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PHASE SPACE DIFFUSION IN TURBULENT PLASMAS

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Abstract

Turbulent diffusion of charged test particles in electrostatic plasma turbulence is analyzed. Two different types of test particles can be distinguished. First passive particles which are subject to the fluctuating electric fields without themselves contributing to the local space charge. The second type are particles introduced at a prescribed phase space position at a certain time and which then self-consistently participate in the phase space dynamics of the turbulence. The latter "active" type of particles can be subject to an effective frictional force due to radiation of plasma waves. In terms of these test particle types, two basically different problems can be formulated. One deals with the diffusion of a particle with respect to its point of release in phase space. Alternatively the relative diffusion between many, or just two, particles can be analyzed. Analytical expressions for the mean square particle displacements in phase space are discussed. More generally equations for the full probability densities are derived and these are solved analytically in special limits.

1. INTRODUCTION

Plasma instabilities often saturate in an enhanced level of turbulent fluctuations which can be described only in statistical terms. An important characteristic of such a turbulence is its ability to interact with the charged particles which constitutes the plasma. A description and understand of the resulting random motion of particles in phase space is important for many practical applications in connection with for instance fusion plasma studies or the ionosphere. A number of basically different formulations of the particle dynamics can be proposed.

i) The simplest from a theoretical point of view is the analysis of a passive test particle, released at a certain phase space position (x_0,v_0) at $t = 0$. Being passive, this particle is subject to the fluctuating electric fields but it does not itself contribute to the local space charge of the plasma. All the relevant statistical properties of the plasma fluctuations are assumed to be prescribed in this case.

ii) A somewhat more complicated problem deals with an **active** test particle which at $t = 0$ is introduced at (x_0,v_0) and its contribution to the charge distribution is taken into account for $t > 0$.

iii) Finally fully selfconsistent particles can be considered, where also the "prehistory" i.e. the spase space trajectory of the particle for $t < 0$ is consistent with the electric field value $E(x = x_0, t = 0)$. The basic difference between test particles of type i) and ii) as compared with the present type iii) is that for the former variant their position (x_0,v_0) at $t = 0$ can be **prescribed** with certainty. For fully selfconsistent test particles their phase space position can be expressed only as a certain probability in a narrow interval. Predictions of particle positions with certainty requires knowledge of the entire particle trajectory, leading ultimately to formulations in terms of conditional probabilities.

In terms of the test particle types described here two basically different problems of turbulent particle diffusion can be formulated. The first one is concerned with **absolute** diffusion of a particle with respect to its point of release in phase space. Alternatively the **relative** diffusion between many, or just two, particles can be considered. In many respects the latter problem is most interesting as it measures for instance the rate of dispersion of a small cloud of contaminants entering a turbulent plasma.

It is interesting to note that many problems within transport in the turbulent atmosphere, as reviewed by for instance Csanady [1] can be recognised also in turbulent plasmas. Relative plasma diffusion across magnetic fields were discussed by Misguich et al. [2] using ideas originating from an investigation of a related atmospheric diffusion problem considered by Mikkelsen et al. [3]. This analogy can be maintained for instance for guiding center diffusion in a plane perpendicular to a homogeneous constant magnetic field. More generally there is a basic difference between particle diffusion in collisionless, or weakly collisional, plasmas and turbulence in a neutral atmosphere. Thus if quite formally the turbulent fluctuations in an atmosphere are frozen at a certain instant, then test particles will remain virtually immobile from then on and any displacement will be due to a very small molecular diffusivity. If in a turbulent collisionless plasma the fluctuating electric field is frozen similarly then test particles will continue its displacement by particle free streaming. In this work it will be argued that in a certain sense the analogy between turbulent diffusion in plasma and neutral flows can be maintained. Partial similarities between phase space diffusion in plasmas and weakly turbulent linear shear flows can be demonstrated. In particular it will be argued that evidence for clump formation in turbulent plasmas can be obtained from the fact that although the turbulence as analyzed by Eulerian sampling very well may be time stationary and spatially homogeneous, it will not be so by Lagrangian sampling along the particle orbits. This difference between the two types of sampling is one of the properties shear flows and plasma phase space dynamics have in common.

In the following we discuss turbulent diffusion in plasma phase space where passive test particles of type i) in the previous summary, are considered in particular.

2. EVOLUTION OF AVERAGED QUANTITIES FOR PASSIVE TEST PARTICLES

In this section the turbulent transport of passive test particles will be considered. Electrostatic turbulence will be assumed. With the test particles being passive in the sense discussed in the Introduction we assume that the electrostatic fluctuations are entirely prescribed by their statistical properties. With this information considered given the transport properties of this turbulence is analysed.

