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Together with the structure interactions and the spatio-temporal chaos properties the wave energy concentration processes have traditionally attracted attention in the nonlinear dynamics investigations. In particular, this attention has passed into the theory of stationary self-focusing and Langmuir wave collapses[1-3], which have become classical examples of nonlinear physics. In any field of knowledge, the information gaining by solution of direct problems is accompanied by the formulation of inverse problems that require a certain level of generalization. For example, one of the most important problems in the physics of nonlinear processes is to determine the properties of a nonlinear medium that could ensure the maximum possible localization of the wave field energy*). Besides the obvious applications, the interest in this problem is due to the fact that in most systems investigated to date upper limits exist on the energy value involved and concentrated by the wave field bunch, and the nonlinear systems demonstrate "flexibility" resisting the localization of an arbitrary given portion of wave energy. This resistance is manifested, as a rule, in the structural instability of wave collapses.

There are several universal types of the behavior of nonlinear systems with collapses, which can conveniently be classified by the character of wave energy trapped into the singularity domain. Figure 1 shows schematically scenarios of the evolution of collapsing field bunches, which we shall call strong, weak, fractal, distributed and complete collapses.

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*) By the energy we mean a conventional energy characteristic the physical meaning of which depends on particular formulation of the problem. This can be the number of quanta involved into singularity, the energy flow localized in the beam cross-section, and so on.

Historically, the case of strong collapse, which occurs at stationary self-focusing of electromagnetic wave beam in media with local cubic nonlinearity, was first studied [3]. In this case, the initial (boundary) field distribution is divided into several secondary collapsing bunches (beams), with a definite, so-called critical portion of energy W_{cr} (critical power of self-focusing) involved into each bunch. Symbolically, this process can be represented as: $W_0 = W_{cr} \times N$, where W_0 is the initial value of the energy parameter and N is the finite number of secondary bunches (beams).

In the case of weak collapses [4,5] the initial distribution is divided, during its evolution, into an infinite number of secondary bunches, with "hollow" field singularity arising in each bunch: a secondary bunch loses the whole of its acquired energy when approaching the singularity. With small-scale dissipation in bunches taken into account, a finite portion of energy $\widetilde{W}$ is absorbed, but with decreasing spatial scale of damping (or increasing threshold field amplitude beginning from which the absorption becomes essential) the locally dissipated energy decreases. Such a behavior makes this case radically different from the case of strong collapse. As a result of its evolution, the initial field distribution undergoes multiple splitting $W_0 = \widetilde{W} \times N$, $\widetilde{W} \to 0$, $N \to \infty$.

Close in result of evolution (but not in dynamic behavior) is the fractal collapse process [6], where each small-scale bunch is, in its turn, unstable and splits into still smaller-scale structures when collapsing. The symbolic representation of this process is analogous to the previous one.

The idea of distributed collapse [7] is that when a "weak" (hollow) singularity is formed, the collapsing pqqK energy does not escape into the background distribution of the wave field but begins to leak into the singularity (the black hole image), so that a finite portion of energy dissipates over a finite time Δt: $W_0 = \int_{\Delta t} \widetilde{W}\, dt$.

A convenient mathematical model for the illustration of the above types of evolution is the nonlinear Schrödinger equation:

$$-i\frac{\partial \psi}{\partial t} + \sum_{i=1}^{d} \alpha_i \frac{\partial^2 \psi}{\partial x_i^2} + |\psi|^s \psi = 0 \qquad (1)$$

In two-dimensional geometry ($d = 2$), at the coefficient values $\alpha_i = 1$ and $S = 2$ Eq. (1) describes a strong collapse with critical (W_{cr}) energy $W_{cr} = \int |\psi|^2 dx_1 dx_2$ involved into each singularity. In a similar three-dimensional situation ($d = 3$) weak collapses take place. A possibility of distributed collapse is also predicted for this case. The fractal collapse process develops in the case of opposite signs of the dispersion coefficients $\alpha_1 = \alpha_2 = 1$; $\alpha_3 = -1$ when the elementary act of evolution consists in two-dimensional compression (along X_1 and X_2) and longitudinal bunch splitting (along X_3).

Thus, in all the examples given above it is not possible to localize arbitrary large energy into the singularity, i.e., to realize its maximum concentration. In what systems a complete collapse is possible then? This is exactly a short reformulation of the inverse problem mentioned above.

