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SPRING COLLEGE ON PLASMA PHYSICS

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LARGE AMPLITUDE PLASMA WAVES

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Lecture Notes for
Trieste ICTP College on
"Large Amplitude Plasma Waves"
by W.B. Mori May, 1989

INTRODUCTION

- For accelerators

  "What is max. longitudinal $\vec{E}$?"

- Simplest estimate:

$$\nabla\cdot\vec{E} \rightarrow ikE \approx 4\pi e n_1 \qquad \text{(GAUSS' LAW)}$$

$$n_1^{max} \lesssim n_0$$

$$\Rightarrow E^{MAX} \approx \frac{4\pi e n_0}{k} = \frac{m\omega_p V_\phi}{e}$$

- For plasma accelerator $\quad V_\phi = \frac{\omega_p}{k} \rightarrow c$

$$\Rightarrow E^{MAX} \approx \frac{m\omega_p c}{e} \approx 1\ \text{GeV/cm} \times \left(\frac{n_0}{10^{18}\,\text{cm}^{-3}}\right)^{1/2}$$

Compared to $E^{MAX} < 1$ MeV/cm for RF linac

WAVEBREAKING OF PLASMA WA

Nonlinear Treatments:

	Non-Relativistic	Relativistic
Cold	$\frac{eE}{m\omega_p V_\phi} = 1$	$\frac{eE}{m\omega_p c} = \sqrt{2}(\gamma_\phi - 1)^{1/2}$
Warm $\left(\beta = \frac{3T}{mV_\phi^2}\right)$	$= \left(1 - \frac{1}{3}\beta - \frac{8}{3}\beta^{1/4} + 2\beta^{1/2}\right)^{1/2}$	?

For [illegible]

$$\frac{eE}{m\omega_p c} = \beta^{-1/4}\left[\ln 2\gamma_\phi^{1/2}\beta^{1/4}\right]^{1/2}$$

If Thermal Effects – but no kinetic effects – are included, how does one proceed?

Need equation of state

In the absence of kinetic effects then for high-frequency oscillations the motion is adiabatic

To see this we resort to a waterbag distribution function – for such a distribution function the heat flux is zero – or in other words it provides a fluid description

Lets Start

Vlasov equation $\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + \frac{F}{m}\frac{\partial}{\partial v}\right) f = 0$

Maxwell's Equations $\frac{\partial E}{\partial t} + 4\pi j = 0$ $\quad \frac{\partial E}{\partial x} + 4\pi e(n_e - n)$

Waterbag $\quad$ f is constant between v_0 and $-v_0$ in equilibrium

$$\int dv f = n_0 = 2v_0 f \qquad f = \frac{n_0}{2v_0}$$

Choose v_0 so that $\int dv\, v^2 f$ is the

Same as for a maxwellian 3

$$\int dv\, v^2 f = \frac{2}{3} v_0^3 f = kT_0 n_0$$

$$v_0^2 = 3kT_0$$

To obtain fluid equations we take moments of vlasov equations

$$V = \frac{\int dv\, v f}{\int dv f} \qquad n = \int dv f = (v_+ - v_-) f$$

$$= (v_+ - v_-)\frac{n_0}{2v_0}$$

$$= \frac{(v_+^2 - v_-^2)\cancel{f}}{2} \Big/ (v_+ - v_-)\cancel{f}$$

$$= \frac{v_+ + v_-}{2}$$

$$2V = v_+ + v_- \qquad 2v_0\frac{n}{n_0} = v_+ - v_- \Rightarrow \boxed{\begin{aligned} v_+ &= V + v_0\frac{n}{n_0} \\ v_- &= V - v_0\frac{n}{n_0} \end{aligned}}$$

$$\int dv \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{eE}{m}\frac{\partial}{\partial v}\right) f = \frac{\partial}{\partial t} n + \frac{\partial}{\partial x} nV = 0$$

$$\int dv\, v \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{eE}{m}\frac{\partial}{\partial v}\right) f$$