A. Single Particle Diffusion

Consider a single charged particle released at x_0, v_0 at $t=0$ in phase-space. The simplest characteristics of the statistical properties of its subsequent trajectory for $t>0$ are $<\Delta v(t)>$, $<\Delta x(t)>$, $<\Delta v^2(t)>$ and $<\Delta x^2(t)>$ where $\Delta v(t) \equiv v(t)-v_0$ and $\Delta x(t) \equiv x(t)-x_0-v_0 t$. For times so short that the electric field can be considered essentially constant we find

$$<\Delta v(t)> \approx \left(\frac{e}{M}\right) <E(t,x_0)> t = 0 \; , \tag{1}$$

$$<\Delta x(t)> \approx <\Delta v(t)> t \; = 0 \; , \tag{2}$$

$$<\Delta v^2(t)> \approx \left(\frac{e}{M}\right)^2 <E^2> t^2 , \tag{3}$$

$$<\Delta x^2(t)> \approx \frac{1}{4}\left(\frac{e}{M}\right)^2 <E^2> t^4 , \tag{4}$$

$$<\Delta x(t)\Delta v(t)> \approx \frac{1}{2}\left(\frac{e}{M}\right)^2 <E^2> t^3 . \tag{5}$$

It is important that the average quantities entering these expressions are the Eulerian averages, i.e. the electric fields are sampled at a fixed position x_0. The turbulence was assumed to be homogeneous and time stationary.

For times so large that the electric field can change appreciably we have more generally

$$\Delta v(t) = \frac{e}{M}\int_0^t E(x_0 + \Delta x(\iota),\iota) d\iota \; , \tag{6}$$

$$\Delta x(t) = \int_0^t \Delta v(\iota) d\iota = \frac{e}{M}\int_0^t (t-\iota)\, E\,(x_0 + \Delta x(\iota),\iota) d\iota \tag{7}$$

where the integration runs along the Lagrangian orbit of the particle as indicated. From (6) we have

$$<\Delta v^2(t)> = 2\left(\frac{e}{M}\right)^2 \int_0^t \int_0^t <E(x(\iota),\iota)\, E(x(s),s)> ds\, d\iota \; , \tag{8}$$

$$<\Delta x^2(t)> = \left(\frac{e}{M}\right)^2 \int_0^t \int_0^t (t-\iota)(t-s) <E(x(\iota),\iota)\, E(x(s),s)> ds\, d\iota \ . \tag{9}$$

B. Clump Formation In Plasmas

The expressions (8)-(9) can be simplified by the assumption of time-stationary Lagrangian electric field correlation functions i.e.

$$<E(x(\iota),\iota)\, E(x(s),s)> = <E^2>\, R_L(\iota - s) \tag{10}$$

where then $R_L(o) = 1$. Although this assumption is often made it can not be generally valid. This can be seen readily from the expression

$$\partial_t <v(t)> = \left(\frac{e}{M}\right) <E(x(t),t)> \ , \tag{11}$$

with x(t) being the Lagrangian orbit. Consider for instance test particles released at random positions all with velocities, say, $v_o = 0$, in weakly turbulent plasmas, where electrostatic fluctuations are excited by an ion beam injected into a stationary plasma. It is well known from a series of investigations, that at least in one dimension this instability saturates in an irregular train of ion phase space vortices, which for symmetry reasons propagate with an average velocity equal to half the ion beam velocity in the case where beam and background ion distributions are identical. It is intuitively clear that a large fraction of the test particles will be trapped in phase space vortices, and be accelerated. Ultimately these particles will propagate at an average velocity close to the characteristic velocity of the vortices. The net result is thus an increase in $<v(t)>$ necessitating $<E(x(t),t)> \neq 0$. (The test particles will of course be decelerated on average for the case where v_o is larger than the average vortex velocity). On the other hand $<E(x_o,t)> = 0$ by assumption for an Eulerian sampling of the turbulent electric field fluctuations.

Consider now systems which are homogeneous and time stationary in a statistical sense. For a class of such systems we may replace the ensemble averaging implied in the foregoing equations by a suitable average over just one realisation. Rather than releasing one particle in each realisation of the ensemble we thus release many (in the limiting case infinitely many) particles uniformly distributed in one realisation. By the assumption of homogeneity we have

$$<E(x(0),0)> = \frac{1}{N}\sum_j^N E(x_j(0),0) = 0 \ , \tag{12}$$

with $x_j(o)$ being the initial position of the j-th particle. The observation

$$<E(x(t),t)> = \frac{1}{N}\sum_j^N E(x_j(t),t) \neq 0 \quad \text{for } t > 0, \tag{13}$$

then necessarily implies that the particle positions $x_j(t > 0)$ can no longer be uniformly distributed. A consequence of this nonuniformity is the necessity for particles to "clump" in certain spatial regions. This conclusion is interesting by providing a proof for plasma clumping which does not involve two particle distributions. Boutros-Ghali and Dupree [4] claimed in their critisism of the work by Dubois and Espedahl [5] and similar works that plasma clumps could be discussed only in terms of this function.

Later on we will argue for a strong resemblance between the phase-space dynamics of turbulent collisionless plasmas and the properties of weakly turbulent linear shear flows. Anticipating this analogy the foregoing conclusions are rather self-evident i.e. the Langrangian velocity of a fluid particle in an inhomogeneous turbulent incompressible flow is not a stationary, random function of time.