An answer (possibly not a single one) to this question is as follows: trapping of an arbitrary portion of energy into singularity occurs in media with nonlinearity inertia. As an illustration, we shall consider a simple example with relaxation nonlinearity obtained from (1) through a modification:

$$i\frac{\partial\psi}{\partial t} + \sum_{i=1}^{2}\frac{\partial^2\psi}{\partial x_i^2} - n\psi = 0 \tag{2.1}$$

$$\frac{\partial n}{\partial t} + n = -|\psi|^2 \tag{2.2}$$

Unlike the case of local coupling ($\frac{\partial}{\partial t} = 0$ in (2.2)) the nonlinear parameter n of the medium (call it for definiteness the matter density perturbation) during the collapse of the distribution $|\psi|$ is too slow to reach the values corresponding to a similar field amplitude at inertialess nonlinearity. The higher the collapse rate the stronger is the retardation of the density perturbations from the locally nonlinear ones, and therefore the greater field amplitude in the bunch is needed to achieve the former level of matter perturbation. It is easy to see that such distributions lead to dependence of the collapse rate on the energy trapped into singularity $\int|\psi|^2 d\vec{x} = W$. On the contrary, for each value of stored energy there is a collapse rate ensuring the trapping of the whole portion of energy. Within the framework of Eqs. (2.1)-(2.2), a complete collapse is described by the following approximate self-similar solutions [8] :

$$\psi = e^{pt}\, u(\vec{r}_\perp e^{pt})\, \exp(i\varphi(t) r^2)$$

$$n = e^{2pt}\, N(\vec{r}_\perp e^{pt}).$$

At $p \to c$, $W \to W_{cr}$; if $p \to \infty$ W increases as p. The structure of a self-similar solution for the case of strongly nonlinear coupling ($p = 100$) is represented in Fig. 2.

Like in the case of local nonlinearity, system (2.1)-(2.2), evolving in the course of time, has its own spatial analog. With a substitution $t \to z$ Eq. (2.1) describes the transverse structure variation of the wave field along the quasioptical beam propagation direction. Equation (2.2) takes into account the spatial nonlocality ("inertia") of the nonlocal response of the medium in the longitudinal coordinate. Such a situation can take place, for example, when intense waves propagate in a medium with a stationary flow of matter (beam self-focusing at longitudinal wind). Evidently, the spatio-temporal analogy is complete with eact transformation of the initial conditions of one problem into the boundary conditions of another problem.

This analogy suggests an idea that the conceptions of complete collapse can be generalized to a wider range of wave systems with bievolution behavior. By the term bievolution we mean the unidirectional process along the temporal and one of the spatial coordinates, i.e., the simultaneous fulfilment of the temporal causality principle and the reflectionless spatial propagation condition for the wave field. The simplest generalization in the wave description is the use of the equation:

$$-i\frac{1}{V_{gr}}\frac{\partial\psi}{\partial t} - i\frac{\partial\psi}{\partial z} + \Delta_\perp\psi - n\psi = 0 \tag{3}$$

for the field complex amplitude, where V_{gr} is the wave group velocity. The medium density (n) perturbations can be found from relations evolving in only one of the independent variables. We shall consider the case of material coupling, inertial in time and local along the wave propagation direction z.

The following typical examples of material coupling can be used:

$$\frac{\partial n}{\partial t} = \nu\left(|\psi|^2\right) n \tag{4}$$

This is the case of ionization nonlinearity. The effects of interest [9] arise with decreasing dependence of the medium ionization frequency on the field amplitude. This occurs, for example, in the presence of superstrong electromagnetic fields in gases when the oscillatory energy of free electrons exceeds noticeably the molecule ionization potential [9]. Without loss of generality, we can assume $\nu \sim 1/|\psi|$:

$$\frac{\partial n}{\partial t} = -n + F\left(|\psi|^2\right) \tag{5}$$

This example of local coupling with respect to all spatial variables simulates the simplest type of nonlinearity relaxation of the medium. It was mentioned when discussing the purely temporal evolution of two-dimensional systems with cubic nonlinearity.

$$\frac{\partial n}{\partial t} = \Delta_\perp n + \Delta_\perp F\left(|\psi|^2\right) \tag{6}$$

At diffusion relaxation the density perturbation onset velocity depends on the beam width. Such a nonlinearity is typical of the medium heating in the field of a powerful electromagnetic wave and is realized, for example, in a weakly ionized collisionless plasma.

