}^{*B}(p_\alpha^B)\mathbf{N}[\boldsymbol{\sigma}^B(p_\beta^B) - \boldsymbol{\sigma}_s^A\nu_s^{A^{-1}}\boldsymbol{\nu}^B(p_\beta^B)] \\ &\quad\cdot \ln[(x_1 - R_1^B) - x_2 p_\alpha^{*B} + R_2^B p_\beta^B]\end{aligned} \quad (A.31)$$

where, in each equation, the real part of the RHS has to be taken.

APPENDIX B: CONTINUITY OF GREEN'S FUNCTION WITH RESPECT TO THE POINT WHERE THE FORCE IS APPLIED

In this appendix we shall show that the $\mathbf{G}^c(\mathbf{x}, \mathbf{x}')$ is continuous as $\mathbf{x}'$ moves across the interface from UHP to LHP or vice versa. For this purpose, first we shall show that the following relations, i.e., the sum rules, are valid:

$$\boldsymbol{\gamma}_s^{A,B} - \boldsymbol{\gamma}_s^{*A,B} = 0 \quad (B.1)$$

and

$$\boldsymbol{\sigma}_s^A - \boldsymbol{\sigma}_s^{*A} = \boldsymbol{\sigma}_s^B - \boldsymbol{\sigma}_s^{*B} = \iota\mathbf{I} \quad (B.2)$$

To prove Eq. (B.1), we use Eqs. (A.7), (A.8), (A.10), and (A.22) to write (for superscripts A or B, $\xi > 0$)

$$\begin{aligned}\boldsymbol{\gamma}_s - \boldsymbol{\gamma}_s^* &= \lim_{\xi\to 0} q_1[\mathbf{g}(q_1, \xi) - \mathbf{g}(q_1, -\xi)] \\ &= \lim_{\xi\to 0}\frac{q_1}{2\pi}\left[\int_{-\infty}^{\infty}\mathbf{G}(\mathbf{q})e^{\iota q_2\xi}dq_2 - \int_{-\infty}^{\infty}\mathbf{G}(\mathbf{q})e^{-\iota q_2\xi}dq_2\right] \\ &= \lim_{\xi\to 0}\frac{\iota q_1}{\pi}\left[\int_{-\infty}^{\infty}\mathbf{G}(\mathbf{q})\sin q_2\xi\, dq_2\right] \\ &= 0\end{aligned} \quad (B.3)$$

since $\sin q_2\xi$ remains finite as $q_2 \to \infty$ and therefore its limit will be zero as $\xi \to 0$. We note that for a general line defect with both line force (Green's function) and dislocation character, the difference $\nu_s^{A,B} - \nu_s^{*A,B}$ would instead relate to the Burgers vector of the dislocation portion.

To prove Eq. (B.2), we first consider its LHS. Using Eq. (A.22) for the definition of $\boldsymbol{\sigma}_s$ along with Eqs. (A.16), (A.17), (A.19), (3.13), and (3.14), we write

$$\boldsymbol{\sigma}_s^A - \boldsymbol{\sigma}_s^{*A} = \lim_{\xi\to 0} J^A(\xi) \quad (B.4)$$

where

$$\mathbf{J}^A(\xi) = \boldsymbol{\eta}^A(q_1, \xi) - \boldsymbol{\eta}^A(q_1, -\xi)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [\mathbf{L}^A(\mathbf{q})\mathbf{G}^A(\mathbf{q})e^{\iota q_2 \xi} - \mathbf{L}^A(\mathbf{q})\mathbf{G}^A(\mathbf{q})e^{-\iota q_2 \xi}]dq_2 \tag{B.5}$$

and as in Eq. (B.3), ξ is taken to be positive.

From Eq. (B.5) we obtain

$$\frac{d\mathbf{J}^A(\xi)}{d\xi} = \frac{\iota}{2\pi}\int_{-\infty}^{\infty} q_2\mathbf{L}^A(\mathbf{q})\mathbf{G}^A(\mathbf{q})[e^{\iota q_2 \xi} + e^{-\iota q_2 \xi}]dq_2 \tag{B.6}$$

From Eqs. (3.14) and (2.12) we see

$$\boldsymbol{\Lambda}^A(\mathbf{q}) = q_2\mathbf{L}^A(\mathbf{q}) + \Delta\mathbf{L}(\mathbf{q}) \tag{B.7}$$

where

$$\Delta L_{ij}(q) = c_{i1j1}q_1^2 + c_{i1j2}q_1q_2 \tag{B.8}$$

since, as mentioned at the end of Sec. II, $q_3 = 0$.

With the help of Eqs. (B.7), (B.8), and (16) we obtain from Eq. (B.6)

$$\frac{d\mathbf{J}^A(\xi)}{d\xi} = \frac{\iota}{2\pi}\int_{-\infty}^{\infty} \mathbf{I}[e^{\iota q_2 \xi} + e^{-\iota q_2 \xi}]dq_2 - \frac{\iota}{2\pi}\int_{-\infty}^{\infty} \Delta\mathbf{L}(\mathbf{q})\mathbf{G}(\mathbf{q})[e^{\iota q_2 \xi} + e^{-\iota q_2 \xi}]dq_2 \tag{B.9}$$

where I is the unit matrix. Using the Fourier representation of the delta function as given by Eq. (8) and the following definition of the step function

$$\int_{-\infty}^{\xi} \delta(\xi) = H(\xi)$$

where the step function $H(\xi)$ is defined by the following ($\xi \geq 0$):

$$H(-\xi) = 0$$
$$H(\xi = 0) = 1/2$$
$$H(\xi) = 1$$

We obtain from Eq. (B.8)

$$\mathbf{J}^A(\xi) = 2\iota\mathbf{I}H(\xi) - \Delta\mathbf{J}^A(\xi)$$

where

$$\Delta\mathbf{J}^A_{ik}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(c_{i1j1}\frac{q_1^2}{q_2} + c_{i1j2}q_1\right)G_{jk}(\mathbf{q}) \cdot [e^{\iota q_2 \xi} - e^{-\iota q_2 \xi}]dq_2 \tag{B.10}$$

Since the limit of $\sin(q_2\xi)$ and $\sin(q_2\xi)/q_2$ is zero for $\xi = 0$ even for $q_2 \to \infty$, we see that $\Delta\mathbf{J}^A(\xi) = 0$ for $\xi = 0$. It may be remarked that we would not put $J^A(\xi) = 0$ in Eq. (B.5) although it contains the factor $\sin(q_2\xi)$ because $\mathbf{L}(\mathbf{q})$ contains the factor q_2 which would be infinite as $q_2 \to \infty$.

Thus, we see

$$\boldsymbol{\sigma}_s^A - \boldsymbol{\sigma}_s^{*A} = \iota\mathbf{I} \tag{B.11}$$

and is independent of the material constants of A. Proceeding in a similar manner, we would obtain the same result *for* $\sigma_s^B - \sigma_s^{*B}$ which proves Eq. (B.2).

Now to prove the continuity of $\mathbf{G}^{CA}(\mathbf{x}, \mathbf{x}')$ we put $x_2' = 0$ in Eq. (59) for UHP and in Eq. (61) for LHP. We obtain

$$\mathbf{G}^{CA}(\mathbf{x}, x_1', x_2' = 0^+) = -\mathbf{R}[\mathbf{I} + \mathbf{M}\{\boldsymbol{\sigma}_s^{*A} - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}}\boldsymbol{\gamma}_s^{*A}\}] \tag{B.12}$$

and

$$\mathbf{G}^{CA}(\mathbf{x}, x_1', x_2' = 0^-) = \mathbf{R}\mathbf{M}[\boldsymbol{\sigma}_s^B - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}}\boldsymbol{\gamma}_s^B] \tag{B.13}$$

where

$$\mathbf{R} = \frac{1}{\pi}\sum_\alpha \boldsymbol{\gamma}^A(p_\alpha)\ \ln[(x_1 - x_1') + x_2p_\alpha^A] \tag{B.14}$$

and we have used Eqs. (A.22) and (A.23). Using Eq. (A.20) we find that

$$\mathbf{G}^{CA}(\mathbf{x}; x_1', x_2' = 0^+) = -\mathbf{R}\mathbf{M}[\boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}}\boldsymbol{\gamma}_s^A - \boldsymbol{\sigma}_s^A + \boldsymbol{\sigma}_s^{*A} - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}}\boldsymbol{\gamma}_s^{*A}] \tag{B.15}$$

Using Eq. (B.1) we find that Eqs. (B.15) and (B.13) reduce, respectively, to the following:

$$\mathbf{G}^{CA}(\mathbf{x}; x_1', x_2 = 0^+) = \mathbf{R}\mathbf{M}(\boldsymbol{\sigma}_s^A - \boldsymbol{\sigma}_s^{*A}) \tag{B.16}$$

and

$$\mathbf{G}^{CA}(\mathbf{x}; x_1', x_2 = 0^-) = \mathbf{R}\mathbf{M}(\boldsymbol{\sigma}_s^B - \sigma_s^{*B}) \tag{B.17}$$

which are equal in view of Eq. (B.2). Similarly, we can prove the continuity of $\mathbf{G}^{CB}(\mathbf{x}, \mathbf{x}')$.

Elastic Green's function for a composite solid with a planar crack in the interface

V. K. Tewary,[a)] R. H. Wagoner, and J. P. Hirth[b)]
Department of Materials Science and Engineering, The Ohio State University, 116 West 19th Avenue, Columbus, Ohio 43210

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The elastic Green's functions for displacements and stresses have been calculated for a composite solid containing a planar crack in a planar interface using the Green's function derived in a previous paper for a line load parallel to the composite interface. The resulting functions can be used to calculate the stress or displacement at any point in the composite for a variety of elastic singularities. As specific applications, the Mode I stress intensity factor of an interfacial crack was calculated as were the Green's functions for the semi-infinite antiplane strain case. The Mode I case shows the near-crack tip oscillations reported by other authors while the Mode III case does not. The newly devised Green's functions are shown to reproduce the results of other authors in the isotropic limit.

I. INTRODUCTION

The interfacial fracture strength of polycrystalline and multiphase solids and also of macroscopic composite materials may be controlled by the properties of cracks embedded in the interfaces in the materials. The mode of fracture, viz., intergranular vs transgranular, is presumably related to the relative fracture toughness of the matrix and the interfaces.

The elastic properties of a homogeneous, continuous solid are contained in the elastic Green's function for that continuum. These functions are valuable for solving a variety of elastic boundary value problems that meet the physical compatibility and equilibrium conditions.[1-3] For a cracked composite body, similar problems can be solved by means of corresponding Green's functions. It is of interest, therefore, to calculate the Green's function of a composite solid containing a crack in the interface: the object of this paper.

A considerable amount of work has already been done on elastic fields of line defects in a cracked body (see, for example, Ref. 4 and other references given therein). The mathematical treatment is generally based upon the formalism developed by Stroh.[5] An alternative treatment based upon an integral representation of Stroh's theory was developed by Barnett[6] and Barnett and Lothe.[7] Excellent reviews of the general theory have been given by Rice[8] and Thomson.[9] These reviews contain many other references to work on fracture.

Interfacial cracks subject to uniform loading of crack surfaces in isotropic solids have been studied by several authors, including England[10] and Rice and Sih.[11] The case of anisotropic solids subject to loading of crack surfaces has been discussed by Willis[12] and Clements.[13] For many practical applications, such as the interaction of dislocations and other defects with cracks, one must calculate the displacement field and the stress distribution in a solid subject to arbitrary loading. The Green's functions as calculated in this paper facilitate the analysis of such applications.

In general, the displacement field and the stress have been found to show strong oscillations very near the crack tip. Our results confirm the existence of such oscillations except, of course, in the simple case of Mode III deformation (antiplane strain mode). In this mode there are no oscillations and the composite solid behaves like a homogeneous solid.