C. Diffusion In Quasi Time Stationary Electric Field Fluctuations

It may, however, be expected that in some cases the error is not significant in assuming time stationarity of the electric field sampled along a Lagrangian orbit. As expected the velocity fluctuations are distinctly nonstationary in all cases. The electric field correlations on the other hand demonstrate that for initial velocities v_0 well outside the resonant region the approximation $<E(t)E(t+\tau)> \approx <E^2>R(\tau)$ is not too bad for limited times. (Note, however, that the normalizing quantity should again here be obtained along Lagrangian orbits. There is no reason to expect that its value coincides with the value obtained by Eulerian sampling. Again the situation differs from that for incompressible flows). With this approximation we may reduce (8) and (9) to

$$<\Delta v^2(t)> = 2\left(\frac{e}{M}\right)^2 <E^2> t \int_0^t (1-\tau/t)R(\tau)d\tau \ , \tag{14a}$$

$$<\Delta x^2(t)> = \frac{1}{3}\left(\frac{e}{M}\right)^2 <E^2> t^3 \cdot \int_0^t (2-3\tau/t+\tau^3/t^3)R(\tau)d\tau \ . \tag{14b}$$

In the limit of large times $t \gg \tau_c$ we find

$$<\Delta v^2(t)> = 2\left(\frac{e}{M}\right)^2 <E^2> t\tau_c \tag{15a}$$

$$<\Delta x^2(t)> = \tfrac{2}{3}\left(\frac{e}{M}\right)^2 <E^2> t^3\tau_c \ , \tag{15b}$$

with $\tau_c \equiv \int_0^\infty R(\tau)d\tau$, where it was assumed that $R(\tau \to \infty) \to 0$. Of interest is also

$$<\Delta x(t)\Delta v(t)> = \left(\frac{e}{M}\right)^2 \int_0^t \int_0^t (t-\tau) <E(x(\tau),\tau)E(x(s),s)> d\tau ds \ , \tag{16}$$

which with the previous assumptions reduces to

$$<\Delta x(t)\Delta v(t)> = \left(\frac{e}{M}\right)^2 <E^2> t^2 \int_0^t (1-\tau/t)R(\tau)d\tau \ , \tag{14c}$$

or for $t \gg \tau_c$

$$<\Delta x(t)\Delta v(t)> = \left(\frac{e}{M}\right)^2 <E^2> t^2\tau_c \ . \tag{15c}$$

The dimensionless ratio formed by use of (15a,b,c) takes a particularly simple form

$$\frac{<\Delta x(t)\Delta v(t)>}{[<\Delta x^2(t)><\Delta v^2(t)>]^{\frac{1}{2}}} = \frac{\sqrt{3}}{2} \approx 0.87 \ . \tag{17}$$

It should be reemphasized that the assumption of time-stationarity has a limited validity only.

The results in this section are expressed in terms of correlation coeeficients for the electric field. This quantity is related to the potential correlation function by

$$<E(x+\xi, t+\tau)E(\xi,\tau)> = -\partial^2 <\phi(x+\xi, t+\tau)\phi(\xi,\tau)>/\partial x^2 \ ,$$

where we explicitly used the spatial homogeneity of the turbulence. Note that the E-field correlation necessarily must take on negative values for <u>some</u> spatial positions since the curvature of the potential correlations <u>must</u> change sign with varying x.

D. Relative Particle Diffusion

In this section we consider the simplest case of just two simultaneously released particles [6], which will here be considered as representative for the properties of the small clouds of test particles. Denoting the positions and velocities of the two particles $x_1(t)$, $x_2(t)$ and $v_1(t)$, $v_2(t)$ respectively, we have with the definition $r(t) = x_1(t)-x_2(t)$

$$\tfrac{1}{2}d^2<r^2(t)>/dt^2 = [v_1(0)-v_2(0)]^2 + \left(\frac{e}{M}\right)^2\left\{\int_0^t d\tau\int_0^\tau ds W(\tau,s) + \int_0^t d\tau\int_0^\tau ds W(s,t)\right\}$$

$$+ 2\left[v_1(0)-v_2(0)\right]\left(\frac{e}{M}\right)\int_0^t <E(x_1(\tau),\tau) - E(x_2(\tau),\tau)>d\tau$$

$$+ \left[x_1(0)+v_1(0)t-x_2(0)-v_2(0)t\right]\left(\frac{e}{M}\right)<E(x_1(t),t)-E(x_2(t),t)> \quad , \tag{18}$$

where $W(\tau,s) \equiv <[E(x_1(\tau),\tau) - E(x_2(\tau),\tau)]\,[E(x_1(s),s) - E(x_2(s),s)]> = 2[<E(x(\tau),\tau)E(x(s),s)> - <E(x_1(\tau),\tau)E(x_2(s),s)>]$, with $W(\tau,s) = W(s,\tau)$, as expected for symmetry reasons. According to the arguments presented in Sec. 2.B the two averages $<E(x_1(t),t>$ and $<E(x_2(t),t>$ will be identical only if they are evaluated for identical initial velocities $v_1(0)$ and $v_2(0)$, for spatially homogeneous turbulence. For short times

$$<r^2(t)> \simeq (r(0)+[v_1(0)-v_2(0)]t)^2 + \tfrac{1}{4}\left(\frac{e}{M}\right)^2\left(<E^2(x_0)> - <E[x_1(0)]\,E[x_2(0)]>\right)t^4 \ . \tag{19a}$$