$$\frac{\partial^2 n}{\partial t^2} = \Delta_\perp n + \Delta_\perp F\left(|\psi|^2\right) \tag{7}$$

This type of coupling is due to the excitation of sound motions in the medium and is characteristic of a wide class of nonlinear processes, for example, in the case of laser radiation self-action in rarefied coronal plasma. In the cases (4)-(7) an introduction of an arbitrary function F enables one to describe various situations, from the simplest dependence $F = |\psi|^2$ to the nonlinearity saturation effects. Note that the transition to new time $\tau = t - z/v_{gz}$ will not change the material coupling structure but simplifies Eq. (3) by excluding the term with a temporal derivative:

$$-i\frac{\partial \psi}{\partial z} + \Delta_\perp \psi - n\psi = 0 \tag{8}$$

The boundary condition at z = 0 retains, evidently, its form while the initial conditions must be set exactly when the pulse operated at the boundary at t = 0 reaches a given point z. Since the nonlinearity is inertial in time, for an unperturbed initial state of the medium the initial (with respect to τ) conditions correspond to a stationary diffractional field pattern in the linear problem with n = 0.

The main idea of what follows is that complete concentration of the energy flow in the beam cross-section is possible in the class of nonlinear systems (4)-(7) with inertial coupling of the medium density perturbations and the wave field amplitude. This can be demonstrated by a scheme based on the search for self-similar solutions with an arbitrary energy flow entrained into singularity, the analysis of stability of these solutions and the numerical illustration of the spatio-temporal dynamics of the wave.

Note, first of all, that a common feature of all the types of inertial nonlinearity being discussed is the class of solutions in the form of homogeneous (in z), collapsing jets along which the trapped electromagnetic wave propagates. The Pointing vector flow is constant along the jet and depends, generally speaking, on the transverse structure of the mode and the collapse rate. The time of singularity formation on a finite-dimension set (either a straight line or a plane depending on the transverse form of the beam) is determined by the type of nonlinearity. Let us formulate some regularities of the jet scale decrease at the self-similar stage of collapse: $a \sim 1/t^2$ for a plane beam in a medium with the ionization defined by the effective frequency $\nu_i \sim \frac{1}{|\psi|}$; $a \sim e^{\rho t} (\rho > 0)$ for the case of local relaxation, cubic nonlinearity and three-dimensional beam. In the analogous case for diffusion relaxation the blow-up behavior takes place: $a \sim \frac{1}{\sqrt{t_0 - t}}$, as well as for sound relaxation: $a \sim \frac{1}{t_c - t}$. For each of these laws it is possible to find a mode localized in the beam cross-section.

Nevertheless, the existence of appropriate self-similarities does not mean that the corresponding solutions will necessarily be realized at arbitrary initial and boundary conditions of the problem For example, if a permanent source is given at the input to the nonlinear medium, then the question arises whether it is possible to match the stationary field distribution at $z = 0$ with the dynamic jet (say, at $z \to \infty$). In a not so general by qualitatively analogous formulation the question is as follows: Is the solution in the form of collapsing jets attracting sets in the class of z-inhomogeneous structures? The simplest analysis under the assumption of given (self-similar) transverse form of the jet, with the width being variable in z. It appears that homogeneous jet distributions with an infinite time of singularity formation ((4)-(5)) are stable manifolds attracting beams, from the close vicinity at least, with the energy flow retained. In a similar analysis, jets with blow-up singularity formation (6), (7) appear to be unstable and, therefore, cannot serve as solutions with complete energy flow concentration in the beam cross-section.

The existence of longitudinal instability of jets with blow-up behavior makes one seek for z-inhomogeneous self-similar solutions that concentrate the energy flow at separate spatial points (which, generally speaking, move in the course of time). Let us discuss in more detail the finding of such solutions in a medium with diffusion relaxation of nonlinearity $F(|\psi|^2) = |\psi|^2$.