The Green's function for a homogeneous anisotropic solid containing a planar crack with straight parallel edges has been obtained by Sinclair and Hirth.[14] In an earlier paper (Ref. 15, henceforth referred to as Paper I) we obtained the elastic Green's function for an anisotropic composite solid with a planar interface. In the present paper we use the results of Paper I to obtain the Green's function for the same model solid containing a planar crack in the interface.

II. EQUATION FOR DISPLACEMENT FIELD AND ITS SOLUTION FOR A CRACK OF FINITE LENGTH

We consider a composite solid with a planar interface containing a crack of length $2c$. The solid is assumed to extend to infinity. The edges of the crack are assumed to be parallel and straight. We choose a system of axes as shown in Fig. 1, such that the interface is along the plane $x_2 = 0$ and the crack extends from $-c$ to c. No variation of the field quantities (stresses, strains, etc.) is allowed in the x_3 direction so that the mathematical problem is essentially a two-dimensional one. As in Paper I, the solid in the upper half plane (UHP: $x_2 > 0$) is denoted by the

[a)] Current address: National Institute of Standards and Technology, Fracture and Deformation Division, Boulder, Colorado 80303.
[b)] Current address: Washington State University, Department of Mechanical and Materials Engineering, Pullman, Washington 99164-2920.

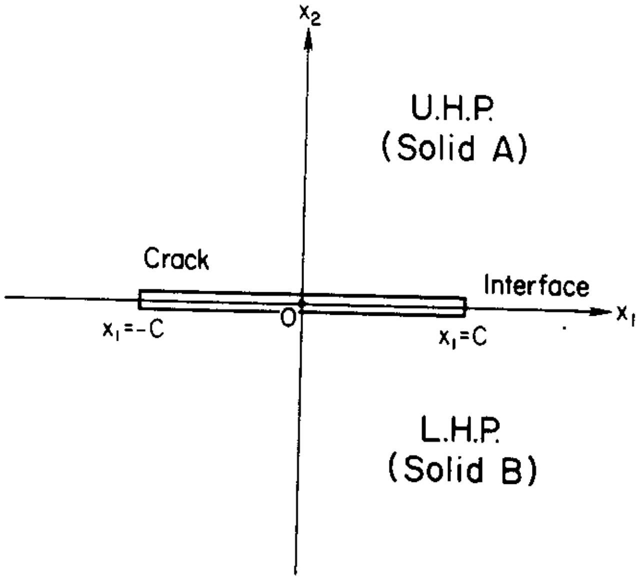

FIG. 1. The coordinate axes. The x_3 axis is normal to the plane of the paper. The interface is along the plane $x_2 = 0$. The crack extends from $x_1 = -c$ to $x_1 = c$ in the interfacial plane.

superscript A and that in the lower half plane (LHP:$x_2 < 0$) by the superscript B.

The equations of elastic equilibrium are as follows:

$$c_{ikjl}\frac{\partial^2 u_j}{\partial x_k\, \partial x_l} = -f_i(\mathbf{x}) \tag{1}$$

where $\mathbf{c}$ is the tensor of elastic constants, $\mathbf{u}$ is the displacement field, $\mathbf{x}$ denotes Cartesian coordinates, and i, j, k, l take the values 1, 2, and 3. In Eq. (1) and throughout this paper we shall follow the summation convention over repeated Roman indices but not over Greek indices which also take values of 1, 2, and 3.

Our object is to calculate the displacement field for a line force, i.e., when $\mathbf{f}(\mathbf{x})$ in Eq. (1) is of the following form:

$$f_i(x) = \phi_i \delta(x_1 - R_1)\delta(x_2 - R_2) \tag{2}$$

where $\boldsymbol{\phi}$ is the constant vector strength of the line force. The line of force application is parallel to the x_3 axis and, as is apparent from Eq. (2), cuts the x_1, x_2 plane at $x_1 = R_1, x_2 = R_2$. The point $(R_1, R_2, 0)$ may be in UHP or LHP and is accordingly labeled by the superscripts A or B. The two-dimensional Green's function G_{ji} is defined by the displacement field u_j for a unit line force, i.e., when $\theta_i = 1$.

We consider the case when the two solids, A and B, are perfectly welded at the interface except, of course, in the region of the crack, where two traction-free surfaces are required. The boundary conditions at the regions of interface outside the crack, as given in Paper I, are that the u_i and the stress component T_{i2} are continuous at the interface. At the surface of the crack the following boundary conditions[14] are specified:

$$T_{i2} = 0 \tag{3}$$

for

$$x_2 = 0 \quad \text{and} \quad -c \leq x_1 \leq c$$

In addition, as in Paper I, we require that the stresses vanish at infinity.

In our present model, as in Paper I, $\mathbf{f}(\mathbf{x})$ is independent of the coordinate x_3. In what follows, therefore, unless otherwise indicated, the vector $\mathbf{x}$ denotes a 2-dimensional vector with components x_1 and x_2. The indices k and l in Eq. (1) take only the values 1 and 2. Of course, the z components of $\mathbf{u}$ and $\mathbf{f}$ need not be zero.

We now discuss the solution of Eq. (1) in four different cases. These four cases correspond to the following four combinations of $\mathbf{x}$ and $\mathbf{R}$ in UHP or LHP: (i) $\mathbf{x}$ and $\mathbf{R}$ in UHP; (ii) $\mathbf{x}$ in LHP, $\mathbf{R}$ in UHP; (iii) $\mathbf{x}$ in UHP, $\mathbf{R}$ in LHP; and (iv) $\mathbf{x}$ and $\mathbf{R}$ in LHP. The Green's functions for these four cases without the crack were obtained in Paper I and are quoted in Appendix A of the present paper. Certain useful relations between the parameters of the Green's function are also given in Appendix A.

We first consider the case when both x and R are in the UHP, i.e., $\mathbf{x} = x_1, x_2$ where $x_2 > 0$ and $\mathbf{R}^A = R_1^A, R_2^A$ where $R_2^A > 0$.

The general solution of Eq. (1), as in Paper I, can be written as follows:

$$u_i^A(\mathbf{x}) = \int G_{ij}^{CA}(\mathbf{x}, \mathbf{x}')f_j(\mathbf{x}')\,dx' + \lim_{\zeta=0^+}\int_{-c}^{c} G_{ij}^{CA}(\mathbf{x}, \mathbf{x}')F_j(x_1')\,\delta(x_2' + \zeta)\,d\mathbf{x}' \tag{4}$$

where

$$f_j(\mathbf{x}') = \phi_j \delta(x_1' - R_1^A)\,\delta(x_2' - R_2^A) \tag{5}$$

$F_j(x_1')$ is an arbitrary function defined over the crack surface and ζ is an arbitrarily small positive number which tends to zero in the limit. The second term on the RHS of Eq. (4) can be considered to arise from a hypothetical distribution of forces which are applied just outside the region A along the surfaces of the crack. This distribution of fictitious forces is required to meet the traction-free surface condition at the crack. As shown in Paper I, the displacement field as given by Eq. (4) satisfies Eq. (1) and those boundary conditions at the interface outside the crack region which correspond to the perfect welding of solids A and B at $x_2 = 0$ for all values of $\mathbf{F}(x_1')$. We therefore determine $\mathbf{F}(x_1')$ so that $\mathbf{u}^A(\mathbf{x})$ satisfies the crack boundary conditions as given by Eq. (3). In addition, we impose the following equilibrium condition that no external net force be applied to the crack:

$$\int_{-c}^{c} \mathbf{F}(x_1')\,dx_1' = 0 \tag{6}$$

Using Eq. (A.1) of Appendix A, we determine the following displacement function for case (i) from Eq. (4):

$$\mathbf{u}^A(\mathbf{x}) = -\frac{1}{\pi}\sum_{\alpha=1}^{3} \gamma^A(p_\alpha^A)\boldsymbol{\phi}^A \ln(z_\alpha^A - \rho_\alpha^A)$$

$$- \frac{1}{\pi} \sum_{\alpha\beta=1}^{3} \boldsymbol{\gamma}^A(p_\alpha^A)\mathbf{Q}_\beta^{\mathrm{I}}\boldsymbol{\phi} \ln(z_\alpha - \rho_\beta^{A*})$$

$$- \frac{1}{\pi} \sum_{\alpha=1}^{3} \boldsymbol{\gamma}^A(p_\alpha^A) \int_{-c}^{c} \ln(z_\alpha^A - t_-)\mathbf{F}(t_+)\,dt$$

$$- \frac{1}{\pi} \sum_{\alpha=1}^{3} \boldsymbol{\gamma}^A(p_\alpha^A)\mathbf{Q}_s^{\mathrm{I}} \int_{-c}^{c} \ln(z_\alpha^A - t_-)\mathbf{F}(t_-)\,dt \tag{7}$$

where

$$Q_s^{\mathrm{I}} = \sum_{\alpha=1}^{3} \mathbf{Q}^{\mathrm{I}}(p_\alpha^A) \tag{8}$$

$$z_\alpha^A = x_1 + p_\alpha^A x_2 \tag{9}$$

$$\rho_\alpha^A = R_1^A + p_\alpha^A R_2^A \tag{10}$$

$$t_+ = t + \iota 0$$

$$t_- = t - \iota 0$$

As in Paper I, the real part of the expression on the RHS of Eq. (7) gives the displacement field. In Eq. (7) the sum over α includes only those three roots p_α of Eq. (A.19) which have positive imaginary parts. Equation (A.19) is a sextic equation in q_i [see Eq. (A.1) of Paper I and Ref. 5 of this paper]. The remaining three roots of that sextic equation are the complex conjugates of p_α (α = 1, 2, and 3). We can, therefore, order the roots such that

$$p_{\alpha+3} = p_\alpha^* \qquad (\alpha = 1 - 3) \tag{11}$$

We now write the stress component T_{i2} in terms of the displacement field in the following:

$$\mathrm{T}_{i2} = c_{i21j}^A \frac{\partial u_j^A}{\partial x_1} + c_{i22j}^A \frac{\partial u_j^A}{\partial x_2}$$

$$= L_{ij}^A(\alpha)\, \partial u_j^A / \partial z_\alpha^A \tag{12}$$

where we have used the fact that u_j^A is a function of z_α^A so that

$$\frac{\partial}{\partial x_1} = \frac{1}{p_\alpha} \frac{\partial}{\partial x_2} = \frac{\partial}{\partial z_\alpha^A} \tag{13}$$

and

$$L_{ij}^A(\alpha) = c_{i21j}^A + p_\alpha^A c_{i22j}^A \tag{14}$$

Using Eqs. (7) and (12), we obtain the following expression for $\mathbf{T}$ in the UHP for $x_2 = 0^+$ where $\mathbf{T}$ is a vector with components T_{c2}:

$$\mathbf{T}(x_1) = -\frac{1}{\pi} \sum_{\alpha=1}^{3} \frac{\boldsymbol{\sigma}^A(p_\alpha^A)}{x_1 - \rho_\alpha^A}\boldsymbol{\phi} - \frac{1}{\pi} \sum_{\beta=1}^{3} \frac{\boldsymbol{\sigma}_s^A \mathbf{Q}_\beta^{\mathrm{I}}}{x_1 - \rho_\beta^{A*}}$$

$$- \frac{1}{\pi} \int_{-c}^{c} \frac{\boldsymbol{\sigma}_s^A \mathbf{F}(t_+)}{x_1 - t_-} - \frac{1}{\pi} \int_{-c}^{c} \frac{\boldsymbol{\sigma}_s^A \mathbf{Q}_s^{\mathrm{I}} F(t_-)}{x_1 - t_-} \tag{15}$$