$$\frac{\partial}{\partial t} Vn + \frac{\partial}{\partial x}\left(\frac{v_+^3 - v_-^3}{3}\right)\frac{n_0}{2v_0} = -\frac{eE}{m} n$$

$$\frac{\partial}{\partial t} nV + \frac{\partial}{\partial x}\frac{\left(\left(V + v_0\frac{n}{n_0}\right)^3 - \left(V - v_0\frac{n}{n_0}\right)^3\right)}{3}\frac{n_0}{2v_0} = -\frac{eE}{m} n$$

$$\frac{\partial}{\partial t} nV + \frac{\partial}{\partial x}\left(nV^2 + \frac{1}{3}v_0^2\frac{n^3}{n_0^2}\right) = -\frac{eE}{m} n$$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}V + V\frac{\partial}{\partial x}V = -\frac{eE}{m} - \frac{v_0^2 n}{n_0^2}\frac{\partial}{\partial x}n && (1)\\ &\frac{\partial}{\partial t}E + 4\pi J = \frac{\partial E}{\partial t} - 4\pi e n V = 0 && (2)\\ &\frac{\partial}{\partial x}E + 4\pi e(n - n_0) = 0 && (3)\end{aligned}}$$

Assume Wave Like Solutions $t - \frac{x}{v_\phi} = \xi$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} \qquad \frac{\partial}{\partial x} = -\frac{1}{v_\varphi}\frac{\partial}{\partial \xi}$$

This gives $\quad (v_0^2 = 3v_{th}^2)$

$$\left(1 - \frac{V}{v_\varphi}\right)\frac{\partial V}{\partial \xi} = -\frac{eE}{m} + \frac{1}{2}\frac{v_0^2}{v_\varphi}\frac{1}{n_0^2}\frac{\partial}{\partial \xi}n^2$$

$$\frac{\partial}{\partial \xi}\left(V - \frac{V^2}{2v_\phi} - \frac{v_0^2 n^2}{2v_\varphi n_0^2}\right) = -\frac{eE}{m} = \frac{e}{m}\frac{\partial}{\partial x}\phi = -\frac{e}{mv_\varphi}\frac{\partial}{\partial \xi}\phi$$

$$\frac{\partial}{\partial \xi}\left(V - \frac{V^2}{2V_\varphi} - \frac{v_0^2 n^2}{2v_\varphi n_0^2} + \frac{e}{mv_\varphi}\phi\right) = 0$$

$$\boxed{\frac{e\phi}{mv_\varphi^2} = -\frac{V}{V_\varphi} + \frac{V^2}{2V_\varphi^2} + \frac{v_0^2}{2v_\varphi^2}\frac{n^2}{n_0^2} + C_1}$$

$$-\frac{\partial E}{v_\varphi \partial \xi} + 4\pi e(n - n_0) = 0 \qquad \frac{\partial E}{\partial \xi} - 4\pi e n V = 0$$

$$v_\varphi(n - n_0) - nV = 0$$

$$n(V_\varphi - V) = n_0 v_\varphi$$

$$\boxed{n = \frac{n_0}{\left(1 - \frac{V}{v_\varphi}\right)}}$$

$$\frac{e\phi}{mv_\varphi^2} = \bar{\Phi} = -\frac{V}{V_\varphi} + \frac{V^2}{2V_\varphi^2} + \frac{v_0^2}{2v_\varphi^2}\frac{1}{\left(1 - \frac{V}{v_\varphi}\right)^2} + C_1$$