To bring out the physical content it is preferable to express the last term in (19a) in terms of the wave number spectrum of the electric field, S(k), which is simply the Fourier transform of the Eulerian correlation function $<E[x_1(o)]\,E[x_2(o)]>$

$$<r^2(t)> = (r(0)+[v_1(0) - v_2(0)]t)^2 + \tfrac{1}{2}t^4\left(\frac{e}{M}\right)^2\int_0^\infty S(k)\left\{1-\cos[kr(0)]\right\}dk \ . \tag{19b}$$

For small initial separations r(0) the term in the curly brackets act as a filter surpressing small k-values, i.e. only wavelengths in the turbulence which are comparable to or smaller than the separation of the two particles contribute to the time-variation of $<r^2(t)>$. <u>Long</u> wavelengths will tend to convect both particles together, along nearby trajectories without any effect on r(t). This quite reasonable physical mechanism is of course effective in fluid type turbulence also, in plasmas as well as atmospheric flows. The central difference is that in phase space the two particles will separate due to the free streaming contribution [the first term in (19)] even in the absence of any turbulence. For $r(0)=0$ and $v_1=v_2$ the two particles of course move together, i.e. $\langle r^2(t)\rangle = 0$. For very large initial separations on the other hand we may assume $\langle E[x_1(0)]\, E[x_2(0)]\rangle = 0$ and the two particles move independently of each other. These features are retained in a more general model presented in the next section. A simple generalization of (19) is readily obtained for the short time evolution. Referring to the previous physical interpretation of (19) we here merely state the result

$$\tfrac{1}{2} d^2\langle r^2\rangle/dt^2 = \left[v_1(0)-v_2(0)\right]^2$$

$$+\, 3\,t^2\left(\frac{e}{M}\right)^2 \int_0^\infty S(k)\left\{1 - e^{-\frac{1}{2}k^2(\langle r^2\rangle - r_0^2(t))} \cos[r_0(t)]\right\} dk \;, \tag{20}$$

where $r_0(t) \equiv r(0)+[v_1(0)-v_2(0)]t$. We used the approximation $\mathrm{Re}\langle \exp(iX)\rangle \approx \cos\langle X\rangle \exp[-\tfrac{1}{2}\langle (X-\langle X\rangle)^2\rangle]$, where we note that $r(0)$ and $r_0(t)$ are statistically sure quantities in the present analysis. The form (20) accounts for the fact that as $\langle r^2\rangle$ increases, a larger and larger fraction of the spectral energy becomes available for the particle separation.

Similarly we find for the short time dispersion in velocity with $u(t) \equiv v_1(t) - v_2(t)$

$$\langle u^2(t)\rangle = u^2(0) + \left(\frac{e}{M}\right)^2 \int_0^t \int_0^t W(\tau,s)\, d\tau\, ds$$

$$+\, 2\,u(0)\left(\frac{e}{M}\right) \int_0^t \langle E(x_1(\tau),\tau) - E(x_2(\tau),\tau)\rangle\, d\tau \;, \tag{21}$$

with W defined in connection with Eq. (18). For short times we have $\langle u^2(t)\rangle \simeq u(0) + (e/M)^2(\langle E^2\rangle - \langle E(x_1)\, E(x_2)\rangle)t^2$ with a physical intepretation as outlined in the discussion of Eq. (20), i.e. the particles disperse in relative velocity as if subject to an effective rms-field amplitude which is obtained by filtering the power spectrum by a filter depending on the actual particle separation.

For extended times the same problems arise here as in the case of absolute diffusion concerning the Lagrangian statistics of the electric field. Assuming again that the randomly varying electric field as detected by the particle is approximately time stationary in a statistical sense, a particularly simple asymptotic result is, however, readily obtained. Thus, for large times we expect the particle separation to become large also. In this limit the two particles will expectedly diffuse independently of each other and of their initial velocities giving the simple correspondence with (14) and (15), e.g.

$$<r^2(t)> \simeq (r(0)+[v_1(0)-v_2(0)]t)^2 + \frac{4}{3}\left(\frac{e}{M}\right)^2 < E^2 > \tau_c t^3 \tag{22}$$

and

$$<u^2(t)> \simeq [v_1(0)-v_2(0)] + 4\left(\frac{e}{M}\right)^2 <E^2> \tau_c t\,. \tag{23}$$

We find the similarities between these results and those obtained for diffusion in weakly turbulent shear flows quite interesting. After all, the underlying physical properties are quite similar. In the absence of turbulent fluctuations two points will separate linearly with time. With a superimposed random motion particles occasionally wander into regions with a higher bulk velocity, with a resulting increase in the rate of mean square separation. Although the analogy is not complete, it can be enlightening also in other contexts as already mentioned.