We have to determine $\mathbf{F}(t_+)$ and $\mathbf{F}(t_-)$ so that the boundary condition given by Eq. (3) is satisfied. However, the stress is given by only the real part of the RHS of Eq. (15). We therefore add to the RHS of Eq. (15) its complex conjugate and multiply by one-half. In accordance with Eq. (11), we obtain the complex conjugate by extending the sums on the RHS of Eq. (15) over α and β = 4 to 6 and also replace t_- and t_- by t_- and t_+, respectively. Proceeding in this manner and using the fact that $x_1 - t_+ = x_{1-} - t$ we find the following equation for $-c \leqslant x_1 \leqslant c$:

$$\int_{-c}^{c} \frac{[\boldsymbol{\sigma}_s^A \mathbf{F}(t_+) + \boldsymbol{\sigma}_s^A \mathbf{Q}_s^{\mathrm{I}} \mathbf{F}(t_-)]\,dt}{t - x_{1+}}$$

$$+ \int_{-c}^{c} \frac{[\boldsymbol{\sigma}_s^{*A} \mathbf{F}(t_-) + \boldsymbol{\sigma}_s^{*A} \mathbf{Q}_s^{*\mathrm{I}} \mathbf{F}(t_+)]\,dt}{t - x_{1-}} = \mathbf{M}(x_1) \tag{16}$$

where

$$\mathbf{M}(x_1) = \sum_{m=1}^{6} \left[\frac{\boldsymbol{\sigma}^A(p_m^A)}{t - \rho_m^A} + \frac{\boldsymbol{\sigma}_s^{Am}\mathbf{Q}_m^{\mathrm{I}}}{x_1 - \rho_m^{A*}} \right] \boldsymbol{\phi} \tag{17}$$

$$\rho_{m+3}^A = \rho_m^{A*} \qquad (m = 1, 2, 3) \tag{18}$$

$$\boldsymbol{\sigma}_s^{Am} = \boldsymbol{\sigma}_s^A \qquad (m = 1, 2, 3)$$

$$= \boldsymbol{\sigma}_s^{*A} \qquad (m = 4, 5, 6) \tag{19}$$

In what follows we use the same notation for other variables and functions, viz.,

$$z_{m+3}^A = z_m^{A*} \qquad (m = 1, 2, 3) \tag{20}$$

$$\boldsymbol{\gamma}(p_{m+3}) = \boldsymbol{\gamma}^*(p_m) \qquad (m = 1, 2, 3) \tag{21}$$

with similar relations for $\boldsymbol{\sigma}$, $\mathbf{Q}$, and $\mathbf{Q}_s$.

Using the same procedure that led to Eq. (16), we obtain the following condition for zero stress in the LHP by using Eq. (A.2):

$$\int_{-c}^{c} \frac{\boldsymbol{\sigma}_s^{*B} \mathbf{Q}_s^{\mathrm{II}} \mathbf{F}(t_-)\,dt}{t - x_{1+}} + \int_{-c}^{c} \frac{\boldsymbol{\sigma}_s^{B} \mathbf{Q}_s^{*\mathrm{II}} \mathbf{F}(t_+)\,dt}{t - x_{1-}} = \mathbf{M}(x_1) \tag{22}$$

where

$$\mathbf{M}(x_1) = \sum_{m=1}^{6} \frac{\boldsymbol{\sigma}_s^{*Bm} \mathbf{Q}_m^{*\mathrm{II}} \boldsymbol{\phi}}{x_1 - \rho_m^{A*}} \tag{23}$$

By means of Eqs. (A.2) and (18)–(21) one can show that the RHS of Eq. (23) for $\mathbf{M}(x_1)$ is the same as that of Eq. (17). Subtracting Eq. (22) from Eq. (16) and using Eq. (A.22), we get

$$\int_{-c}^{c} \frac{[\boldsymbol{\sigma}_s^A \mathbf{F}(t_+) - \boldsymbol{\sigma}_s^{*A} \mathbf{F}(t_-)]\,dt}{t - x_{1+}}$$

$$+ \int_{-c}^{c} \frac{[\boldsymbol{\sigma}_s^{*A} \mathbf{F}(t_-) - \boldsymbol{\sigma}_s^A \mathbf{F}(t_+)]\,dt}{t - x_{1-}} = 0 \tag{24}$$

In view of Eq. (24), we can choose

$$\boldsymbol{\sigma}_s^A \mathbf{F}(t_+) = \boldsymbol{\sigma}_s^{*A} \mathbf{F}(t_-) = \mathbf{f}(t) \tag{25}$$

Using Eq. (25), we can write Eq. (16) in the following form ($-c \leqslant x_1 \leqslant c$):

$$\mathbf{T}_U \mathbf{H}(x_{1+}) + \mathbf{T}_U^* \mathbf{H}(x_{1-}) = \mathbf{M}(x_1) \tag{26}$$

where

$$\mathbf{T}_U = \mathbf{I} + \boldsymbol{\sigma}_s^A \mathbf{Q}_s^{\mathrm{I}} (\boldsymbol{\sigma}_s^{*A})^{-1} \tag{27}$$

$$\mathbf{H}(z) = \int_{-c}^{c} \frac{\mathbf{f}(t)\,dt}{t - z} \tag{28}$$

$$z = x_1 + \iota x_2$$

and $\mathbf{I}$ is the unit matrix.

Equation (26) is in the standard form of the Hilbert problem. Its solution[16] is given below:

$$\mathbf{H}(z) = \frac{\mathbf{T}_U^{-1}}{2\pi\iota}\mathbf{J}_\lambda(z)Y(z)\int_{-c}^{c}\frac{\mathbf{J}_\lambda^{-1}(t_+)}{Y(t_+)}\frac{\mathbf{M}(t)}{t - z}dt \tag{29}$$

where

$$\mathbf{J}_\lambda(z) = \left(\frac{z + c}{z - c}\right)^{\iota\boldsymbol{\lambda}} \tag{30}$$

$$Y(z) = 1/(z^2 - c^2)^{1/2} \tag{31}$$

and

$$\exp(-2\pi\boldsymbol{\lambda}) = \mathbf{T}_U(\mathbf{T}_U^*)^{-1} \tag{32}$$

Here, both $\boldsymbol{\lambda}$ and therefore $\exp(-2\pi\boldsymbol{\lambda})$ as defined by Eq. (32) are matrices. The matrix $\exp(-2\pi\boldsymbol{\lambda})$, which is a scalar raised to a matrix power, is defined in terms of the eigenvectors and the eigenvalues of $\mathbf{T}_U(\mathbf{T}_U^*)^{-1}$ as follows:

$$\exp(-2\pi\boldsymbol{\lambda}) = \mathbf{E}\boldsymbol{\Delta}_\lambda\mathbf{E}^{*T}$$

where $\mathbf{E}$ is the matrix of the eigenvectors of $\mathbf{T}_U(\mathbf{T}_U^*)^{-1}$, $\mathbf{E}^T$ is the transpose of $\mathbf{E}$, and $\boldsymbol{\Delta}_\lambda$ a diagonal matrix. The elements of $\boldsymbol{\Delta}_\lambda$ are given below:

$$\boldsymbol{\Delta}_{\lambda\alpha\beta} = \exp(-2\pi\boldsymbol{\lambda}_\alpha)\,\delta_{\alpha\beta} \qquad \text{(no sum over } \alpha)$$

where λ_α is an eigenvalue of $\boldsymbol{\lambda}$ and is obtained in terms of ξ_α, an eigenvalue of $\mathbf{T}_U(\mathbf{T}_U^*)^{-1}$, as follows:

$$\lambda_\alpha = -\frac{1}{2\pi}\ln\xi_\alpha \tag{33}$$

Willis[12] has shown that $\mathbf{T}_U$ and $\mathbf{T}_U^*$ are Hermitian matrices so that ξ_α are real. The eigenvectors are taken to be orthonormal such that

$$\mathbf{E}E^{*T} = \mathbf{I} \tag{34}$$

In Eq. (29), $\mathbf{J}_\lambda(t_+)$ and $Y(t_+)$ denote the values of $\mathbf{J}_\lambda(z)$ and $Y(z)$ in the UHP for $z = t^+$. Taking $z - c = (c - t)\exp(\iota\pi)$ and $z + c = c + t(-c \leq t \leq c)$ in the UHP, we have

$$\mathbf{J}_\lambda(t_+) = \mathbf{R}_\lambda(t)\exp(\pi\boldsymbol{\lambda}) \tag{35}$$

$$Y(t_+) = -\iota Y_0(t) \tag{36}$$

$$\mathbf{J}_\lambda(t_-) = \mathbf{R}_\lambda(t)\exp(-\pi\boldsymbol{\lambda}) \tag{37}$$

and

$$Y(t_-) = \iota Y_0(t) \tag{38}$$

where

$$\mathbf{R}_\lambda(t) = \left(\left|\frac{c + t}{c - t}\right|\right)^{\iota\boldsymbol{\lambda}} \tag{39}$$

and

$$Y_0(t) = 1/(|c^2 - t^2|)^{1/2} \tag{40}$$

The matrices $\mathbf{J}_\lambda$ and $\mathbf{R}_\lambda$ are also defined in a manner analogous to $\exp(-2\pi\boldsymbol{\lambda})$. For example,

$$\mathbf{R}_\lambda(t) = \mathbf{E}\boldsymbol{\Delta}_R\mathbf{E}^{*T} \tag{41}$$

where $\boldsymbol{\Delta}_R$ is a diagonal matrix whose elements are given by

$$\boldsymbol{\Delta}_{R\alpha\beta} = \left(\frac{c + t}{c - t}\right)^{\iota\lambda_\alpha}\delta_{\alpha\beta} \tag{42}$$

The matrices $\mathbf{J}_\lambda$, $\mathbf{R}_\lambda$, $\exp(\pm\pi\boldsymbol{\lambda})$, and $\mathbf{T}_U(\mathbf{T}_U^*)^{-1}$ commute with each other since they all have the same eigenvectors. Their order in a product can therefore be interchanged.

Now we come to the solution of Eq. (28). This is a standard singular integral equation with a known solution.[16,17] The solution for $x = x_1$ on the real axis is given below:

$$\begin{aligned}\mathbf{f}(x_1) &= \frac{1}{2\pi\iota}[\mathbf{H}(z = x_{1+}) - \mathbf{H}(z = x_{1-})]\\ &= -\frac{1}{4\pi^2}[\mathbf{T}_U^{-1} + \mathbf{T}_U^{*-1}]\mathbf{R}_\lambda(x_1)Y_0(x_1)x\\ &\quad\cdot\int_{-c}^{c}\frac{\mathbf{R}_\lambda^{-1}(t)}{Y_0(t)}\frac{\mathbf{M}(t)\,dt}{t - x_1}\end{aligned} \tag{43}$$

Using Eq. (17) for $\mathbf{M}(t)$, we obtain

$$\begin{aligned}\mathbf{f}(x_1) = -\frac{1}{2\pi}[\mathbf{T}_U^{-1} + \mathbf{T}_U^{*-1}]\mathbf{R}_\lambda(x_1)Y_0(x_1)\sum_{m=1}^{6}\\ \cdot[\mathbf{I}(x_1, p_m^A)\boldsymbol{\sigma}^A(p_m^A) + \mathbf{I}(x_1, p_m^{A*})\boldsymbol{\sigma}_s^{Am}\mathbf{Q}_m^I]\boldsymbol{\phi}\end{aligned} \tag{44}$$

where

$$\mathbf{I}(x, p) = \frac{1}{2\pi}\int_{-c}^{c}\frac{\mathbf{R}_\lambda^{-1}(t)}{Y_0(t)}\frac{dt}{(t - x_1)(t - p)} \tag{45}$$

The integral in Eq. (45) is evaluated in Appendix B. Using the result given by Eq. (B.5) we find

$$\begin{aligned}\mathbf{f}(x_1) = -\frac{1}{2\pi}\mathbf{T}_U^{-1}\exp(-\pi\boldsymbol{\lambda})\mathbf{R}_\lambda(x_1)Y_0(x)\sum_{m=1}^{6}\\ \cdot[\mathbf{P}'(x_1, p_m^A)\boldsymbol{\sigma}^A(p_m^A) + \mathbf{P}'(x_1, p_m^{A*})\boldsymbol{\sigma}_s^{Am}\mathbf{Q}_m^I]\boldsymbol{\phi}\end{aligned} \tag{46}$$

where

$$\mathbf{P}'(x, p) = \frac{\mathbf{J}_\lambda^{-1}(p)Y^{-1}(p)}{p - x} - 1 \tag{47}$$

We can now determine $\mathbf{F}(x_{1+})$ and $\mathbf{F}(x_{1-})$ from Eq. (46) by using Eq. (25) as given below:

$$\mathbf{F}(x_+) = (\boldsymbol{\sigma}_s^A)^{-1}\mathbf{f}(x) \tag{48}$$

and

$$\mathbf{F}(x_-) = (\boldsymbol{\sigma}_s^{*A})^{-1}\mathbf{f}(x) \tag{49}$$

where $f(x)$ is given by Eq. (46).