$X = \frac{V}{V_\varphi}$

$$\boxed{\bar{\Phi} = -X + \frac{X^2}{2} + \frac{\bar{v}_0^2}{2}\frac{1}{(1-X)^2} + C_1}$$

from

$$\frac{\partial}{\partial \xi}\left(X - \frac{X^2}{2} - \frac{\bar{v}_0^2}{2}\frac{1}{(1-X)^2}\right) = -\frac{eE}{mv_\phi}$$

$$\frac{\partial^2}{\partial \xi^2}\left(X - \frac{X^2}{2} - \frac{\bar{v}_0^2}{2}\frac{1}{(1-X)^2}\right) = -\frac{e}{mv_\phi}\frac{\partial E}{\partial \xi} = -\frac{e}{mv_\phi}\frac{4\pi e n_0 V}{1 - \frac{V}{V_\varphi}}$$

$$= -\omega_p^2\frac{X}{1-X}$$

Normalize Time to ω_p^{-1}

$$\frac{\partial^2}{\partial \bar{\xi}^2}\left(x - \frac{x^2}{2} - \frac{\bar{v}_0^2}{2}\frac{1}{(1-x)^2}\right) = \frac{x}{1-x}$$

$$\frac{\partial}{\partial \bar{\xi}}\left(x - \frac{x^2}{2} - \frac{\bar{v}_0^2}{2}\frac{1}{(1-x)^2}\right) = -\bar{E} \qquad \bar{E} = \frac{eE}{m v_e \omega_p}$$

$$\frac{\partial}{\partial \bar{\xi}}(\;)\frac{\partial^2}{\partial \bar{\xi}^2}(\qquad) = \frac{x}{1-x}\frac{\partial}{\partial \bar{\xi}}(\;)$$

$$\frac{1}{2}\frac{\partial}{\partial \bar{\xi}}\left(\frac{\partial}{\partial \bar{\xi}}(\;)\right)^2 = \frac{1}{2}\frac{\partial}{\partial \bar{\xi}}\bar{E}^2 = \frac{x}{(1-x)}\left(\frac{\partial x}{\partial \bar{\xi}} - x\frac{\partial x}{\partial \bar{\xi}} - \bar{v}_0^2\frac{1}{(1-x)^3}\frac{\partial x}{\partial \bar{\xi}}\right)$$

$$\frac{\partial}{\partial \bar{\xi}}\frac{\bar{E}^2}{2} = x\frac{\partial x}{\partial \bar{\xi}} - \bar{v}_0^2\frac{x}{(1-x)^4}\frac{\partial x}{\partial \bar{\xi}}$$

$$\int dx\,\frac{x}{(1-x)^4} = \int dx\left\{\frac{(x-1)}{(1-x)^4} + \frac{1}{(1-x)^4}\right\}$$

$$= -\frac{1}{2}\frac{1}{(1-x)^2} + \frac{1}{3}\frac{1}{(1-x)^3}$$

$$\frac{\partial}{\partial \bar{\xi}}\frac{\bar{E}^2}{2} = \frac{\partial}{\partial \bar{\xi}}\left(\frac{x^2}{2} - \bar{v}_0^2\left(\frac{1}{3}\frac{1}{(1-x)^3} - \frac{1}{2}\frac{1}{(1-x)^2}\right)\right)$$

$$\frac{\bar{E}^2}{2} = \frac{x^2}{2} - \bar{v}_0^2\left(\frac{1}{3}\frac{1}{(1-x)^3} - \frac{1}{2}\frac{1}{(1-x)^2}\right) + C_2$$

Need to evaluate C_2

Note: In order to calculate E it is not necessary to evaluate C_1

$E=0$ when ϕ is an extremum, i.e.,

$$\frac{\partial\phi}{\partial\bar{\xi}} = 0 \qquad \frac{\partial\phi}{\partial\bar{\xi}} = \frac{\partial x}{\partial\bar{\xi}}\frac{\partial\phi}{\partial x}.$$

For the moment lets assume $\frac{\partial\phi}{\partial\bar{\xi}} = 0 \Rightarrow \frac{\partial\phi}{\partial x} = 0$