Observed variations $<\Delta x^2> \sim t^3$ and $<\Delta v^2> \sim t$ can be interpreted as a verification of the hypothesis of approximately time-stationary statistics of the Lagrangian electric field fluctuations. The method may have diagnostic applications in numerical simulations, since the required analysis is much less demanding than the analysis of the Lagrangian correlation functions. The results (22) and (23) ultimately break down if a significant number of particles have diffused into a vortex trapping region where the assumption of time stationarity break down.

3. TIME EVOLUTION OF PROBABILITY DENSITIES

A discussion, which is more detailed than the one presented in Sec. 2 for passive test particle diffusion in phase space, can be presented in terms of Boltzmann's collisionless transport equation, or the Vlasov equation, valid for an individual realization, here written in dimensionless units for simplicity

$$\partial_t f + v\partial_x f + E\partial_v f = 0 \ , \tag{24}$$

where f = f(x,v,t) is the test particle phase space distribution, while E is the fluctuating electric field. We again used the notation $\partial_t f \equiv \partial f/\partial t$ etc. for brevity. In general we consider f to be a continous function corresponding to a smeared-out charge distribution (the Vlasov limit). A δ-function distribution is considered as the limiting case of just one charged test particle, subject to the (dimensionless) random force E(x,t).

A. Single Particle Or Absolute Diffusion

The probability density for finding a particle in a narrow interval around (x,v) at a time t is denoted P(x,v,t). This quantity is related to the solution of (24) with the initial condition $f(x,v,0) = \delta(x-x_o)\delta(v-v_o)$ by $P(x,v,t) = \langle f(x,v,t)\rangle$, where again the averaging is performed over all realizations of E(x,t). In order to obtain a differential equation for P we decompose $f = P+\tilde{f}$ and assume $\langle E\rangle = 0$, with f(x,v,0) being statistically independent of E for all t. Obtaining

$$\partial_t P + v\partial_x P + \langle E\partial_v \tilde{f}\rangle = 0 \ , \tag{25}$$

from (24) we approximate

$$\partial_t \tilde{f} + v\partial_x \tilde{f} + E\partial_v P = 0 \ ,$$

giving

$$\tilde{f}(x,v,t) = -\int_0^t ds \int\int_{-\infty}^{\infty} dy du G_0(x,v,t;y,u,s) E(y,s)\partial_u P(y,u,s) \ , \tag{26}$$

since $\tilde{f}(x,v,0) = 0$. In this approximation, interaction at fluctuations with fluctuations in the individual realizations are neglected completely. Fluctuations are directly coupled only to average field (which is incidentially zero here by assumption) and P. For later reference we intorduced the "free-streaming" Green function $G_0(x,v,t;x',v',t') = \delta(x-x'-v[t-t'])\delta(v-v')$, which is a solution to $(\partial_t + v\partial_x)G_0(x,v,t;x',v',t') = 0$ with the initial condition $G_0(x,v,t';x',v',t') = \delta(x-x')\delta(v-v')$. Following Orszag and Kraichnan [7] we insert (26) in (25) to obtain the result corresponding to the first order smoothing approximation:

$$\partial_t P(x,v,t) + v\partial_x P(x,v,t) - \partial_v \int_0^t ds \int\int_{-\infty}^{\infty} dy du G_0(x,v,t;y,u,s)\langle E(x,t)E(y,s)\rangle \partial_u P(y,u,s) = 0 \ ,$$

(27)

or, using the explicit expression for G_0

$$\partial_t P(x,v,t) + v\partial_x P(x,v,t) - \partial_v \int_0^t ds < E(x,t)E(\chi,s) > \partial_v P(v,\chi,s)$$

$$+ \partial_v \int_0^t ds(t-s) < E(x,t)E(\chi,s) > \partial_\chi P(v,\chi,s) = 0$$

where $\chi \equiv x-v(t-s)$.

The free-streaming propagator relates the Eulerian correlation function to an approximation for its Lagrangian counterpart needed in the exact solution for f. As in Sec. 2 we now assume that $<E(x,t)E(x-v(t-s),s)>$ is at least to an approximation time stationary and in addition let the correlation time τ_c be small. With these approximations we reduce (27) to a Fokker-Planck equation of the form

$$\partial_t P + v\partial_x P = D\partial_v^2 P \ , \ t \gg \tau_c \tag{28}$$

where D is obtained from the integral in (27) in the limit $t \to \infty$. We assume that D can be considered constant. It is readily demonstrated that a Gaussian form for P is a solution to (28) with

$$<x^2> = \tfrac{2}{3} Dt^3 \tag{29a}$$

$$<v^2> = 2\,Dt \tag{29b}$$

$$<xy> = Dt^2 \tag{29c}$$

giving in particular $<xy>/[<x^2><y^2>]^{\frac{1}{2}} = \frac{1}{2}\sqrt{3}$. The results of Sec. 2 are thus recovered from the formulation (27) which is more general. In particular it accounts for the nonstationary Lagrangian autocorrelation function expected for the present type of problems. With the assumption that P is a slowly varying function as compared to the integral kernel in (27), we may thus obtain the approximation [8]