The displacement field is given by Eqs. (7), (48), and (49). In a similar manner we obtain the displacement fields when the force $\boldsymbol{\phi}$ is applied in the LHP, by using

Eqs. (A.3) and (A.4). The Green's functions are then found by replacing the force vector $\boldsymbol{\phi}$ by a unit matrix. The calculated Green's functions are given in the next section.

III. GREEN'S FUNCTIONS

In this section we present the displacement as well as the stress Green's functions for a composite solid containing an interfacial crack. The displacement Green's function $\mathbf{G}(\mathbf{x},\mathbf{x}')$, as defined in Paper I, gives the displacement field at the point $\mathbf{x}$ when a unit line force is applied at the point $\mathbf{x}'$. Similarly, the stress Green's function $\mathbf{S}(\mathbf{x},\mathbf{x}')$ gives the stress at the point $\mathbf{x}$ when a unit line force is applied at the point $\mathbf{x}'$.

As in Paper I, we distinguish four cases corresponding to $\mathbf{x}$ and $\mathbf{x}'$ being, respectively, in (i) UHP, UHP; (ii) LHP, UHP; (iii) UHP, LHP; and (iv) LHP, LHP. In the expressions given below, the Greek summation variables α and β have the range 1–3 whereas the Roman variables m and n have the range 1–6. In each case only the real part of the final expression is to be taken.

A. Displacement Green's functions

1. UHP, UHP ($x_2 \geq 0$, $x_2' \geq 0$)

$$\begin{aligned}\mathbf{G}(\mathbf{x},\mathbf{x}') = &-\frac{1}{2\pi}\sum_m \boldsymbol{\gamma}^A(p_m^A)\ln(z_m^A - z_m'^A) \\ &-\frac{1}{2\pi}\sum_{\alpha\beta}[\boldsymbol{\gamma}^A(p_\alpha^A)\mathbf{Q}_\beta^{\mathrm{I}}\ln(z_\alpha^A - z_\beta'^{A*}) \\ &\qquad + \boldsymbol{\gamma}^{*A}(p_\alpha^A)\mathbf{Q}_\beta^{*\mathrm{I}}\ln(z_\alpha^{A*} - z_\beta'^A)] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\gamma}^A(p_m^A)(\boldsymbol{\sigma}_s^{Am})^{-1}\mathbf{T}_U^m\mathbf{T}_U^{-1} \\ &\cdot\exp(-\pi\boldsymbol{\lambda})\sum_n[\mathbf{P}(z_m^A, z_n'^A)\boldsymbol{\sigma}^A(p_n^A) \\ &\qquad + \mathbf{P}(z_m^A, z_n'^{A*})\boldsymbol{\sigma}_s^{An}\mathbf{Q}_n^{\mathrm{I}}]\end{aligned} \tag{50}$$

2. LHP, UHP ($x_2 \leq 0$, $x_2' \geq 0$)

$$\begin{aligned}\mathbf{G}(\mathbf{x},\mathbf{x}') = &-\frac{1}{2\pi}\sum_{\alpha\beta}[\boldsymbol{\gamma}^{*B}(p_\alpha^B)\mathbf{Q}_\beta^{\mathrm{II}}\ln(z_\alpha^{B*} - z_\beta'^{A*}) \\ &\qquad + \boldsymbol{\gamma}^B(p_\alpha^B)\mathbf{Q}_\beta^{*\mathrm{II}}\ln(z_\alpha^B - z_\beta'^A)] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\gamma}^{*B}(p_m^B)\mathbf{Q}_s^{\mathrm{II}m}(\boldsymbol{\sigma}_s^{*Am})^{-1}\mathbf{T}_U^{-1} \\ &\cdot\exp(-\pi\boldsymbol{\lambda})\sum_n \mathbf{P}(z_m^{B*}, z_n'^{A*})\boldsymbol{\sigma}_s^{*Bn}\mathbf{Q}_n^{\mathrm{II}}\end{aligned} \tag{51}$$

3. UHP, LHP ($x_2 \geq 0$, $x_2' \leq 0$

$$\begin{aligned}\mathbf{G}(\mathbf{x},\mathbf{x}') = &\frac{1}{2\pi}\sum_{\alpha\beta}[\boldsymbol{\gamma}^A(p_\alpha^A)\mathbf{Q}_\beta^{\mathrm{III}}\ln(z_\alpha^A - z_\beta'^B) \\ &\qquad + \boldsymbol{\gamma}^{*A}(p_\alpha^A)\mathbf{Q}_\beta^{*\mathrm{III}}\ln(z_\alpha^{A*} - z_\beta'^{B*})] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\gamma}^A(p_m^A)\mathbf{Q}_s^{\mathrm{III}m}(\boldsymbol{\sigma}_s^{Bm})^{-1}\mathbf{T}_L^{-1} \\ &\cdot\exp(-\pi\boldsymbol{\lambda})\sum_n \mathbf{P}(z_m^A, z_n'^B)\boldsymbol{\sigma}_s^{An}\mathbf{Q}_n^{\mathrm{III}}\end{aligned} \tag{52}$$

4. LHP, LHP ($x_2 \leq 0$, $x_2' \leq 0$)

$$\begin{aligned}\mathbf{G}(\mathbf{x},\mathbf{x}') = &-\frac{1}{2\pi}\sum_m \boldsymbol{\gamma}^{*B}(p_m^B)\ln(z_m^{B*} - z_m'^{B*}) \\ &+\frac{1}{2\pi}\sum_{\alpha\beta}[\boldsymbol{\gamma}^{*B}(p_\alpha^B)\mathbf{Q}_\beta^{\mathrm{IV}}\ln(z_\alpha^{B*} - z_\beta'^B) \\ &\qquad + \boldsymbol{\gamma}^B(p_\alpha^B)\mathbf{Q}_\beta^{*\mathrm{IV}}\ln(z_\alpha^B - z_\beta'^{B*})] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\gamma}^{*B}(p_m^B)(\boldsymbol{\sigma}_s^{*Bm})^{-1}\mathbf{T}_L^m\mathbf{T}_L^{-1} \\ &\cdot\exp(-\pi\boldsymbol{\lambda})\sum_n[\mathbf{P}(z_m^{B*}, z_n'^{B*})\boldsymbol{\sigma}^{*B}(p_n^B) \\ &\qquad - \mathbf{P}(z_m^{B*}, z_n'^B)\boldsymbol{\sigma}_s^{*Bn}\mathbf{Q}_n^{\mathrm{IV}}]\end{aligned} \tag{53}$$

In the above equations, factors with the following definitions are employed:

$$\mathbf{P}(z,\rho) = \frac{\mathbf{J}_\lambda^{-1}(\rho)}{Y(\rho)}\mathbf{I}_\lambda(z,\rho) - \mathbf{K}_\lambda(\rho) \tag{54}$$

$$\mathbf{I}_\lambda(z,\rho) = \frac{1}{2\pi}\int_{-c}^{c}\ln(z-t)\frac{\mathbf{R}_\lambda(t)Y_0(t)}{\rho - t}\,dt \tag{55}$$

$$\mathbf{K}_\lambda(z) = \frac{1}{2\pi}\int_{-c}^{c}\ln(z-t)\mathbf{R}_\lambda(t)Y_0(t)\,dt \tag{56}$$

$$\mathbf{T}_L = \mathbf{I} - \boldsymbol{\sigma}_s^{*B}\mathbf{Q}_s^{\mathrm{IV}}(\boldsymbol{\sigma}_s^B)^{-1} \tag{57}$$

$$\mathbf{T}_L(\mathbf{T}_L^*)^{-1} = \mathbf{T}_U(\mathbf{T}_U^*)^{-1} = \exp(-2\pi\boldsymbol{\lambda}) \tag{58}$$

$$\begin{aligned}\mathbf{T}_U^m &= \mathbf{T}_U(m = 1,2,3) \\ &= T_U^*(m = 4,5,6)\end{aligned} \tag{59}$$

with a similar relation for $\mathbf{T}_L^m$ in analogy with that defined by Eqs. (18)–(21), where $\mathbf{I}$ is the unit matrix. The evaluation of the integrals in Eqs. (55) and (56) is described in Appendix B.

In most cases of practical interest, one needs the stress distribution in the solid rather than the displacement field. We, therefore, give below the stress Green's functions $\mathbf{S}(\mathbf{x},\mathbf{x}')$. The element $S_i(\mathbf{x},\mathbf{x}')$ of the stress Green's function gives the stress T_{i2} at $\mathbf{x}$ when a unit line force is applied at $\mathbf{x}'$. In accordance with Eq. (12), we can obtain $\mathbf{S}(\mathbf{x},\mathbf{x}')$ from the displacement Green's function $\mathbf{G}(\mathbf{x},\mathbf{x}')$ by replacing $\boldsymbol{\gamma}$ by $\boldsymbol{\sigma}$ and differentiating it with respect to z, maintaining the appropriate indices A or B and α or β.

From Eqs. (54)–(56) we see that

$$\frac{d\mathbf{P}(z,\rho)}{dz} = \frac{\mathbf{J}_\lambda^{-1}(\rho)}{Y(\rho)}\mathbf{I}_\lambda'(z,\rho) - \mathbf{K}_\lambda'(z) \tag{60}$$

where

$$\mathbf{I}'_\lambda(z,\rho) = \frac{1}{2\pi}\int_{-c}^{c} \frac{\mathbf{R}_\lambda(t)Y_0(t)}{(t-z)(t-\rho)}dt \tag{61}$$

and

$$\mathbf{K}'_\lambda(z) = -\frac{1}{2\pi}\int_{-c}^{c} \frac{\mathbf{R}_\lambda(t)Y_0(t)}{t-z}dt \tag{62}$$

The evaluation of $\mathbf{I}'_\lambda$ and $\mathbf{K}'_\lambda$ is given in Appendix B. Using the results given by Eqs. (B.6) and (B.7), we find

$$\frac{d\mathbf{P}(z,\rho)}{dz} = [\exp(-\pi\boldsymbol{\lambda}) + \exp(\pi\boldsymbol{\lambda})]^{-1} \times \left[\frac{1}{z-\rho} + \mathbf{V}(z,\rho)\right] \tag{63}$$

where

$$\mathbf{V}(z,\rho) = -\left[\frac{\mathbf{J}_\lambda^{-1}(\rho)}{Y(\rho)}\frac{1}{z-\rho} + \mathbf{I}\right]\mathbf{J}_\lambda(z)Y(z) \tag{64}$$

and $\mathbf{I}$ is the unit matrix.