Note, first of all, the general assertion that the structurally stable mode of collapse in a reference frame collapsing together with the field distribution and renorming its amplitude must correspond to a stable stationary localized structure. We now apply the inhomogeneous compression transformation to system (7)-(8):

$$\psi = \frac{1}{a(\eta)} \bar{\Psi}\left(\frac{\vec{r}_\perp}{a}, \eta, t\right) \exp\left(-i\, \frac{a_\eta}{4a} \vec{r}_\perp^{\,2}\right),$$

$$n = \frac{1}{a^2} N\left(\frac{\vec{r}_\perp}{a}, \eta, t\right); \quad \eta = z + \int v(t)\,dt, \quad \vec{\xi} = \frac{\vec{r}_\perp}{a} \tag{9}$$

where $v(t)$ is the arbitrary velocity at which the singularity propagates towards the incident radiation. For the self-similar functions $\bar{\Psi}$ and N we have:

$$i a^2 \frac{\partial \bar{\Psi}}{\partial \eta} = \Delta_{\vec{\xi}} \bar{\Psi} - N\bar{\Psi} - \frac{a^3 a_{\eta\eta}}{4} \xi^2 \bar{\Psi}, \tag{10}$$

$$a^2(N_t + vN_\eta) - v a a_\eta (\vec{\xi}\, \nabla_{\vec{\xi}} N + 2N) - \Delta_{\vec{\xi}} N = \Delta_{\vec{\xi}} |\Psi|^2 \tag{11}$$

Evidently, the presence of the last (lens) term in Eq. (10) excludes the existence of strictly localized modes (except for the case $a_{\eta\eta} = 0$ but then a complete self-similarity is not reached in (11). It is possible, however, taking into account the known stabilising properties of the lens term, to construct quasilocalized solutions with exponentially weak leakage of RF field quanta from the mode to the surrounding background. In these solutions

$$a = \sqrt{\eta_0 - \eta}, \quad \bar{\Psi} = u(\vec{\xi}) \exp\left[-i\Gamma(t)\int \frac{d\eta}{a^2}\right],$$
$$N = N(\vec{\xi})$$

and the transverse structure is completely analogous to the distribution in the homogeneous dynamic jet and can be found from the system:

$$\Delta_{\vec{\xi}} u - \Gamma u - Nu = 0$$
$$\frac{v}{2}\left(\vec{\xi}\, \nabla_{\vec{\xi}} N + 2N\right) - \Delta_{\vec{\xi}} N = \Delta_{\vec{\xi}} u^2 \tag{12}$$

To illustrate this, Fig. 3 shows the main axisymmetric mode for strongly nonlocal nonlinearity (the high rate of collapse v/Γ). Note that in such a situation the energy flow concentrated into singularity and the mode structure depend on the parameter v/Γ. The supercritical (as compared to stationary nonlinearity) localization requires the relations $v \gg \Gamma \gg 1$ to be satisfied. If the singularity propagates at a velocity $v \sim 1$, then the energy flow into it is close to a critical one.

A similar consideration is possible in the case of sound relaxation of nonlinearity. Unlike the previous case, the field structure near the singularity is characterized by a conical form: $a = \eta_0 - \eta$, and the corresponding mode appears to be strictly localized because of the absence of the lens term for this type of self-similarity. As previously, an increase in collapse rate $p \sim v/\Gamma$ leads to an increase in energy flow into the singularity propagating towards the radiation source.

Thus, the dynamic pattern of wave energy concentration can be radically different from the known self-focusing processes in media with stationary nonlinearity. In this context, it should be reasonable to ascertain whether the correpondence between the dynamic and the stationary regime of self-action is retained during the long-term evolution of the electromagnetic wave emitted by a given source at the boundary with the medium. Evidently, the energy concentration peculiarities are fully manifested in the case of sharp switching on the source for the time less than the field linear relaxation time in the medium. Exactly at this stage complete concentration of energy is possible with the appearance of singularities such as jets or moving foci. We should emphasize that under such conditions, supercritical localization of the energy flow in the wave beam is possible. At times much larger than the linear relaxation duration the system must inevitably evolve to the stationary limit. It should be interesting to consider a situation with a transition from complete dynamic concentration of the wave beam to the regime of its multiple splitting in the stationary case with supercritical power in the wave cross-section. This process was simulated in the case of self-action of a two-dimensional wave beam $\Delta_\perp = \frac{\partial^2}{\partial x^2}$ using the local relaxation of nonlinearity to a level determined by its saturation $F(|\psi|^2) = |\psi|^4/(1+\alpha|\psi|^4)$. At the initial dynamic stage (see Fig. 4) (the source was switched on instantaneously and then maintained at a fixed level) there was the formation of an exponentially compressing plane jet entraining the bulk of the electromagnetic energy flow passed through the medium. The maximum level of the field reached $|\psi| \sim 1/\alpha^{1/4}$ and then the interaction passed into a quasistationary regime with an almost unchanging form of the jet and slow propagation of its start towards the source. Only at times much larger than the linear relaxation time, the structural instability of the jet developed, which led to the multiple splitting of the field distribution and the turbulization of the interaction region. The completely stationary interaction pattern with the filamentational structure was established at the expense of the displacement of the dynamic turbulence region towards larger z.