$$\partial_t P + v\partial_x P = \partial_v C_0(x,v,t)\partial_v P + \partial_v C_1(x,v,t)\partial_x P \tag{30}$$

with

$$C_0(x,v,t) \equiv \int_0^t <E(x,t)E(x-v(t-s),s)>ds$$

and

$$C_1(x,v,t) \equiv \int_0^t (t-s)<E(x,t)E(x-v(t-s),s)>ds \ .$$

Integration over x gives, with the assumption of short correlation times for the fluctuations in electric field,

$$\partial_t P = \partial_v C_0(v)\partial_v P \ , \tag{31}$$

implying $d_t<v> = <d_v C_0(v)>$ and $d_t<v^2> = 2<d_v[vC_0(v)]>$ where $<v>$ indicates the average velocity of a particle released at $t=0$ and $<v^2>$ its velocity spread. The latter expression reproduces (29b) when C_0 is independent of v. For the more general case covered by (27), (30) and (31) we find $<v> \neq 0$ as discussed in Sec. 2. Referring to the discussion of Sec. 2.B we may thus take the velocity variation of C_0 as evidence for plasma clumping. The quasi time stationary limit of Sec. 2.C is obtained for velocity regions where the v-dependence of C_0 is weak or absent.

The probability density P can be considered as the transition probability $P(x,v,t;x_0,v_0,0)$ for a particle released at (x_0,v_0) i.e. when the particle is known with certainty to be at x_0,v_0 at $t=o$ then $P(x,v,t)$ gives the probability of finding it in a narrow interval around x,v at a time t. With this interpretation the result (27) is readily generalized to include the random coupling approximation of Orszag and Kraichnan [7] giving the renormalized nonlinear differential equation

$$\partial_t P(x,v,t;x_0,v_0,0) + v\partial_x P(x,v,t;x_0,v_0,0)$$

$$- \partial_v \int_0^t ds \int\int_{-\infty}^{\infty} dydu P(x,v,t;y,u,s)<E(x,t)E(y,s)>\partial_u P(y,u,s;x_0,v_0,0) = 0 \ . \tag{32}$$

Physically this expression amounts to replace the Green function corresponding to the straight line phase-space orbit in (27) with the one describing the average orbit giving an expected increase in accuracy at the resulting equation for the probability density. Whether this increase in accuracy is sufficient to compensate the drastic increase in complexity of the equations is still an open question.

By use of (26) an expression for the quantity $<f(x_1,v_1,t_1)f(x_2,v_2,t_2)>$ can also be given. Here we consider the simpler case

$$<|f(x,v,t)|^2> = \int_0^t \int_0^t ds\, d\iota < E(x-v(t-s),s)E(x-v(t-\iota),\iota)>$$

$$\times [\partial_v P(x-v(t-s),s)][\partial_v P(x-v(t-\iota),\iota)] \ , \tag{33}$$

where a more general expression was derived by Orszag and Kraichnan [7]. The simple expression (33) sufficies for the approximation in the diffusion limit of quasi time stationary turbulence. Here we have

$$\frac{<f^2>}{P^2} \approx 2D\left[\frac{v}{2Dt} + 2\frac{x}{Dt^2}\right] = \frac{v}{t} + 4\frac{x}{t^2} \ . \tag{34}$$

The normalized value of $<f^2>$ thus decreases as $1/t$ for $t \to \infty$. More generally we have according to (33) that $<f^2>$ is large where $\partial_v P$ is large which would be also intuitively expected.

B. Two Particle Relative Diffusion

Joint probability densities as $P(x_1,v_1,\ldots\ldots.x_N,v_N,\ t)$ can be analyzed in a very similar way. As an illustration consider $P(x_1,v_1,x_2,v_2,t)$ for two simultaneously released particles, and obtain

$$\partial_t P(x_1,v_1,x_2,v_2,t) + v_1\partial_{x_1} P(x_1,v_1,x_2,v_2,t) + v_2\partial_{x_2} P(x_1,v_1,x_2,v_2,t) =$$

$$\partial_{v_1}\int_0^t ds < E(x_1,t)E(x_1-v_1(t-s),s) > \partial_{v_1} P(x_1-v_1(t-s),v_1,x_2,v_2,s)$$

$$+\ \partial_{v_1}\int_0^t ds < E(x_1,t)E(x_2-v_2(t-s),s > \partial_{v_2} P(x_1,v_1,x_2-v_2(t-s),v_2,s)$$

$$+\ \partial_{v_2}\int_0^t ds < E(x_2,t)E(x_1-v_1(t-s),s) > \partial_{v_1} P(x_1-v_1(t-s),v_1,x_2,v_2,s)$$

$$+\ \partial_{v_2}\int_0^t ds < E(x_2,t)E(x_2-v_2(t-s),s > \partial_{v_2} P(x_1,v_1,x_2-v_2(t-s),v_2,s) \ . \tag{35}$$