The expressions for the stress Green's function obtained from Eq. (63) by the procedure described above are given in Eqs. (67)–(70). These expressions give the stress component T_{i2}. The corresponding Green's function for T_{i1} can be obtained in an exactly analogous manner except that $\boldsymbol{\gamma}$ in $\mathbf{G}(\mathbf{x},\mathbf{x}')$ has to be replaced by $\boldsymbol{\sigma}^1$ which is defined as follows [cf. Eq. (12)]:

$$\sigma^1_{ij} = L^1_{ik}\gamma_{kj} \tag{65}$$

where

$$L^1_{ij}(\alpha) = c_{i11j} + p_\alpha c_{i12j} \tag{66}$$

B. Stress Green's functions (for T_{i2})

1. UHP, UHP ($x_2 \geq 0$, $x'_2 \geq 0$)

$$\begin{aligned}\mathbf{S}(\mathbf{x},\mathbf{x}') = &-\frac{1}{2\pi}\sum_m \frac{\boldsymbol{\sigma}^A(p^A_m)}{z^A_m - z'^A_m} \\ &-\frac{1}{2\pi}\sum_{\alpha\beta}\left[\frac{\boldsymbol{\sigma}^A(p^A_\alpha)\mathbf{Q}^{\mathrm{I}}_\beta}{z^A_\alpha - z'^{A*}_\beta} + \frac{\boldsymbol{\sigma}^{*A}(p^A_\alpha)\mathbf{Q}^{*\mathrm{I}}_\beta}{z^{A*}_\alpha - z'^A_\beta}\right] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\sigma}^A(p^A_m)(\boldsymbol{\sigma}^{Am}_s)^{-1}\mathbf{T}^m_U(\mathbf{T}^*_U + \mathbf{T}_U)^{-1} \\ &\cdot\sum_n\Bigg[\frac{\boldsymbol{\sigma}^A(p^A_n)}{z^A_m - z'^A_n} + \frac{\boldsymbol{\sigma}^{An}_s\mathbf{Q}^{\mathrm{I}}_n}{z^A_m - z'^{A*}_n} \\ &\quad + \mathbf{V}(z^A_m, z'^A_n)\boldsymbol{\sigma}^A(p^A_n) \\ &\quad + \mathbf{V}(z^A_m, z'^{A*}_n)\boldsymbol{\sigma}^{An}_s\mathbf{Q}^{\mathrm{I}}_n\Bigg]\end{aligned} \tag{67}$$

2. LHP, UHP ($x_2 \leq 0$, $x'_2 \geq 0$)

$$\begin{aligned}\mathbf{S}(\mathbf{x},\mathbf{x}') = &-\frac{1}{2\pi}\sum_{\alpha\beta}\left[\frac{\boldsymbol{\sigma}^{*B}(p^B_\alpha)\mathbf{Q}^{\mathrm{II}}_\beta}{z^{B*}_\alpha - z'^{A*}_\beta} + \frac{\boldsymbol{\sigma}^B(p^B_\alpha)\mathbf{Q}^{*\mathrm{II}}_\beta}{z^B_\alpha - z'^A_\beta}\right] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\sigma}^{*B}(p^B_m)\mathbf{Q}^{\mathrm{II}m}_s(\boldsymbol{\sigma}^{*Am}_s)^{-1} \\ &\cdot(\mathbf{T}^*_U + \mathbf{T}_U)^{-1} \\ &\cdot\sum_n\left[\frac{\boldsymbol{\sigma}^{*Bn}_s\mathbf{Q}^{\mathrm{II}}_n}{z^{B*}_m - z'^{A*}_n} + \mathbf{V}(z^{B*}_m, z'^{A*}_n)\boldsymbol{\sigma}^{*Bn}_s\mathbf{Q}^{\mathrm{II}}_n\right]\end{aligned} \tag{68}$$

3. UHP, LHP ($x_2 \geq 0$, $x'_2 \leq 0$)

$$\begin{aligned}\mathbf{S}(\mathbf{x},\mathbf{x}') = &\frac{1}{2\pi}\sum_{\alpha\beta}\left[\frac{\boldsymbol{\sigma}^A(p^A_\alpha)\mathbf{Q}^{\mathrm{III}}_\beta}{z^A_\alpha - z'^B_\beta} + \frac{\boldsymbol{\sigma}^{*A}(p^A_\alpha)\mathbf{Q}^{*\mathrm{III}}_\beta}{z^{A*}_\alpha - z'^{B*}_\beta}\right] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\sigma}^A(p^A_m)\mathbf{Q}^{\mathrm{III}m}_s(\boldsymbol{\sigma}^{Bm}_s)^{-1}(\mathbf{T}^*_L + \mathbf{T}_L)^{-1} \\ &\cdot\sum_n\left[\frac{\boldsymbol{\sigma}^{An}_s\mathbf{Q}^{\mathrm{III}}_n}{z^A_m - z'^B_n} + \mathbf{V}(z^A_m, z'^B_n)\boldsymbol{\sigma}^{An}_s\mathbf{Q}^{\mathrm{III}}_n\right]\end{aligned} \tag{69}$$

4. LHP, LHP ($x_2 \leq 0$, $x'_2 \leq 0$)

$$\begin{aligned}\mathbf{S}(\mathbf{x},\mathbf{x}') = &-\frac{1}{2\pi}\sum_m \frac{\boldsymbol{\sigma}^{*B}(p^B_m)}{z^{B*}_m - z'^{B*}_m} \\ &+\frac{1}{2\pi}\sum_{\alpha\beta}\left[\frac{\boldsymbol{\sigma}^{*B}(p^B_\alpha)\mathbf{Q}^{\mathrm{IV}}_\beta}{z^{B*}_\alpha - z'^B_\beta} + \frac{\boldsymbol{\sigma}^B(p^B_\alpha)\mathbf{Q}^{*\mathrm{IV}}_\beta}{z^B_\alpha - z'^{B*}_\beta}\right] \\ &+\frac{1}{2\pi}\sum_m \boldsymbol{\sigma}^{*B}(p^B_m)(\boldsymbol{\sigma}^{*Bm}_s)^{-1}\mathbf{T}^m_L(\mathbf{T}_L^* + \mathbf{T}_L)^{-1} \\ &\cdot\sum_n\Bigg[\frac{\boldsymbol{\sigma}^{*B}(p^B_n)}{z^{B*}_m - z'^{B*}_n} \\ &\quad - \frac{\boldsymbol{\sigma}^{*Bn}_s\mathbf{Q}^{\mathrm{IV}}_n}{z^{B*}_m - z'^B_n} + \mathbf{V}(z^{B*}_m, z'^{B*}_n)\boldsymbol{\sigma}^{*B}(p^B_n) \\ &\quad - \mathbf{V}(z^{B*}_m, z'^B_n)\boldsymbol{\sigma}^{*Bn}_s\mathbf{Q}^{\mathrm{IV}}_n\Bigg]\end{aligned} \tag{70}$$

IV. STRESS INTENSITY FACTOR

In this section we use the Green's functions as derived in the previous section to calculate the Mode I stress intensity factor when the crack is introduced into a body initially loaded only by uniform external stresses. (For reviews of the physical significance of the stress intensity factor, see, for example, Refs. 8 and 9.) For brevity, we refer to this case as a crack subjected to a "uniform load". This boundary value problem has been discussed earlier for isotropic solids by England[10] and Rice and Sih.[11] Our results, as given below, are applicable to general anisotropic solids.

Let $\mathbf{f}_0$ and $-\mathbf{f}_0$ denote, respectively, the forces on the upper and the lower surfaces of the crack. The applied force $\boldsymbol{\phi}(x_1)$ is thus represented as follows:

$$\begin{aligned}\boldsymbol{\phi}(x_1) &= \mathbf{f}_0 \qquad (x_2 = 0^+; -c \leq x_1 \leq c)\\ &= -\mathbf{f}_0 \qquad (x_2 = 0^-; -c \leq x_1 \leq c)\\ &= 0 \qquad (|x_1| > c) \end{aligned} \tag{71}$$

The stress $T_{i2}(\mathbf{x})$ at a point (x_1, x_2) in the UHP can be written as:

$$\mathbf{T}(\mathbf{x}) = \int_{-c}^{c} \mathbf{S}^{\mathrm{I}}(\mathbf{x},\mathbf{x}')\mathbf{f}_0 dx_1' - \int_{-c}^{c} \mathbf{S}^{\mathrm{II}}(\mathbf{x},\mathbf{x}')\mathbf{f}_0 dx_1' \tag{72}$$

where $\mathbf{T}$ is a vector with components T_{i2}, $\mathbf{f}_0$ is a constant vector as defined in Eq. (71), and $\mathbf{x}$ is in UHP ($x_2 \geq 0$). The stress Green's functions $\mathbf{S}^{\mathrm{I}}$ and $\mathbf{S}^{\mathrm{II}}$ correspond to cases (i) and (iii), respectively, as given by Eqs. (67) and (69).

Using Eqs. (67), (69), (57), (58), (A.23), and (A.24), we derive the following equation from Eq. (72):

$$\mathbf{T}(\mathbf{x}) = \frac{\iota}{2\pi} \sum_m \boldsymbol{\sigma}^A(p_m^A)(\boldsymbol{\sigma}_s^{Am})^{-1}\mathbf{T}_U^m(\mathbf{T}_U^* + \mathbf{T}_U)^{-1} \cdot \int_{-c}^{c} [\mathbf{V}(z_m, x_{1+}') - \mathbf{V}(z_m, x_{1-}')] dx_1' \mathbf{f}_0 \tag{73}$$

The integrals in Eq. (73) have been evaluated in Appendix B. Using Eqs. (B.21) and (B.22), we determine the following result from Eq. (73) for the stress distribution in UHP when the crack is subjected to a uniform load:

$$\begin{aligned}\mathbf{T}(\mathbf{x}) = -\sum_m \boldsymbol{\sigma}^A(p_m^A)(\boldsymbol{\sigma}_s^{Am})^{-1}\mathbf{T}_U^m(\mathbf{T}_U^* + \mathbf{T}_U)^{-1} \cdot \lim_{\eta\to 0}\Bigg[&\{\mathbf{I} + \exp(2\pi\boldsymbol{\lambda})\}^{-1}\mathbf{J}_\lambda^{-1}\frac{(z_m - \iota\eta)}{Y(z_m - \iota\eta)}\\ &+ \{\mathbf{I} + \exp(-2\pi\boldsymbol{\lambda})\}^{-1}\frac{\mathbf{J}_\lambda^{-1}(z_m + \iota\eta)}{Y(z_m + \iota\eta)}\\ &- (z_m\mathbf{I} - 2\iota c\lambda)\Bigg]\mathbf{J}_\lambda(z_m)Y(z_m)\mathbf{f}_0\end{aligned} \tag{74}$$

For $\mathbf{x}$ on the real axis between $-c$ and c, the first two terms inside the square brackets on the RHS of Eq. (74) cancel. In this case $\mathbf{T}(\mathbf{x})$ is zero, as expected, when summed over m. In other cases, i.e., for $\mathbf{x}$ not in the range between $-c$ and c, we obtain

$$\mathbf{T}(\mathbf{x}) = -\sum_m \boldsymbol{\sigma}^A(p_m^A)(\boldsymbol{\sigma}_s^{Am})^{-1}\mathbf{T}_U^m(\mathbf{T}_U^* + \mathbf{T}_U)^{-1} \cdot [\mathbf{I} - (z_m\mathbf{I} - 2\iota c\lambda)\mathbf{J}_\lambda(z_m)Y(z_m)]\mathbf{f}_0 \tag{75}$$

The stress distribution on the real axis for $\mathbf{x} = x_1$ with $|x_1| > c$, as found from Eq. (75) using Eqs. (35–40) is

$$\mathbf{T}(x_1) = -\mathbf{f}_0 + (x_1^2 - c^2)^{-1/2}(x_1\mathbf{I} - 2\iota c\boldsymbol{\lambda}) \cdot \exp\left[\iota\boldsymbol{\lambda} \ln\frac{x_1 + c}{x_1 - c}\right]\mathbf{f}_0 \tag{76}$$

Equation (76) has the same form as the corresponding equation for isotropic solids derived by England[10] and Rice and Sih.[11] As shown in Appendix C, this equation reduces to those derived by these authors in the isotropic limit. Also, Eqs. (75) and (76) reveal that as $|x|$ approaches ∞, $\mathbf{T}(x_1)$ approaches zero.