To summarize, we stress one again that the use of media with inertial nonlinearity is attractive as a possible way to reach, in dynamic regimes, high levels of wave energy concentration, exceedin the corresponding values for stationary action of radiation on matter.

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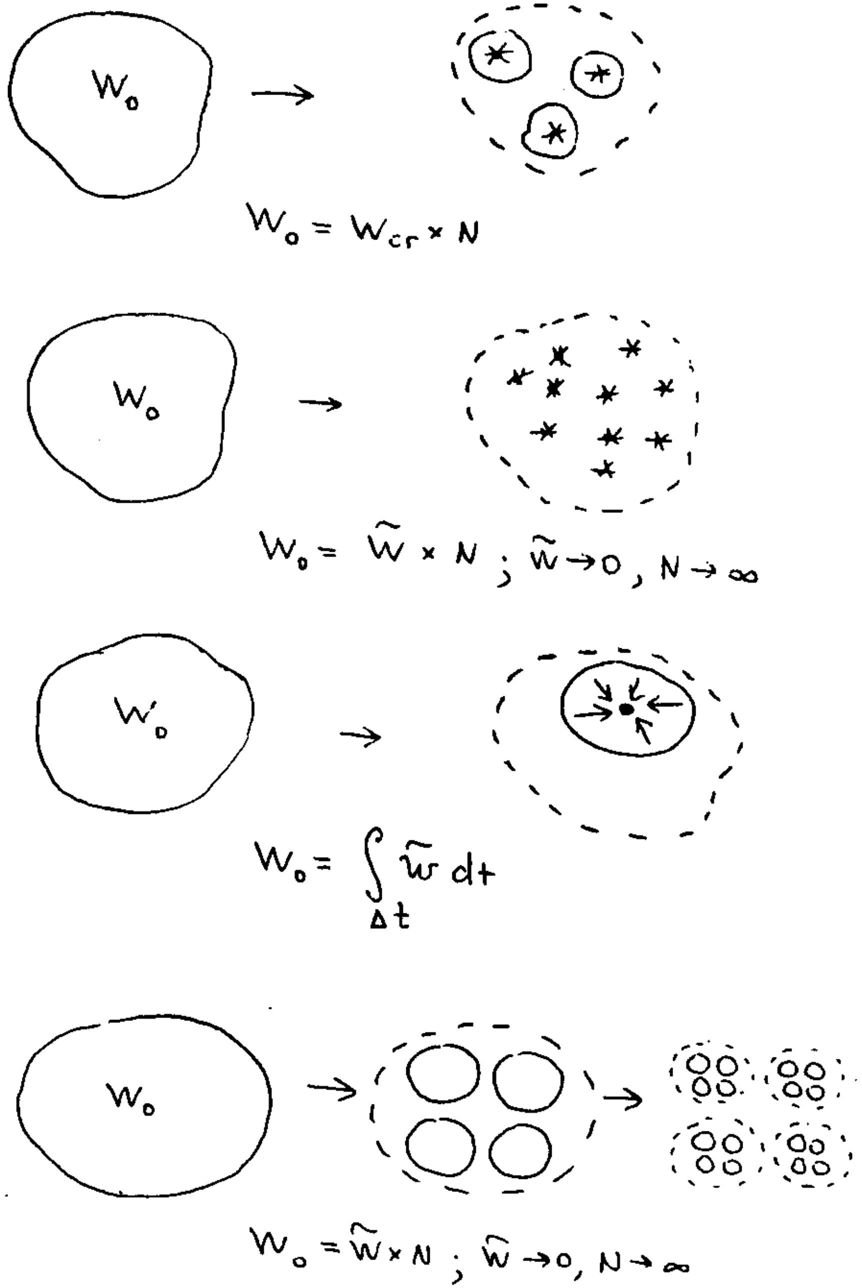