Then

$$\frac{\partial\phi}{\partial x} = 0 \Rightarrow (1-x) - \bar{v}_0^2\frac{1}{(1-x)^3} = 0$$

$$(1-x)^4 = \bar{v}_0^2 \equiv \beta$$

$$(1-x) = \beta^{1/4} \qquad x = 1-\beta^{1/4}$$

so

$$0 = \frac{(1-\beta)^{1/4}}{2} - \frac{\beta}{2}\left(\frac{1}{3}\beta^{-3/4} - \frac{1}{2}\beta^{-1/2}\right) + C_2$$

so

$$\bar{E} = \pm\left(x^2 - 2\beta\left(\frac{1}{3}\frac{1}{(1-x)^3} - \frac{1}{2}\frac{1}{(1-x)^2}\right) - (1-\beta^{1/4})^2 + 2\beta\left(\frac{1}{3}\beta^{-3/4} - \frac{1}{2}\beta^{-1/2}\right)\right.$$

Amperes Law and Gauss' Law give

$$\frac{\partial E}{\partial \xi} = 0 \Rightarrow V = 0 \Rightarrow X = 0$$

Hence
(choose − sign)

$$E_{max} = \left((1-\beta^{1/4})^2 - 2\beta\left(\tfrac{1}{3}\beta^{-3/4} - \tfrac{1}{2}\beta^{-1/2}\right) + 2\beta\left(\tfrac{1}{3} - \tfrac{1}{2}\right)\right)$$

$$= \left(1 - 2\beta^{1/4} + \beta^{1/2} - \tfrac{2}{3}\beta^{1/4} + \beta^{1/2} - \tfrac{\beta}{3}\right)^{1/2}$$

$$\boxed{\tilde{E}_{max} = \left(1 - \tfrac{8}{3}\beta^{1/4} + 2\beta^{1/2} - \tfrac{\beta}{3}\right)^{1/2}}$$

Coffey 1971

OUTLINE OF FLUID/WATERBAG THEORETICAL MODEL

1. Start from Energy-Momentum Tensor with unspecified $\bar{e}$ and $\bar{p}$
2. Derive Euler's equation and a wave equation for ϕ from conservation equations and Maxwell's equations
3. Derive expressions for $\bar{p}$ and $\bar{e}$ as functions of n_p and T for Waterbag fq
4. Obtain a wave equation in the single variable V (Numerically integrate
5. Let $V_\phi \to c$ and Analytically integrat
6. Evaluate integration constant from trapping argument and solve for E_{max}

STARTING POINT IS THE ENERGY-MOMENTUM TENSOR (Momentum Flux)

In Fluids Rest Frame:

$$\begin{bmatrix} \bar{p} & 0 \\ 0 & \bar{e} \end{bmatrix}$$

$\bar{p}$ is pressure

$\bar{e}$ is internal energy

Lorentz Transform to Lab Frame.

$$\begin{bmatrix} \bar{p} + \frac{(\bar{e}+\bar{p})v^2}{1-v^2} & \frac{(\bar{e}+\bar{p})v}{1-v^2} \\ \frac{(\bar{e}+\bar{p})v}{1-v^2} & -\bar{p} + \frac{\bar{e}+\bar{p}}{1-v^2} \end{bmatrix}$$

$$T^{\alpha\beta} = \bar{p}\, g^{\alpha\beta} + (\bar{e}+\bar{p})\, U^\alpha U^\beta$$

$$U^1 = \frac{v}{(1-v^2)^{1/2}} \qquad U^0 = \frac{1}{(1-v^2)^{1/2}}$$

CONSERVATION EQUATIONS AND MAXWELL'S EQUATIONS

$$\frac{\partial}{\partial t} T^{01} + \frac{\partial}{\partial x} T^{11} + \frac{enE}{m} = 0$$

$$\frac{\partial}{\partial t} T^{00} + \frac{\partial}{\partial x} T^{01} + \frac{envE}{m} = 0$$

$$\frac{\partial}{\partial t} E - 4\pi env = 0$$

$$\frac{\partial}{\partial x} E + 4\pi e(n-n_0) = 0$$

The first two equations are identical to:

$$\int dp\, p \left\{ \frac{\partial}{\partial t} f + v \frac{\partial f}{\partial x} - \frac{eE}{m} \frac{\partial f}{\partial p} \right\}$$

$$\int dp\, \gamma \left\{ \qquad \right\}$$

RELATIVISTIC EULER'S EQUATION

$$① \ \frac{\partial}{\partial t} T^{01} + \frac{\partial}{\partial x} T^{11} + nE = 0$$

$$② \ \frac{\partial}{\partial t} T^{00} + \frac{\partial}{\partial x} T^{01} + nvE = 0$$

① $- v$ ②, the identities

$$T^{01} - vT^{00} = \bar{p} v$$
$$T^{11} - vT^{01} = \bar{p}$$

and some algebra gives:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\gamma v = -\frac{eEnc^2}{\gamma(\bar{e}+\bar{p})} - \frac{\gamma c^2}{\bar{e}+\bar{p}}\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$\bar{e} + \bar{p} = n\gamma^{-1}mc^2$ for $c \to \infty$ or $\bar{p} \to 0$

WAVE EQUATION (Second order differential equation for

Assume Wave-like solutions $(x,t) \longrightarrow \xi \equiv x - v_\phi t$. The conservation laws become:

$$① \ -v_\phi \frac{\partial}{\partial \xi} T^{01} + \frac{\partial}{\partial \xi} T^{11} = -nE$$

$$② \ -v_\phi \frac{\partial}{\partial \xi} T^{00} + \frac{\partial}{\partial \xi} T^{01} = -nvE$$

① $- \frac{1}{v_\phi}$ ② gives

LHS:

$$\frac{\partial}{\partial \xi}\left\{T^{00} + T^{11} - v_\phi T^{01} - \frac{1}{v_\phi} T^{01}\right\}$$

$$= \frac{\partial}{\partial \xi}\left\{\frac{\bar{p}+\bar{e}}{1-v^2}\left[1 + v^2 - \frac{v}{v_\phi}(1+v_\phi^2)\right]\right\}$$

RHS:

$$-n\left(1-\frac{v}{v_\phi}\right)E$$

To simplify we utilize

Ampere's Law and Gauss' Law

$$-v_\phi \frac{\partial}{\partial \xi} E \quad - n v = 0 \qquad \text{Ampere's}$$

$$\frac{\partial}{\partial \xi} E \quad + n - 1 = 0 \qquad \text{Gauss'}$$

$\frac{1}{v_\phi}$ Ampere's + Gauss' gives

$$n = \left(1-\frac{v}{v_\phi}\right)^{-1} \quad \text{or} \quad n\left(1-\frac{v}{v_\phi}\right) = 1$$

Hence:

$$\frac{\partial}{\partial \xi}\underbrace{\left\{\frac{\bar{p}+\bar{\varepsilon}}{1-v^2}\left[1+v^2-\frac{v}{v_\phi}(1+v_\phi^2)\right]\right\}}_{\phi\ +\ \text{constant}} = -E$$

To obtain the nonlinear wave equation we differentiate this last equation:

$$\frac{\partial^2}{\partial \xi^2}\left\{\frac{\bar{p}+\bar{\varepsilon}}{1-v^2}\left[1+v^2-\frac{v}{v_\phi}(1+v_\phi^2)\right]\right\}$$

$$= -\frac{\partial}{\partial \xi} E = \frac{v}{v_\phi - v} \qquad \left\{\begin{array}{l}\text{From Ampere's}\\ \text{and Gauss' Law}\end{array}\right.$$

To close this equation we need $\bar{p}+\bar{\varepsilon}$ as a function of v

HOW TO EVALUATE $\bar{p}$ AND $\bar{e}$

Use the definitions of the components of the energy-momentum tensor ([illegible])

$$T^{11} = \int dp \, \frac{p^2}{\gamma} f(p)$$

$$T^{00} = \int dp \, \gamma \, f(p)$$

(comes from Vlasov equation as well)