Near the crack tip, i.e., near $x_1 = c$, the stress as given by Eq. (76) has a strongly oscillatory behavior in addition to the square root singularity. These oscillations are characteristic of interfacial cracks in the elastic theory. Following Rice and Sih[11] (see also Ref. 12), we define the stress intensity factor, which is a vector in this case,[12] as

$$\mathbf{K} = 2\sqrt{2}[\exp(\pi\boldsymbol{\lambda}) + \exp(-\pi\boldsymbol{\lambda})]^{-1} \cdot \lim_{x_1\to c} (x_1 - c)^{\iota\boldsymbol{\lambda}+1/2}\mathbf{T}(x_1) \tag{77}$$

From Eqs. (76) and (77) we obtain

$$\mathbf{K} = 2\sqrt{c}\,[\exp(\pi\boldsymbol{\lambda}) + \exp(-\pi\boldsymbol{\lambda})]^{-1} \cdot [\mathbf{I} - 2\iota\boldsymbol{\lambda}]\exp[\iota\boldsymbol{\lambda} \ln 2c]\mathbf{f}_0 \tag{78}$$

As shown in Appendix C, Eq. (78) reduces to the result derived by Rice and Sih[11] in the isotropic limit.

The displacement field follows directly in the present case of a uniformly loaded crack. For this calculation, following the "inverse" of the procedure described in Sec. III for the calculations of $\mathbf{S}(\mathbf{x},\mathbf{x}')_n$ and as implied in Eq. (21), we replace $\boldsymbol{\sigma}^A(p_m^A)$ in Eq. (74) by $\boldsymbol{\gamma}^A(p_m^A)$ and integrate it with respect to z_m. This gives

$$\mathbf{U}(\mathbf{x}) = -\sum_m \boldsymbol{\gamma}^A(p_m^A)(\boldsymbol{\sigma}_s^{Am})^{-1}\mathbf{T}_U^m(\mathbf{T}_U^* + \mathbf{T}_U)^{-1} \cdot [z_m\mathbf{I} - \mathbf{J}_\lambda(z_m)/Y(z_m)] \tag{79}$$

The displacement field vanishes as $\mathbf{x} \to \infty$ since in this limit $Y(z_m) \to 1/z_m$ and $\mathbf{J}_\lambda(z_m) \to \mathbf{I}$. Equation (79) agrees with that derived by England[11] in the isotropic limit. The presence of the $\mathbf{J}_\lambda(z_m)$ term on the RHS of Eq. (79) gives rise to the characteristic oscillatory behavior of the displacement field. The stress distribution and the displacement field as given by Eqs. (74) and (79), respectively, are derived for $\mathbf{x}$ in UHP. In a similar manner, we can derive those for $\mathbf{x}$ in LHP. Both the stress T_{i2} and the displacement field are, of course, continuous at the interface on the real axis for $|x| > c$.

V. SEMI-INFINITE CRACK–ANTIPLANE STRAIN PROBLEM

As a simple illustration of the formulae derived in Sec. III, we consider in this section the antiplane strain problem associated with a semi-infinite interfacial crack. The oscillations in the displacement field and the stress associated with the interfacial crack are absent in the antiplane strain mode. In this mode the composite solid behaves like a homogeneous solid.

As given in Sec. IV of Paper I, in the antiplane strain mode α and β take only the value 3. The indices m and n in Eqs. (50)–(53) and (67)–(70) for the Green's functions therefore assume only the two values 3 and 6, which correspond to complex conjugates of one another. Further, all the matrices in this case reduce to pure numbers as given below:

$$\gamma_\alpha^{A,B} = \gamma_s^{A,B} = 1/2d^{A,B} \tag{80}$$

$$\sigma_\alpha^{A,B} = \sigma_s^{A,B} = \iota/2 \tag{81}$$

$$Q_\alpha^{\mathrm{I}} = Q_s^{\mathrm{I}} = Q_s^{\mathrm{IV}} = d^A - d^B \tag{82}$$

$$Q_s^{\mathrm{II}} = 2d^B \tag{83}$$

$$Q_s^{\mathrm{III}} = -2d^A \tag{84}$$

where

$$d^{A,B} = \frac{c_{44}^{A,B}}{(c_{44}^A + c_{44}^B)} \tag{85}$$

and $\gamma_s^{A,B}$ are expressed in units of $(c_{44}^A + c_{44}^B)^{-1}$.

The constants $d^{A,B}$ obviously obey the relation

$$d^A + d^B = 1 \tag{86}$$

From Eqs. (27) and (57) we obtain

$$\mathbf{T}_U = \mathbf{T}_U^* = 2d^B \tag{87}$$

and

$$\mathbf{T}_L = \mathbf{T}_L^* = 2d^A \tag{88}$$

The matrix $\boldsymbol{\lambda}$ is zero in view of Eq. (87). Hence $J_\lambda = 1$ and therefore the oscillations in the displacement field and the stress distribution are absent in this case.

Since the only root of Eq. (A.19) which is relevant for the antiplane strain problem is $p_3 = \iota$, we have in the present case

$$z \equiv z_\alpha = x_1 + \iota x_2 \tag{89}$$

For further simplification, we assume that the crack is of infinite length. We shift the origin to one end of the crack so that the crack extends from $-\infty$ to 0. We achieve this by defining a new variable Z as follows:

$$Z = z - c \tag{90}$$

and then take the limit of large c.

The functions $P(z,\rho)$ and $V(z,\rho)$ as defined by Eqs. (54) and (64), respectively, can now be calculated and are given below:

$$P(Z,\rho) = \ln(\sqrt{Z} + \sqrt{\rho}) \tag{91}$$

and

$$V(Z,\rho) = -\left[\frac{\sqrt{\rho}}{Z-\rho}\right]\frac{1}{\sqrt{Z}} \tag{92}$$

where

$$Z = X + \iota Y$$

and

$$\rho = X' + \iota Y'$$

The variables Z and ρ refer to the crack tip as the origin is defined by Eq. (90), and we have neglected the terms in P which correspond to rigid body displacements.

The calculated values of $G(\mathbf{x},\mathbf{x}')$ and $S(\mathbf{x},\mathbf{x}')$ obtained from Eqs. (80)–(88) and (91)–(92) are given below. The expressions for $G(\mathbf{x},\mathbf{x}')$ are in units of $(c_{44}^A + c_{44}^B)^{-1}$.

A. Displacement Green's functions

1. UHP, UHP ($X_2 \geq X_2' \geq 0$)

$$G(\mathbf{X},\mathbf{X}') = -\frac{1}{4\pi d^A}[\ln(Z-Z') + \ln(Z^*-Z'^*) + (d^A-d^B)\cdot\{\ln(Z-Z'^*) + \ln(Z^*-Z')\}] + \frac{1}{2\pi}\frac{d^B}{d^A}L(Z,Z') \tag{93}$$

2. LHP, UHP ($X_2 \leq 0$, $X_2'J \geq 0$)

$$G(\mathbf{X},\mathbf{X}') = -\frac{1}{2\pi}[\ln(Z-Z') + \ln(Z^*-Z'^*)] + \frac{1}{2\pi}L(Z,Z') \tag{94}$$

3. UHP, LHP ($X_2 \geq 0$, $X_2' \leq 0$)

$$G(\mathbf{X},\mathbf{X}') = -\frac{1}{2\pi}[\ln(Z-Z') + \ln(Z^*-Z'^*)] + \frac{1}{2\pi}L(Z,Z') \tag{95}$$

4. LHP, LHP ($X_2F \leq 0$, $X_2'F \leq 0$)

$$G(\mathbf{X},\mathbf{X}') = -\frac{1}{4\pi d^B}[\ln(Z-Z') + \ln(Z^*-Z'^*) - (d^A-d^B)\cdot\{\ln(Z-Z'^*) + \ln(Z^*-Z')\}] + \frac{1}{2\pi}\frac{d^A}{d^B}L(Z,Z') \tag{96}$$

where

$$L(Z,Z') = \ln(\sqrt{Z} + \sqrt{Z'}) + \ln(\sqrt{Z^*} + \sqrt{Z'^*}) - \ln(\sqrt{Z} + \sqrt{Z'^*}) - \ln(\sqrt{Z^*} + \sqrt{Z'}) \tag{97}$$

B. Stress Green's functions (for $\mathbf{T}_{32}$)

1. UHP, UHP ($X_2 \geq 0$, $X_2' \geq 0$)

$$S(\mathbf{X},\mathbf{X}') = -\iota\frac{d^A}{4\pi}A_1(Z,Z') - \iota\frac{d^B}{4\pi}A_2(Z,Z') \tag{98}$$

2. LHP, UHP ($X_2 \leq 0$, $X_2' \geq 0$)

$$S(\mathbf{X},\mathbf{X}') = -\iota\frac{d^B}{4\pi}A_1(Z,Z') - \iota\frac{d^B}{4\pi}A_2(Z,Z') \tag{99}$$

3. UHP, LHP ($X_2 \geqslant 0$, $X_2' \leqslant 0$)

$$S(\mathbf{X},\mathbf{X}') = -\iota\frac{d^A}{4\pi}A_1(Z,Z') - \iota\frac{d^A}{4\pi}A_2(Z,Z') \tag{100}$$

4. LHP, LHP ($X_2 \leqslant 0$, $X_2' \leqslant 0$)

$$S(\mathbf{X},\mathbf{X}') = -\iota\frac{d^B}{4\pi}A_1(Z,Z') - \iota\frac{d^A}{4\pi}A_2(Z,Z') \tag{101}$$

where

$$A_1(Z,Z') = \frac{1}{Z - Z'} - \frac{1}{Z^* - Z'^*} + \frac{1}{Z - Z'^*} - \frac{1}{Z^* - Z'} \tag{102}$$

and

$$A_2(Z,Z') = \frac{1}{\sqrt{Z}}\left[\frac{\sqrt{Z'}}{Z - Z'} - \frac{\sqrt{Z'^*}}{Z - Z'^*}\right] + \frac{1}{\sqrt{Z^*}}\left[\frac{\sqrt{Z'}}{Z^* - Z'} - \frac{\sqrt{Z'^*}}{Z^* - Z'^*}\right] \tag{103}$$

VI. SUMMARY

We have derived the displacement as well as the stress Green's functions for a composite solid containing a planar crack of finite or infinite length in the interface. The results can be applied in the fracture mechanics of a macroscopic composite solid in which the interface is between two different solids or a multiphase or a polycrystalline solid in which the interface is between two phases or orientations of the same solid (for the fracture mechanics of interfacial cracks; see, e.g., Ref. 12).

The advantage of the Green's function is that it gives the displacement field arising from a unit line force which satisfies all the prescribed boundary and equilibrium conditions. By using these functions, one can therefore derive the displacement fields and stress distributions near a crack caused by any elastic singularity in the solid which can be represented in terms of a distribution of line forces. Examples of singularities include such defects as dislocations, misfitting inclusions, or externally applied stresses. Some of these applications will be reported in a forthcoming paper.

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APPENDIX A: GREEN'S FUNCTION FOR A COMPOSITE SOLID

The Green's function for a composite solid as derived in Paper I is given below. The displacement field is given by the real part of the Green's function.