Fig. 1

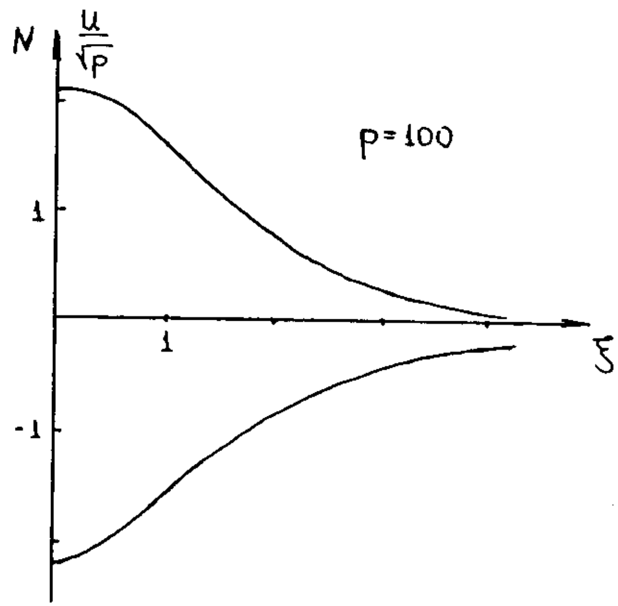


Fig. 2

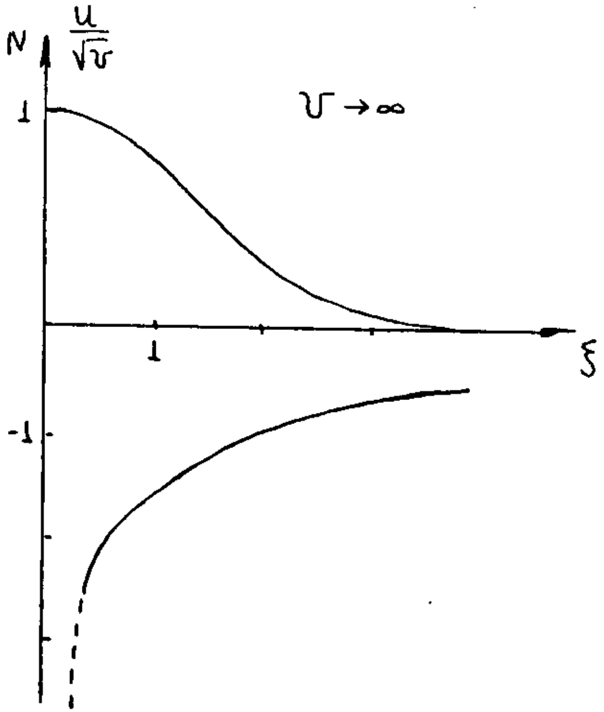