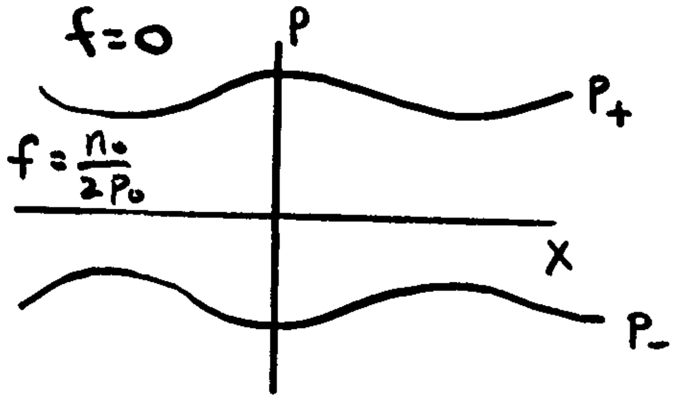
Recall that:

$$\bar{p} = T^{11} \text{ in rest frame } \Big|_{v=0}$$

$$\bar{e} = T^{00} \text{ in rest frame } \Big|_{v=0}$$

It is hopeless to evaluate the integrals for arbitrary-dynamical $f(p)$. However for the special case of a Waterbag the integrals become possible.

WHAT IS A WATERBAG ?

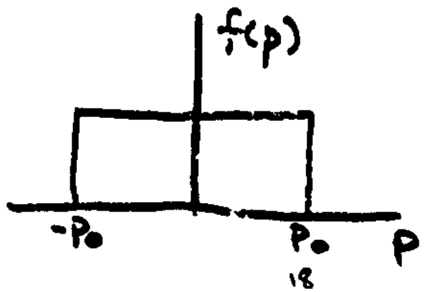


$$f(p) = \frac{n_0}{2P_0} \quad \text{for} \quad P_- < p < P_+$$

$$= 0 \quad \text{otherwise}$$

$$P_0 = \sqrt{3Tm}$$

Unperturbed $f(p)$



AS A RESULT

$$\frac{T^{11}}{T^{00}} = \frac{1}{2p_0}\left\{\frac{p}{2}(1+p^2)^{1/2} \mp \sinh^{-1} p\right\}\Bigg|_{P_-}^{P_+}$$

In addition

$$V = \frac{\int dp \frac{p}{(1+p^2)^{1/2}} f(p)}{\int dp\, f(p)} = \frac{(1+P_+^2)^{1/2} - (1+P_-^2)^{1/2}}{P_+ - P_-}$$

$$\therefore \quad V=0 \implies P_+ = -P_-$$

Hence

$$\frac{\bar{p}}{\bar{e}} = \frac{1}{2p_0}\left\{P_+ (1+P_+^2)^{1/2} \mp \sinh^{-1} P_+\right\}$$

Furthermore

$$n(V=0) \equiv n_p = \int dp f(p)\Big|_{V=0} = \frac{P_+}{P_0}$$

SO FINALLY

$$\frac{\bar{p}}{\bar{e}} = \frac{1}{2p_0}\left\{n_p p_0 (1+(n_p p_0)^2)^{1/2} \mp \sinh^{-1} n_p p_0\right\}$$

Note for $n_p p_0 \ll 1$ this gives

$$\bar{p} \simeq \frac{1}{3} p_0^2 n_p^3 \quad \text{and} \quad \bar{e} = n_p + \frac{1}{2}\bar{p}$$

1-D adiabatic gas Law

We needed

$$\bar{p} + \bar{e} = n_p (1 + \beta n_p^2)^{1/2}$$

where $\beta \equiv \frac{3T}{mc^2}$

Note that n_p is

$$n_p = \frac{n}{\gamma} = \frac{(1-V^2)^{1/2}}{1-V/\ldots}$$

VALIDITY OF 1-D DESCRIPTION

$$T^{xx} = \int dP_z\, dP_y\, dP_x \, \frac{P_x^2}{(1+P_x^2+P_y^2+P_z^2)} f(\vec{p})$$