(i) <u>**x** and **x**' in UHP ($x_2 \geqslant 0$, $x_2' \geqslant 0$)</u>

$$G^{CA}(\mathbf{x},\mathbf{x}') = -\frac{1}{\pi}\sum_{\alpha=1}^{3}\boldsymbol{\gamma}^A(p_\alpha^A)\ln(z_\alpha^A - z_\alpha'^A) - \frac{1}{\pi}\sum_{\alpha\beta=1}^{3}\boldsymbol{\gamma}^A(p_\alpha^A)\mathbf{Q}_\beta^{\mathrm{I}}\ln(z_\alpha^A - z_\beta'^{A*}) \tag{A.1}$$

(ii) <u>**x** in LHP and **x**' in UHP ($x_2 \leqslant 0$, $x_2' \geqslant 0$)</u>

$$G^{CB}(\mathbf{x},\mathbf{x}') = -\frac{1}{\pi}\sum_{\alpha\beta=1}^{3}\boldsymbol{\gamma}^{*B}(p_\alpha^B)\mathbf{Q}_\beta^{\mathrm{II}}\ln(z_\alpha^{B*} - z_\beta'^{A*}) \tag{A.2}$$

(iii) <u>**x** in UHP, **x**' in LHP ($x_2 \geqslant 0$, $x_2' \leqslant 0$)</u>

$$G^{CA}(\mathbf{x},\mathbf{x}') = \frac{1}{\pi}\sum_{\alpha\beta=1}^{3}\boldsymbol{\gamma}^A(p_\alpha^A)\mathbf{Q}_\beta^{\mathrm{III}}\ln(z_\alpha^A - z_\beta'^B) \tag{A.3}$$

(iv) <u>**x** and **x**' in LHP ($x_2 \leqslant 0$, $x_2' \leqslant 0$)</u>

$$G^{CB}(\mathbf{x},\mathbf{x}') = -\frac{1}{\pi}\sum_{\alpha=1}^{3}\boldsymbol{\gamma}^{*B}(p_\alpha^B)\ln(z_\alpha^{B*} - z_\alpha'^{B*}) + \frac{1}{\pi}\sum_{\alpha\beta=1}^{3}\boldsymbol{\gamma}^{*B}(p_\alpha^B)\mathbf{Q}_\beta^{\mathrm{IV}}\ln(z_\alpha^{B*} - z_\beta'^B) \tag{A.4}$$

where (for superscript A or B)

$$z_{\alpha,\beta} = x_1 + p_{\alpha,\beta}x_2 \tag{A.5}$$

$$z'_{\alpha,\beta} = x_1' + p_{\alpha,\beta}x_2' \tag{A.6}$$

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$$\mathbf{Q}_\beta^{\mathrm{I}} = \mathbf{M}[\boldsymbol{\sigma}^{*A}(p_\beta^A) - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}}\boldsymbol{\gamma}^{*A}(p_\beta^A)] \quad (A.7)$$

$$\mathbf{Q}_\beta^{\mathrm{II}} = \mathbf{N}[\boldsymbol{\sigma}^{*A}(p_\beta^A) - \boldsymbol{\sigma}_s^A\boldsymbol{\gamma}_s^{A^{-1}}\boldsymbol{\gamma}^{*A}(p_\beta^A)] \quad (A.8)$$

$$\mathbf{Q}_\beta^{\mathrm{III}} = \mathbf{M}[\boldsymbol{\sigma}^B(p_\beta^B) - \boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}}\boldsymbol{\gamma}^B(p_\beta^B)] \quad (A.9)$$

$$\mathbf{Q}_\beta^{\mathrm{IV}} = \mathbf{N}[\boldsymbol{\sigma}^B(p_\beta^B) - \boldsymbol{\sigma}_s^A\boldsymbol{\gamma}_s^{A^{-1}}\boldsymbol{\gamma}^B(p_\beta^B)] \quad (A.10)$$

$$\mathbf{M} = \boldsymbol{\gamma}_s^{A^{-1}}[\boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}} - \boldsymbol{\sigma}_s^A\boldsymbol{\gamma}_s^{A^{-1}}]^{-1} \quad (A.11)$$

$$\mathbf{N} = \boldsymbol{\gamma}_s^{*B^{-1}}[\boldsymbol{\sigma}_s^{*B}\boldsymbol{\gamma}_s^{*B^{-1}} - \boldsymbol{\sigma}_s^A\boldsymbol{\gamma}_s^{A^{-1}}]^{-1} \quad (A.12)$$

$$\boldsymbol{\gamma}_s = \sum_\alpha \boldsymbol{\gamma}(p_\alpha) \quad (A.13)$$

$$\boldsymbol{\sigma}_s = \sum_\alpha \boldsymbol{\sigma}(p_\alpha) \quad (A.14)$$

$$\boldsymbol{\sigma}(p_\alpha) = \mathbf{L}(p_\alpha)\boldsymbol{\gamma}(p_\alpha) \quad (A.15)$$

$$L_{ik}(p_\alpha) = c_{i2k1} + p_\alpha c_{i2k2} \quad (A.16)$$

$$\boldsymbol{\gamma}(p_\alpha) = \frac{\iota}{aq_1^4}\frac{\Gamma_{ij}(q_2 = q_1 p_\alpha)}{(p_\alpha - p_\alpha^*)\prod_{\beta\neq\alpha}(p_\alpha - p_\beta)(p_\alpha - p_\beta^*)} \quad (A.17)$$

$$\Gamma_{ij}(\mathbf{q}) = \text{Cofactor of } \boldsymbol{\Lambda}(\mathbf{q})$$

$$\Lambda_{ij}(q) = c_{ikjl}q_k q_l \quad (A.18)$$

q_1, q_2 are components of the wave vector $\mathbf{q}$ and p_α is obtained such that $q_2 = p_\alpha p_1$ is a root of the following equation:

$$\|\boldsymbol{\Lambda}(\mathbf{q})\| = 0 \quad (A.19)$$

a is the coefficient of q_2^6 in $\|\Lambda(q)\|$ and $\|\Lambda\|$ denotes the determinant of the matrix Λ.

The index α takes values 1 to 3 such that the imaginary part of p_α is positive.

Certain useful relations among the various parameters of the Green's function as defined above are given below:

$$\boldsymbol{\gamma}_s^{*B}\mathbf{Q}_\alpha^{\mathrm{II}} - \boldsymbol{\gamma}_s^A\mathbf{Q}_\alpha^{\mathrm{I}} = \boldsymbol{\gamma}^{*A}(p_\alpha^A) \quad (A.20)$$

$$\boldsymbol{\sigma}^A(p_\alpha^A) = \boldsymbol{\sigma}_s^A(\boldsymbol{\gamma}_s^A)^{-1}\boldsymbol{\gamma}^A(p_\alpha^A) \quad (A.21)$$

$$\boldsymbol{\sigma}_s^{*B}\mathbf{Q}_\alpha^{\mathrm{II}} - \boldsymbol{\sigma}_s^A\mathbf{Q}_\alpha^{\mathrm{I}} = \boldsymbol{\sigma}^{*A}(p_\alpha^A) \quad (A.22)$$

$$\boldsymbol{\gamma}_s^A\mathbf{Q}_\alpha^{\mathrm{III}} - \boldsymbol{\gamma}_s^{*B}\mathbf{Q}_\alpha^{\mathrm{IV}} = -\boldsymbol{\gamma}^B(p_\alpha^B) \quad (A.23)$$

$$\boldsymbol{\sigma}_s^A\mathbf{Q}_\alpha^{\mathrm{III}} - \boldsymbol{\sigma}_s^{*B}\mathbf{Q}_\alpha^{\mathrm{IV}} = -\boldsymbol{\sigma}^B(p_\alpha^B) \quad (A.24)$$

$$\mathbf{Q}_s^{\mathrm{II}} + \mathbf{Q}_s^{\mathrm{IV}} = -(Q_s^{\mathrm{I}} + Q_s^{\mathrm{III}}) = \mathbf{I} \quad (A.25)$$

$$\mathbf{Q}_s^{\mathrm{I}} = -\mathbf{I} - \iota\boldsymbol{M} \quad (A.26)$$

$$\mathbf{Q}_s^{\mathrm{IV}} = \mathbf{I} - \iota\mathbf{N} \quad (A.27)$$

$$\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_s^* = \iota\mathbf{I} \quad (A.28)$$

APPENDIX B: EVALUATION OF INTEGRALS

In this appendix we evaluate certain integrals required for the formulae given in Sec. II and Sec. III. For the evaluation of all these integrals, we choose the contour[16,17] as shown in Fig. 2. This contour consists of a large outer circle of radius R and an inner rectangle ABDE which encloses the real axis between $x_1 = -c$ and c.

First we consider the following integral given in Eq. (45):

$$\mathbf{I}(\rho) = \int_{-c}^{c}\frac{\mathbf{R}_\lambda^{-1}(t)}{Y_0(t)}\frac{dt}{(t - x_1)(t - \rho)} \quad (B.1)$$

where ρ is complex and x_1 is on the real axis lying between $-c$ and c. In order to evaluate ρ, we replace t by the complex variable z and consider the following integral over the contour given in Fig. 2.

$$\mathbf{I}_c = \oint \mathbf{I}'(z)\,dz \quad (B.2)$$

where

$$\mathbf{I}'(z) = \frac{\mathbf{J}_\lambda^{-1}(z)}{Y(z)}\frac{1}{(z - x_1)(z - \rho)} \quad (B.3)$$

The only pole enclosed by the contour is the one at $z = \rho$. Hence, by Cauchy's theorem

$$\mathbf{I}_c = 2\pi\iota\frac{\mathbf{J}_\lambda^{-1}(\rho)}{y(\rho)}\frac{1}{\rho - x_1} \quad (B.4)$$

The integral over the outer circle, in the limit $\mathbf{R} \to \infty$, is $2\pi\iota$. Using Eqs. (35)–(38) for the values of $\mathbf{I}'(z)$ over AB and DE, we obtain

$$\mathbf{I}(\rho) = 2\pi[\exp(-\pi\boldsymbol{\lambda}) + \exp(\pi\boldsymbol{\lambda})]^{-1}\left[\frac{\mathbf{J}_\lambda^{-1}(\rho)}{Y(\rho)}\frac{1}{\rho - x_1} - \mathrm{I}\right] \quad (B.5)$$

where I on the RHS is the unit matrix.

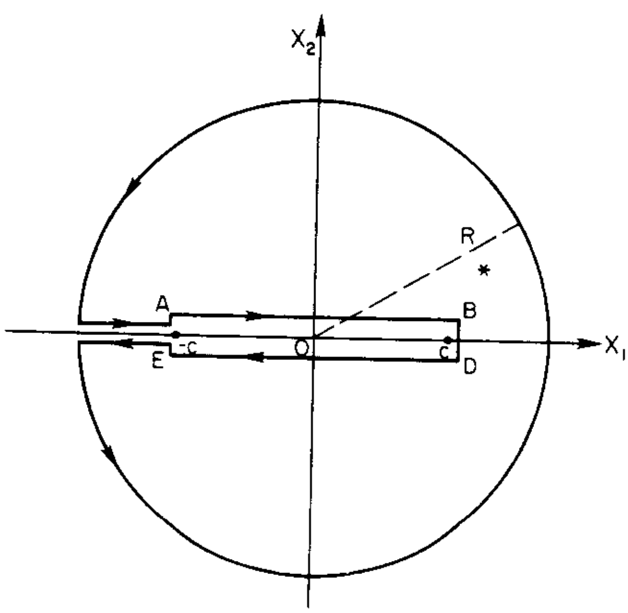

FIG. 2. The contour used for integration in Appendix B. The cut is on the x axis from $-c$ to c. The star denotes the position of a pole.