Fig 3

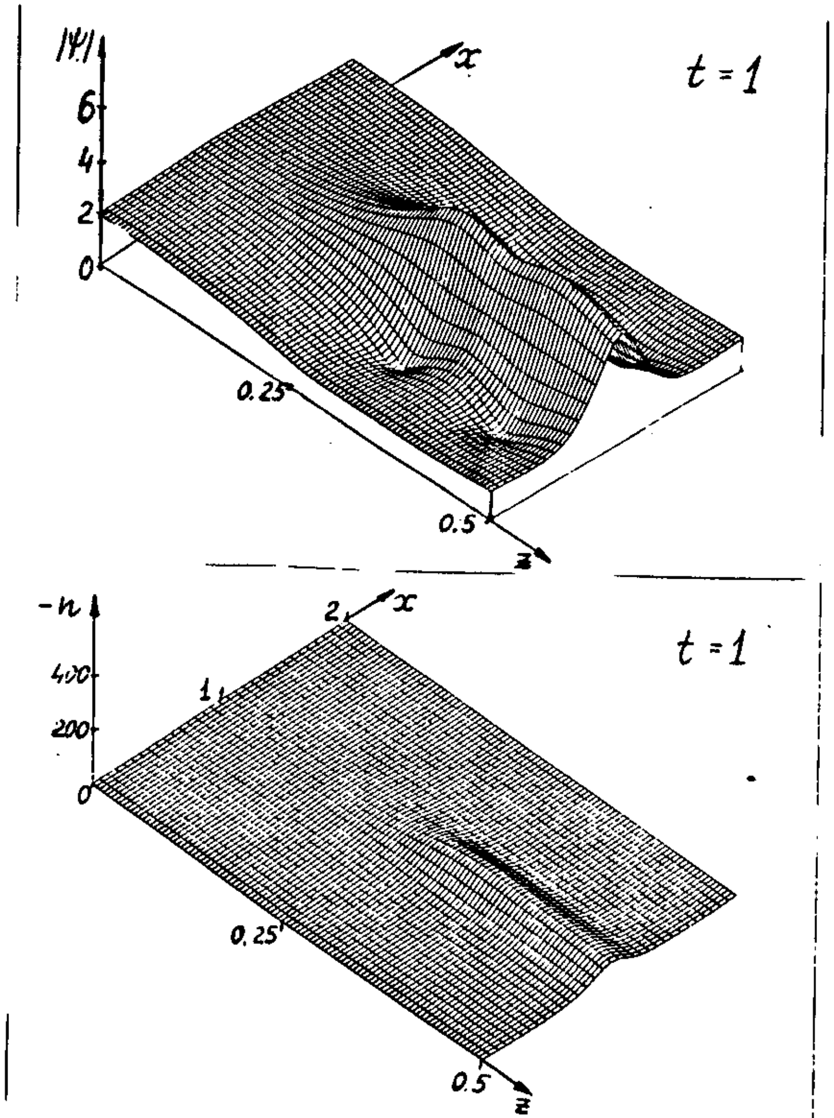
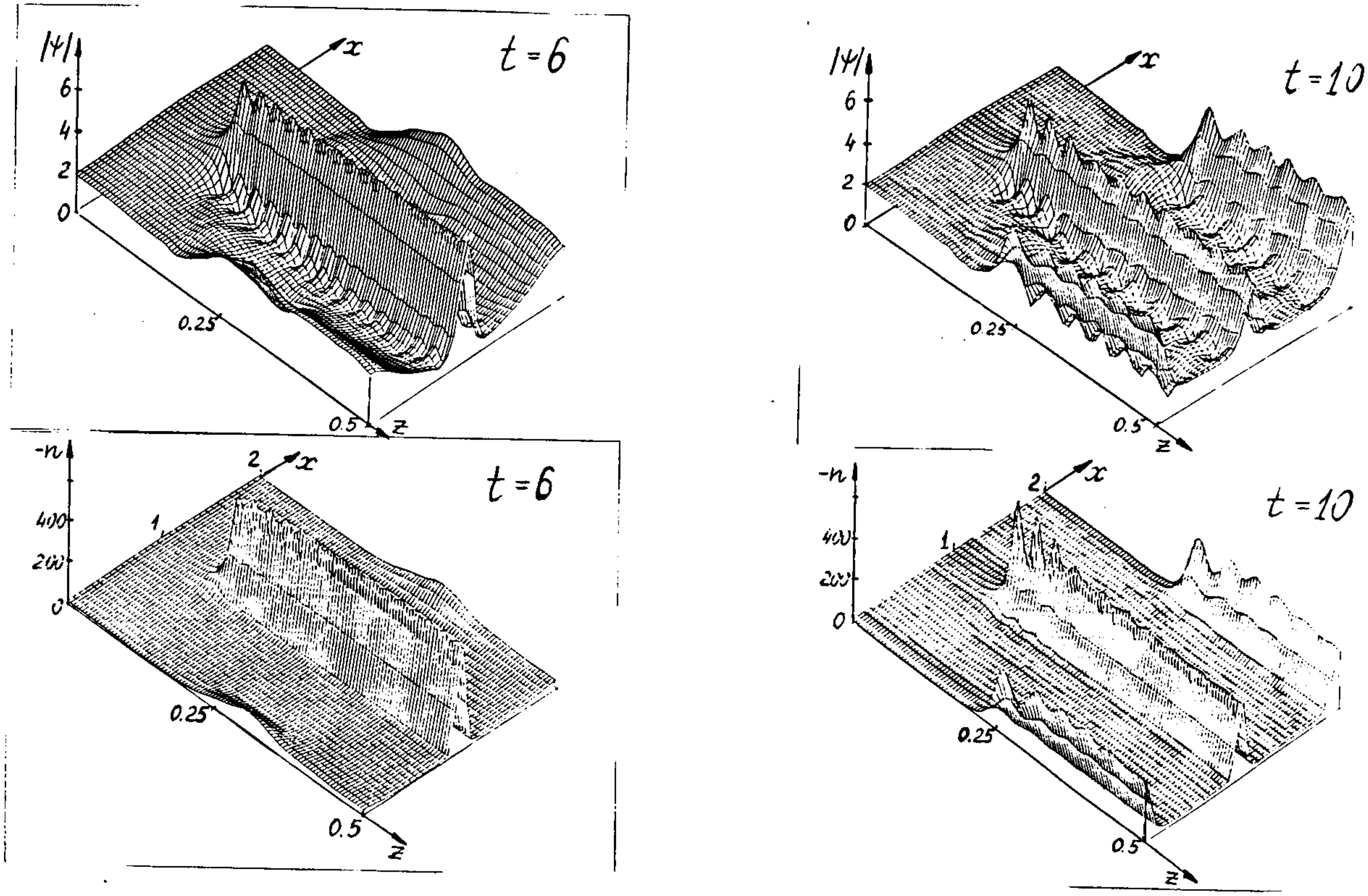