If $P_y^2 + P_z^2 << P_x^2$ then the integrations are decoupled

Non-relativistic isotropic temperature and relativistic 1-D bulk fluid motion

$P_y = P_z = P_0$ while

$P_x >> P_0$

THE EXACT NONLINEAR WAVE EQUATION IS THEREFORE

$$\frac{\partial^2}{\partial \xi^2} \left\{ \frac{1-vv_\phi}{(1-v^2)^{1/2}} \left(1+\beta \frac{(1-v^2)}{(1-v/v_\phi)^2}\right)^{1/2} \right\} = \frac{v}{v_\phi - v}$$

Can be solved numerically

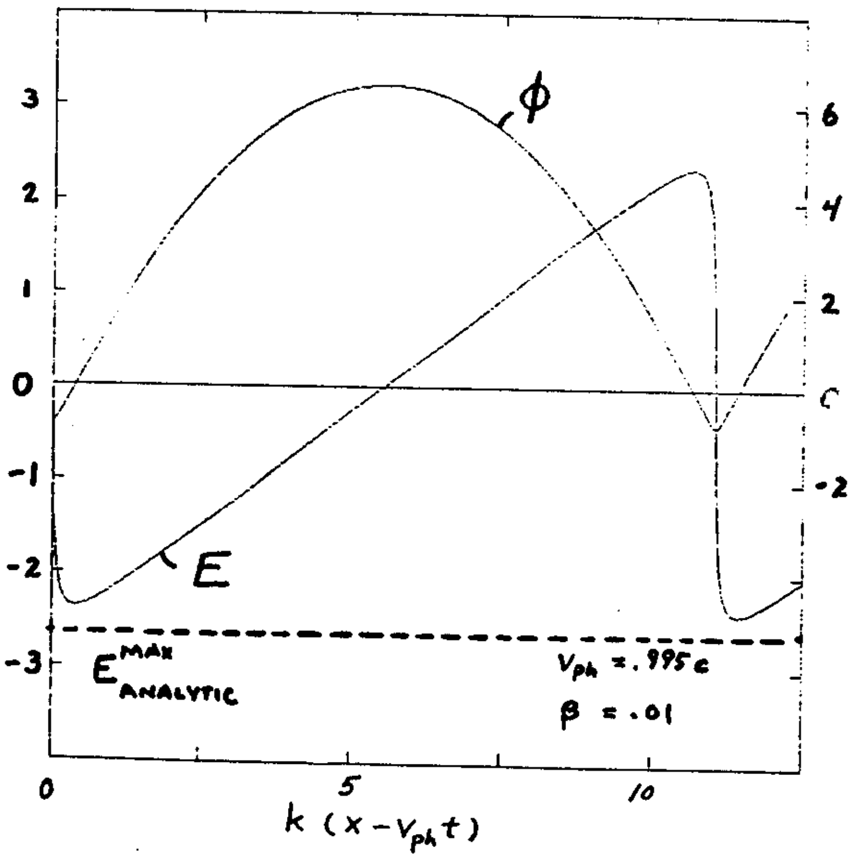
$$\frac{\partial^2}{\partial \xi^2} \phi = \frac{V(\phi)}{V_\phi - V(\phi)}$$

To proceed further analytically we let $V_\phi = C$ identically in the wave equation and get

$$\frac{\partial^2}{\partial \xi^2} (x^2+\beta)^{1/2} = \frac{1}{2}\left(\frac{1}{x^2} - 1\right)$$

where $x \equiv \left(\frac{1-v}{1+v}\right)^{1/2}$ and

$$\frac{\partial}{\partial \xi} (x^2+\beta)^{1/2} = -E$$



THE FIRST INTEGRAL TO THE APPROXIMATE WAVE EQUATION BY MULTIPLYING BOTH SIDES BY

$$\frac{\partial}{\partial \xi}(x^2+\beta)^{1/2} = -E$$

Resulting in

$$\frac{1}{2}\left[\frac{\partial}{\partial \xi}(x^2+\beta)^{1/2}\right]^2 = \frac{1}{2}E^2$$