In a similar manner, we can determine the integrals $\mathbf{I}'_\lambda$ and $\mathbf{K}_\lambda$ as defined by Eqs. (61) and (62), respectively. In these cases the integral over the outer circle vanishes in the limit of large R. Thus we find:

$$\mathbf{I}'_\lambda(z,\rho) = [\exp(-\pi\lambda) + \exp(\pi\lambda)]^{-1} \cdot \left[\frac{\mathbf{J}_\lambda(\rho)Y(\rho)}{z-\rho} - \frac{\mathbf{J}_\lambda(z)Y(z)}{z-\rho}\right] \tag{B.6}$$

and

$$\mathbf{K}'_\lambda(z) = [\exp(-\pi\lambda) + \exp(\pi\lambda)]^{-1}\mathbf{J}_\lambda(z)Y(z) \tag{B.7}$$

Now we consider the integrals I_λ and K_λ as given in Eqs. (55) and (56), respectively. These integrals can be obtained by integrating Eqs. (B.6) and (B.7). Alternatively, we can obtain $\mathbf{I}_\lambda$ by contour integration. For this purpose we use the same contour as given by Sinclair and Hirth,[14] which accounts for the logarithmic term in the integrand. The final result, after neglect of the singular terms which only contribute a rigid displacement, is

$$\begin{aligned}\mathbf{I}'_\lambda(z,\rho) = {} & [\exp(-\pi\lambda) + \exp(\pi\lambda)] \\ & \cdot \Bigg[\mathbf{J}_\lambda(\rho)Y(\rho)\ln(z-\rho) + \frac{1}{\sqrt{\rho^2 - c^2}}\frac{1}{b^p} \\ & \cdot \Bigg\{\ln\frac{1-b}{1-bg} - p[(1-b) - (1-bg)] \\ & + \frac{p(p-1)}{2} \cdot \frac{[(1-b)^2 - (1-bg)^2]}{2} - \cdots\Bigg\} \\ & - \frac{1}{\sqrt{\rho^2 - c^2}}\frac{1}{(-b)^p} \\ & \cdot \Bigg\{\ln\frac{1+b}{1+bg} - p[(1+b) - (1+bg)] \\ & + \frac{p(p-1)}{2} \cdot \frac{[(1+b)^2 - (1+bg)^2]}{2} - \cdots\Bigg\}\Bigg]\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}\mathbf{K}_\lambda(z) = {} & [\exp(-\pi\lambda) + \exp(\pi\lambda)]^{-1} \\ & \cdot \Bigg[\frac{1}{(-1)^p}\Bigg\{\ln(1+g) - p(1-g) \\ & + \frac{p(p-1)}{2}\frac{(1+g)^2}{2} - \cdots\Bigg\} \\ & - \Bigg\{\ln(1-g) - p(1-g) \\ & + \frac{p(p-1)}{2}\frac{(1-g)^2}{2} - \cdots\Bigg\}\Bigg]\end{aligned} \tag{B.9}$$

where

$$p = 2\iota\lambda \tag{B.10}$$

$$b^2 = \frac{\rho - c}{\rho + c} \tag{B.11}$$

and

$$g^2 = \frac{z + c}{z - c} \tag{B.12}$$

The series given in Eqs. (B.8) and (B.9) mainly serve the purpose of showing the qualitative nature of the Green's functions. For computational purposes, it may be more convenient to obtain I_λ and K_λ directly by numerical integration. However, as remarked in the text, in most cases of practical interest, one needs the stress Green's functions for which analytical results in closed form have been given in the text.

In the special case that $\lambda = 0$, the expressions for I_λ and K_λ are considerably simplified. This case arises, for example, for homogeneous solids or for antiplane strain in composite solids. In this case we obtain the following result from Eqs. (B.8) and (B.9).

For $\lambda = 0$

$$I_\lambda(z,\rho) = \frac{1}{2\sqrt{\rho^2 - c^2}} \cdot \ln\left[\frac{\rho z - c^2 + (\rho^2 - c^2)^{1/2}(z^2 - c^2)^{1/2}}{\rho + (\rho^2 - c^2)^{1/2}}\right] \tag{B.13}$$

$$K_\lambda(z) = \frac{1}{2}\ln[z + (z^2 - c^2)^{1/2}] \tag{B.14}$$

These results, as expected, agree with those obtained by Sinclair and Hirth.[14]

Finally, we evaluate the integral on the RHS of Eq. (73). This integral can be written as follows:

$$\int_{-c}^{c} [\mathbf{V}(z,t_+) - \mathbf{V}(z,t_-)]dt = \mathbf{I}_{V1} - \mathbf{I}_{V2} \tag{B.15}$$

where

$$I_{V1} = \lim_{\eta\to 0}\int_{-c}^{c} \frac{\Delta(\rho_+)}{\rho + \iota\eta - z}dt \tag{B.16}$$

$$I_{V2} = \lim_{\eta\to 0}\int_{-c}^{c} \frac{\Delta(\rho_-)}{\rho - \iota\eta - z}dt \tag{B.17}$$

$$\Delta(\rho) = J_\lambda^{-1}(\rho)Y(\rho) \tag{B.18}$$

and t is the real part of ρ.

To evaluate I_{V1}, we consider the following integral over the contour given in Fig. 2.

$$I_{CV} = \oint \frac{\Delta(\rho)}{\rho - (z - \iota\eta)} \tag{B.19}$$

The only pole enclosed by the contour is at $\rho = z - \iota\eta$. The value of the integrand in the LHP, as obtained from

Eqs. (35)–(38) is $-\exp(-2\pi\boldsymbol{\lambda})$ times that in the UHP. The integral over the outer circle in the limit $R \to \infty$ is given by

$$\int_{R\to\infty} \frac{\boldsymbol{\Delta}(\rho)\, d\rho}{\rho - z} = 2\pi\iota(z\mathbf{I} - 2\iota c\boldsymbol{\lambda}) \tag{B.20}$$

where $\mathbf{I}$ is the unit matrix.

Thus we obtain

$$\mathbf{I}_{V^1} = 2\pi\iota[\mathbf{I} + \exp(-2\pi\boldsymbol{\lambda})]^{-1} \cdot [\lim_{\eta\to 0} \boldsymbol{\Delta}(z - \iota\eta) - (z\mathbf{I} - 2\iota c\boldsymbol{\lambda})] \tag{B.21}$$

In a similar manner we can evaluate I_{V^2}. The result is

$$\mathbf{I}_{V^2} = -2\pi\iota[\mathbf{I} + \exp(2\pi\boldsymbol{\lambda})]^{-1} \cdot [\lim_{\eta\to 0} \boldsymbol{\Delta}(z + \iota\eta) - (z\mathbf{I} - 2\iota c\boldsymbol{\lambda})] \tag{B.22}$$

APPENDIX C: STRESS DISTRIBUTION IN THE ISOTROPIC LIMIT

In this appendix we apply the formulae of Sec. IV to a crack at the interface of two isotropic solids. This would provide an illustrative example for the use of the formulae in the plane strain problem as well as their verification by comparison with the results derived by earlier authors.

In the isotropic case the roots p_α of Eq. (A.19) are degenerate. Consequently, the matrices $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}$ become singular [see Eqs. (73)–(79) of Paper I]. However, $\boldsymbol{\gamma}_s$ and $\boldsymbol{\sigma}_s$ tend to finite limits as the anisotropy parameter [Eq. (64) of Paper I] goes to zero. The limiting values of $\boldsymbol{\gamma}_s$ and $\boldsymbol{\sigma}_s$ are given below:

$$\boldsymbol{\gamma}_s = \frac{1+\zeta}{c_{44}\zeta}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{C.1}$$

$$\boldsymbol{\sigma}_s = \frac{\iota}{2}\begin{pmatrix}1 & -\iota/\zeta\\ \iota/\zeta & 1\end{pmatrix} \tag{C.2}$$

where, as defined in Eq. (70) of Paper I,

$$\zeta = c_{11}/c_{44}$$

In the above equations we have omitted the superscripts A or B for notational brevity. These superscripts, when inserted, identify the appropriate parameter for the solid A (UHP) or B (LHP), as in Paper I.

With the use of Eqs. (C.1) and (C.2), all other matrices, viz., $\mathbf{M}$, $\mathbf{N}$, Q_s, etc., can be easily calculated with the help of the various formulae given in Appendix A and the text. All of these matrices can be diagonalized by the transformation $\mathbf{E}$ where $\mathbf{E}$ is the matrix of orthonormal eigenvectors as defined below:

$$\mathbf{E} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \iota\\ \iota & 1\end{pmatrix} \tag{C.3}$$

For example, we see from Eqs. (C.2) and (C.3)

$$\mathbf{E}^*\boldsymbol{\sigma}_s\mathbf{E} = \boldsymbol{\sigma}_d \tag{C.4}$$

where

$$\boldsymbol{\sigma}_d = \begin{pmatrix}1 + 1/\zeta & 0\\ 0 & 1 - 1/\zeta\end{pmatrix} \tag{C.5}$$

and $\mathbf{E}^*$ is the Hermitian conjugate of $\mathbf{E}$ which obeys the orthonormality condition.

$$\mathbf{E}^*\mathbf{E} = \mathbf{I} \tag{C.6}$$

$\mathbf{I}$ being the unit matrix.

In a similar manner we can diagonalize the other matrices using the transformation $\mathbf{E}$. After some algebraic manipulation we obtain

$$\mathbf{E}^*\mathbf{T}_U\mathbf{E} = \frac{2t}{1+t}\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix} \tag{C.7}$$

and

$$\mathbf{E}^*\mathbf{T}_U^*\mathbf{E} = \frac{2t}{1+t}\begin{pmatrix}t_2 & 0\\ 0 & t_1\end{pmatrix} \tag{C.8}$$

where

$$t_1 = \frac{1 - 1/\zeta^B}{1 - 1/\zeta^A}\cdot\frac{1}{1-\mu} \tag{C.9}$$

$$t_2 = \frac{1 + 1/\zeta^B}{1 + 1/\zeta^A}\cdot\frac{1}{1+\mu} \tag{C.10}$$

$$\mu = \frac{1}{1+t}\left[\frac{t}{\zeta^B} - \frac{1}{\zeta^A}\right] \tag{C.11}$$

$$t = \frac{\beta(1 + 1/\zeta^A)}{1 + 1/\zeta^B} \tag{C.12}$$

and

$$\beta = c_{44}^B/c_{44}^A$$

The same transformation also diagonalizes the matrices $\exp \pm 2\pi\boldsymbol{\lambda}$, $J_\lambda(z)$, etc., as defined in the text. From Eq. (33) we obtain the following result for the eigenvalues of $\boldsymbol{\lambda}$:

$$\lambda_1 = -\lambda_2 = -\frac{1}{2\pi}\ln t_1/t_2 = -\frac{1}{2\pi}\ln\frac{1 + \beta(1 + 1/\zeta^A)/(1 - 1/\zeta^A)}{\beta + (1 + 1/\zeta^B)/(1 - 1/\zeta^B)} \tag{C.13}$$

Equation (C.13) agrees with the value of the bielastic constant as derived by England[10] and Rice and Sih[11] for the plane strain problem.

We now operate on both sides of Eq. (72) by the operator E^* which gives

$$\mathbf{T}'(\mathbf{x}) = \int_{-c}^{c} \mathbf{S}^d(\mathbf{x}, \mathbf{x}')\, dx_1'\, \mathbf{f}_0' \tag{C.14}$$

where

$$\mathbf{T}'(\mathbf{x}) = E^*\mathbf{T}(\mathbf{x}) \tag{C.15}$$

$$\mathbf{f}_0' = \mathbf{E}^*\mathbf{f}_0 \tag{C.16}$$

and

$$\mathbf{S}^d = \mathbf{E}^*[\mathbf{S}^{\mathrm{I}} - \mathbf{S}^{\mathrm{II}}]\mathbf{E} \tag{C.17}$$

As mentioned before, the matrix $\mathbf{S}^d$ is diagonal. Using the diagonal elements, one can treat the matrix equations of Sec. IV as scalar equations. Taking the first diagonal element of each matrix, we obtain the following relation from Eq. (76) by using Eqs. (C.14)–(C.17):

$$\mathrm{T}_1(x_1) - \iota \mathrm{T}_2(x_1) = -[1 - (x_1^2 - c^2)^{-1/2} \cdot \{x_1 \cos\theta + 2c\lambda_1 \sin\theta + \iota(x_1 \sin\theta - 2c\lambda_1 \cos\theta)\}] \cdot (f_{01} - \iota f_{02}) \quad \text{(C.18)}$$

where

$$\theta = \lambda_1 \ln \frac{x_1 + c}{x_1 - c} \quad \text{(C.19)}$$

The components f_{01} and f_{02} of the vector $\mathbf{f}_0$ can be identified as the applied shear and the normal load, respectively. Similarly, T_1 and T_2, being the components of the stress T_{i2} for $i = 1$ and 2, respectively, can be identified with σ_{xy} and σ_{yy}, respectively, in the notation of England.[10] The separation of the real and the imaginary parts of Eq. (C.18) would lead to Eq. (27) of England.[10] Proceeding further, we see that the stress intensity factor derived from Eq. (C.18) using Eq. (77) gives the result obtained by Rice and Sih.[11]