With the assumption of quasi-stationarity and short correlation times as before, (35) is reduced to

$$\partial_t P + v_1 \partial_{x_1} P + v_2 \partial_{x_2} P = D[\partial^2_{v_1} P + \partial^2_{v_2} P] + 2W(x_1 - x_2)\partial_{v_1}\partial_{v_2} P \tag{36}$$

where

$$D = \int_0^\infty ds \langle E(x,t)E(x,s)\rangle$$

as before and

$$W(x_1 - x_2) = \int_0^\infty ds \langle E(x_1,t)E(x_2,s)\rangle \quad ,$$

where we made use of the spatial homogeneity of the turbulence. Introducing $R = \frac{1}{2}(x_1 + x_2)$, $r = x_1 - x_2$, $V = \frac{1}{2}(v_1 + v_2)$ and $u = v_1 - v_2$ we obtain

$$\partial_t P + V\partial_R P + u\partial_r P = \tfrac{1}{2}[D + W(r)]\partial^2_V P + 2[D - W(r)]\partial^2_u P \quad . \tag{37}$$

Integration with respet to R and V yields [9]

$$\partial_t P + u\partial_r P = 2[D - W(r)]\partial^2_u P \quad , \tag{38}$$

where now $P = P(u,r,t)$. For very large particle separations we expect $W(r \to \infty) \to 0$. In this limit the expression (28) is recovered with the double value for the diffusion coefficient, thus reproducing the known result that two independent particles diffuse with respect to each other with twice the diffusion coefficient for single particle diffusion. For very close particles we have on the other hand $W(r \to 0) \to D$ according to the definition of W, demonstrating a very slow relative diffusion, i.e. the two particles follow almost identical trajectories in this limit. To describe this limit in more detail we approximate $W(r) = D\text{-}Br^2$ for small r. For nearby particles the relative diffusion is controlled by the micro-length scale which is derived from the curvature of the correlation function at the origin. In this limit we reduce (38) to

$$\partial_t P + u\partial_r P = 2Br^2\partial^2_u P \quad . \tag{39}$$

From this equation we find that $<r^2>$, $<u^2>$ and $<ru>$ all follow the same differential equation e.g

$$\partial_t^3 <r^2> = 8B <r^2>$$

with

$$<r^2> \equiv \int_{-\infty}^{\infty} r^2 P(u,r,t) dr du \ .$$

Consequently we have

$$\begin{matrix} <r^2> \\ <u^2> \\ <ur> \end{matrix} \quad = C_1 e^{\beta t} + C_2 e^{-\frac{1}{2}\beta t} \cos\left(\frac{\sqrt{3}}{2}\beta t\right) + C_3 e^{-\frac{1}{2}\beta t} \sin\left(\frac{\sqrt{3}}{2}\beta t\right) \qquad (40)$$

with $\beta^3 = 8B$. Differences in the time evolutions of $<r^2>$, $<u^2>$ and $<ru>$ thus originate from the inital conditions which determine C_1, C_2 and C_3. For instance with $<r^2> = r_o^2$ at $t=0$ while $<u^2> = <ur> = 0$ one has [9]

$$<r^2> = \tfrac{1}{3} r_o^2 \left\{ e^{\beta t} + 2e^{-\frac{1}{2}\beta t} \cos\left(\frac{\sqrt{3}}{2}\beta t\right) \right\}$$

$$<u^2> = \frac{1}{6}\beta^3 r_o^2 \left\{ e^{\beta t} - e^{-\frac{1}{2}\beta t} \left[\cos\left(\frac{\sqrt{3}}{2}\beta t\right) - \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\beta t\right) \right] \right\}$$

$$<ru> = \frac{\beta}{6} r_o^2 \left\{ e^{\beta t} - e^{-\frac{1}{2}\beta t} \left[\cos\left(\frac{\sqrt{3}}{2}\beta t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\beta t\right) \right] \right\} \ .$$

In all cases we find the contribution $e^{\beta t}$ to dominate $t \to \infty$. This result for the exponential trajectory separation provides a quantity β which may be interpreted as a Lyapunov exponent characterizing the turbulence as discussed e.g. by Misguich and Balescu [6]. The importance of this result should, however, not be overemphasized. First of all (37) - (39) were obtained under rather idealized conditions. In addition we expect the renormalization to be particularly important for the problem of relative diffusion. For r_o much smaller than the Eulerian integral length scale we expect, however, (40) to be rather accurate in a transient time period.

The results for multiple joint probability densities $P(x_1,v_1, \ldots\ldots x_N,v_N,t)$ can, as already mentioned, be obtained by a straightforward generalization of these results. Also the case where a continuous cloud of test fluid (rather than particles) is released initially in phase-space can be analyzed. The result for $P(x_1,v_1,x_2,v_2,t)$ can only be considered as representative for this problem in a qualitative way. The problems are principally different. Thus the N-particle problem depends essentially only on the correlations between N phase-space positions. For a continuous cloud the result is in principle sensitive to correlations between any set of phase-space positions. We shall not discuss this problem further here. The present summary is restricted to one spatial dimension. The results are, however, readily generalized to higher dimensions.