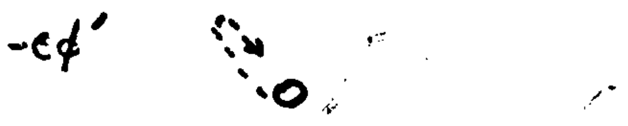
$$= C_1 - \frac{1}{2}(x^2+\beta)^{1/2} - \frac{1}{2\sqrt{\beta}}\ln\left|\frac{\sqrt{\beta}(x^2+\beta)^{1/2}+\beta}{x}\right|$$

Need to evaluate C_1

Evaluate C_1 at an extremum of ϕ $(E=0)$ by the following physical arguments

THE MOST NEGATIVE VALUE OF ϕ IS ACHIEVED WHEN THE UPPER WATERBAG SURFACE IS TRAPPED

Go to wave frame



$$-e\phi'_{tr} \geq (\gamma'-1)mc^2$$

$$\gamma' = \gamma_\phi \left(\gamma - v_\phi \frac{p}{mc^2}\right)$$

Since $p = \beta^{1/2}$, $\gamma = (1+\beta)^{1/2}$ and $\phi'_{tr} = \gamma_\phi \phi_{tr}$, it follows that

$$-\phi_{tr} = \left[(1+\beta)^{1/2} - \beta^{1/2} v_\phi\right] - \frac{1}{\gamma_\phi}$$

EQUATE ϕ_{tr} TO ϕ AND SOLVE FOR CRITICAL v_0

Recall:

$$\phi = \frac{1 - vv_\phi}{(1-v^2)^{1/2}}\left[1 + \beta\frac{(1-v^2)}{(1-v/v_\phi)^2}\right]^{1/2} - (1+\beta)^{1/2}$$

Setting $\phi = \phi_{tr}$ gives

$$v_0 = v_\phi - \frac{2\beta^{1/2}}{\gamma_\phi}$$

[Same result is obtained from $\frac{\partial\phi}{\partial v} = 0$

for $\gamma_\phi \beta^{1/2} \gg 1$. Hence

$$X_0 \equiv \left[\frac{1-v_0}{1+v_0}\right]^{1/2} \simeq \frac{\beta^{1/4}}{\gamma_\phi^{1/2}}$$

ANALYTIC EXPRESSION FOR E

$$E^2 = (X_0^2+\beta)^{1/2} - (X^2+\beta)^{1/2} + \frac{1}{\sqrt{\beta}} \ln \frac{X}{X_0} \left(\frac{(X_0^2+\beta)^{1/2}+\sqrt{\beta}}{(X^2+\beta)^{1/2}+\sqrt{\beta}} \right)$$

Maximum of E occurs when
V=0 or equivalently X=1 as
seen from Ampere's and Gauss' Law

$$E_{max}^2 = (X_0^2+\beta)^{1/2} - (1+\beta)^{1/2} + \frac{1}{\sqrt{\beta}} \ln \frac{1}{X_0} \left(\frac{(X_0^2+\beta)^{1/2}+\sqrt{\beta}}{(1+\beta)^{1/2}+\sqrt{\beta}} \right)$$

or

$$\boxed{E_{max} \simeq \frac{mc\omega_p}{e} \beta^{-1/4} \left[\ln 2\gamma_\phi^{1/2} \beta^{1/4} \right]^{1/2}}$$

So long as

$$E_{max} < \sqrt{2}\,(\gamma_\phi - 1)^{1/2} \Rightarrow \gamma_\phi >> \frac{1}{2\beta^{1/2}}$$