Fig. 4



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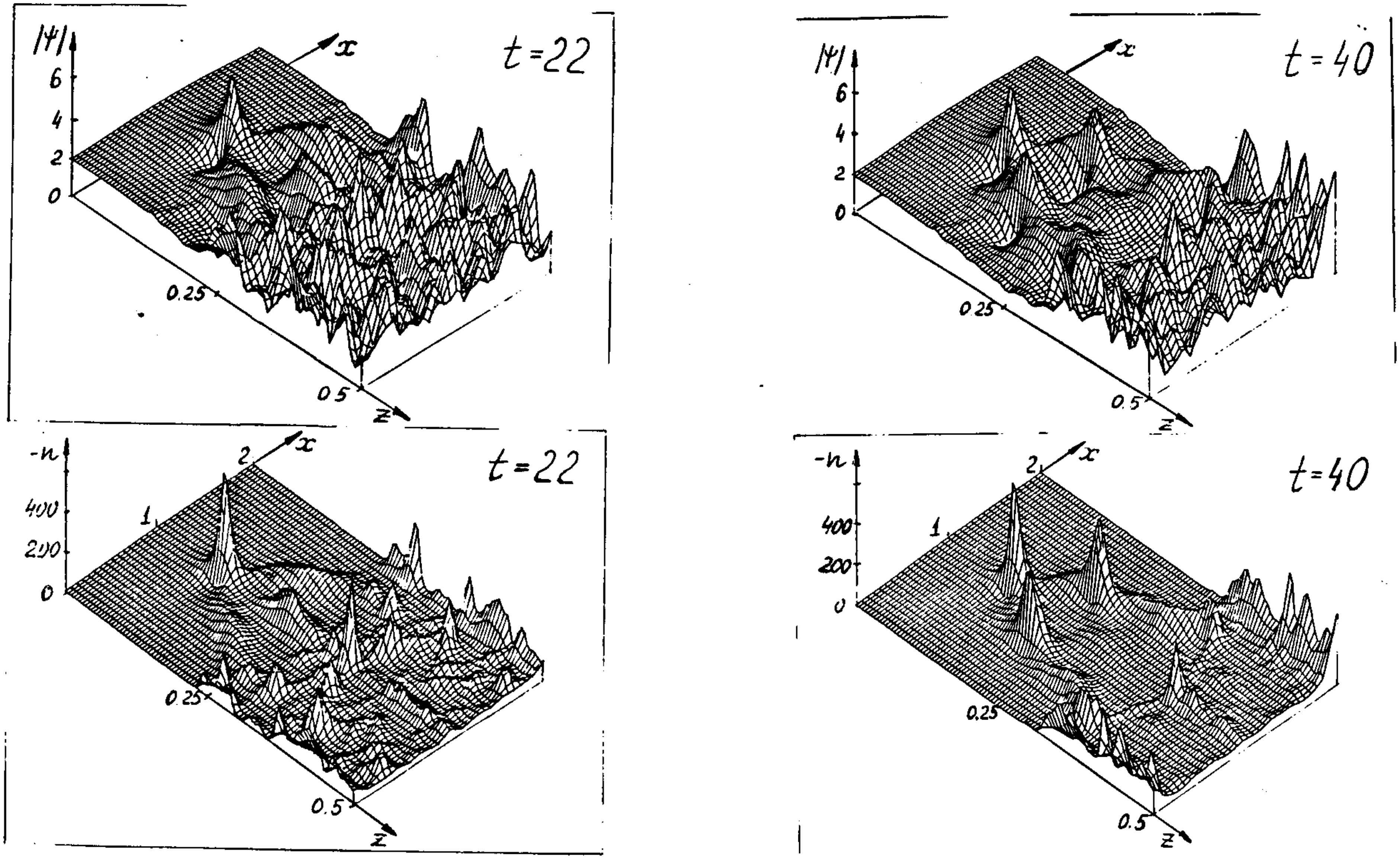
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