4. CONDITIONALLY RELEASED TEST PARTICLES

It can be argued that the standard analysis of test particle diffusion in the form discussed here is too crude for many problems of interest for plasma physics. The previous results are thus not particularly informative for discussions of particle trapping in turbulent plasmas since for instance the expressions (27) or (32) do not discriminate particles released in the vicinity of a local potential maximum or minimum. Likewise it can be argued that the relative motion of two particles released at each side of a local potential maximum is not particularly interesting, i.e. it is rather selfevident that they will move apart. It is much more interesting to learn how long a potential well is able to guide two particles along nearby orbits. As this short discussion may indicate there are good reasons to consider the turbulent diffusion of conditionally released particles, i.e. to give a statistical analysis of particle dynamics in a conditionally chosen subensemble subject to certain conditions on the electric field at the initial particle position x_o at $t = 0$ such as $E(x_o,0) = 0$ and $\partial_x E(x = x_o, 0) < 0$, which defines a local potential well.

The analysis summarized in the foregoing sections referred solely to unconditionally released particles. The interesting point is however that the procedure can be directly applied to conditionally released particles also. When considering this latter problem one thus selects a subensemble satisfying certain conditions. Although the full ensemble is assumed to be homogeneous and time stationary, then the subensemble will in general be inhomogeneous and nonstationary. Thus for the unconditional problem we have $\langle E(x_o,t)\rangle = 0$ with x_o being a fixed but arbitrary position, while this is not necessarily so for averages performed over the subensemble. The generalization will here be illustrated by the problem of single particle diffusion. Repeating the procedure giving (27) we write the electric field E = $<E>_c$ + E, where in this case $<E>_c$ $\neq 0$, and obtain

$$[\partial_t + v\partial_x + <E(x,t)>_c\partial_v]G_0(x,v,t,x',v',t') = 0 \qquad (41)$$

to be solved with the initial condition $G_0(x,v,t';x',v't') = \delta(x-x')\delta(v-v')$.With the aid of G_0 determined by (41) we obtain the equation for the transition probability $P_c \equiv <f>_c$,

$$[\partial_t + v\partial_x + <E(x,t)>_c\partial_v]P_c(x,v,t;x_0,v_0,0) =$$

$$\partial_v \int_0^t ds \int\int_{-\infty}^{\infty} dydu G_0(x,v,t;y,u,s)<E(x,t)E(y,s)>_c\partial_u P_c(y,u,s;x_0,v_0,0) \qquad (42)$$

The conditionally averaged electric fields and their correlations entering (41) and (42) are all readily measurable functions. In the present discussion we have not specified the actual condition imposed on the signal, it is not even logically necessary that the conditions are imposed in the position x_0. Arguing as before we may generalize (42) to included the features of the random coupling model by replacing G_0 by P_c itself. The result is thus a nonlinear differential equation for P_c with coefficients expressed in terms of Eulerian averages $<E(x,t)>_c$ and $<E(x,t)E(y,s)>_c$. Note that the latter quantity is explicitly a function of x,y,t and s in contrast to its unconditional counterpart, which can be assumed to be a function of |x-y| and |t-s| only. Analytical expressions in terms of unconditional averages can be obtained also for the conditional correlation function. The generalization of the present analysis to the evolution of joint probability functions is straight forward and need not be summarized here.

5. CONCLUSIONS AND DISCUSSION

These notes review certain basic problems of phase space diffusion in turbulent collisionless plasmas. The analysis was restricted to one dimension, but the analysis is easily generalised. Diffusion in magnetised, i.e. anisotropic, but still homogeneous plasmas, can be analysed by a generalisation of the expressions, where the unperturbed straight line orbits, inherent in the unperturbed Green function G_0, are replaced by the helical orbits of charged particles along straight magnetic field lines. The resulting expressions become rather cumbersome. A difficult but important problem is the analysis of inhomogeneous and nonstationary plasma turbulence. In a way this problem is closely related to that of conditionally released test particles discussed in Sec. 4 and at least formally the results can be readily obtained.

The actual solution of the equations is probably only possible by numerical methods, and is even then rather difficult. A particularly interesting problem is the evaluation of the results for the random coupling model. Intuitively a significant improvement of the results is expected when applied to a strongly turbulent plasma. This conjecture seems however yet unproven and a comparison between theoretical results based on this analysis and particle diffusion in for instance numerical simulations of plasma turbulence would be most valuable.

Finally, it is reemphasized that the present review deals with test particles released in a turbulent field which is assumed to be known in a statistical sense and the analysis relies on e.g. the correlation functions as a priori given. Analytical theories predicting such funtions are outside the scope of this work. It should be mentioned however that the average Green function, or transition probability P in (32), is an essential ingredient of such theoretical studies of plasma turbulence as the random coupling model of Orszag and Kraichman [7] and related works.

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