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SPRING COLLEGE ON PLASMA PHYSICS

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THE MODULATIONAL INSTABILITY OF COLINEAR WAVES

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The Modulational Instability of Colinear Waves

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Brief Derivation of Equations

Consider $\omega(k, |A|^2) A = 0$

Expand about linear frequency ω_0

$$\omega \approx \omega_0(k_0) + \left(\frac{\partial \omega}{\partial k}\right)_0 (k - k_0) + \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial k^2}\right)_0 (k - k_0) + \left(\frac{\partial \omega}{\partial |A|^2}\right)_0 |A|^2$$

Let $\omega - \omega_0 \longrightarrow i\partial_t$, $k - k_0 \longrightarrow -i\partial_z$ and obtain the nonlinear Schroedinger equation

$$\left[i(\partial_t + v\partial_z) + \mu \partial_{zz}^2 + \lambda |A|^2\right] A = 0$$

$$\mu = \frac{1}{2}\left(\frac{\partial^2 \omega}{\partial k^2}\right)_0 \quad , \quad \lambda = -\left(\frac{\partial \omega}{\partial |A|^2}\right)_0$$

For multidimensional waves,

$$\frac{1}{2}\left(\frac{\partial^2 \omega}{\partial k_i \partial k_j}\right)_0 \frac{\partial^2}{\partial x_i \partial x_j}$$

Colinear Waves

Consider $\omega_1(k_1, |A_1|^2, |A_2|^2)\, A_1 = 0$

Proceed as before.

$$\left[i(\partial_t + v_1\partial_z) + \mu_1 \partial^2_{zz} + \lambda_{11}|A_1|^2 + \lambda_{12}|A_2|^2\right] A_1 = 0$$

$$\left[i(\partial_t + v_2\partial_z) + \mu_2 \partial^2_{zz} + \lambda_{21}|A_1|^2 + \lambda_{22}|A_2|^2\right] A_2 = 0$$

Two Coupling Mechanisms

$$\omega_p^2 = \frac{4\pi n_0 e^2}{m} \qquad \omega = (\omega_p^2 + c^2k^2)^{1/2}$$

$$\omega \simeq \omega_p\left(1 + \tfrac{1}{2}k^2\lambda_0^2\right)$$

relativistic nonlinearity

$$m = \frac{m_0}{(1 - v^2/c^2)^{1/2}}, \quad v = v_1 + v_2$$

$$\delta\omega_1 = -\frac{3}{16}\left|\frac{v_1}{c}\right|^2$$

ponderomotive nonlinearity

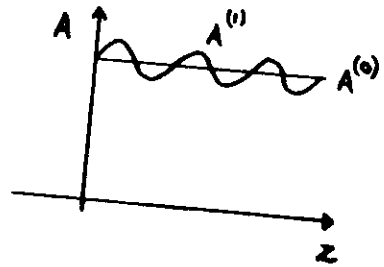
$$\delta n \propto I \propto |A_1|^2 + |A_2|^2$$

$$(\partial^2_{tt} - c_s^2\nabla^2)\,\delta n = \frac{Z m_e}{m_i}\nabla^2\langle A^2\rangle \quad ; \quad \delta n_1 = -\left|\frac{v_1}{v_e}\right|^2$$

Single Wave

Linearise NS equation around a plane-wave equilibrium and assume an amplitude perturbation of the form

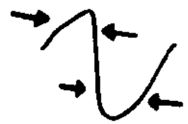
$$A_1^{(1)} \propto e^{\gamma(t - z/v)}\, e^{i\vec{k}\cdot\vec{x}}$$

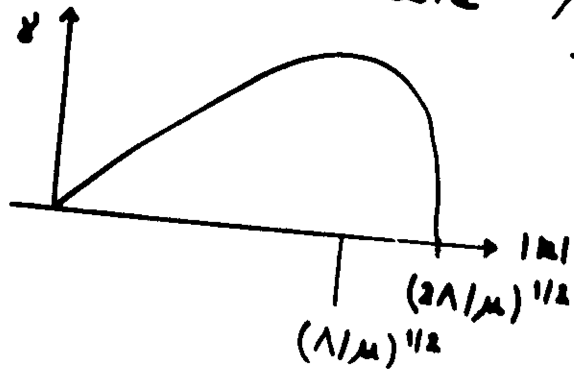


The dispersion relation is

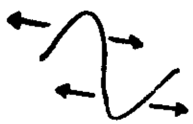
$$\boxed{\gamma^2 - \mu k^2(2\Lambda - \mu k^2) = 0}\ ; \quad \Lambda = \lambda|A|^2$$

If $\mu\lambda > 0$, wave is unstable





If $\mu\lambda < 0$, wave is stable



Coupled Waves

Proceed as before :

$$\begin{cases} \left[\gamma^2 - k^2(2\Lambda_{11} - k^2)\right] A_1^{(1)} = 2\Lambda_{12}\, k^2 A_2^{(1)} \\ \left[\gamma^2 - k^2(2\Lambda_{22} - k^2)\right] A_2^{(1)} = 2\Lambda_{12}\, k^2 A_1^{(1)} \end{cases}$$

which can be rewritten as

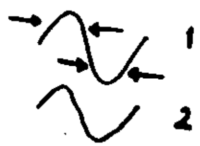
$$\boxed{\left[\gamma^2 - k^2(2\Delta_+ - k^2)\right]\left[\gamma^2 - k^2(2\Delta_- - k^2)\right] A_{1,2}^{(1)} = 0 ,}$$

where

$$\boxed{2\Delta_\pm = (\Lambda_{11} + \Lambda_{22}) \pm \sqrt{(\Lambda_{11} - \Lambda_{22})^2 + 4\Lambda_{12}^2}}$$

Suppose $\Lambda_{11} \gg \Lambda_{22} > 0$. Then

$$\Delta_+ \simeq \Lambda_{11} + \Lambda_{12}^2/\Lambda_{11}$$

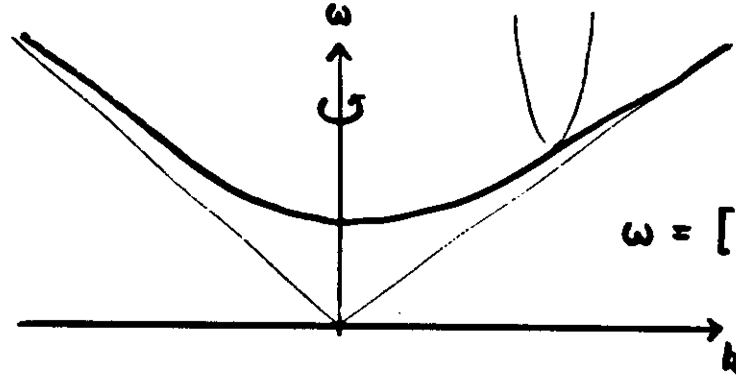


ie larger wave drives smaller wave unstable.

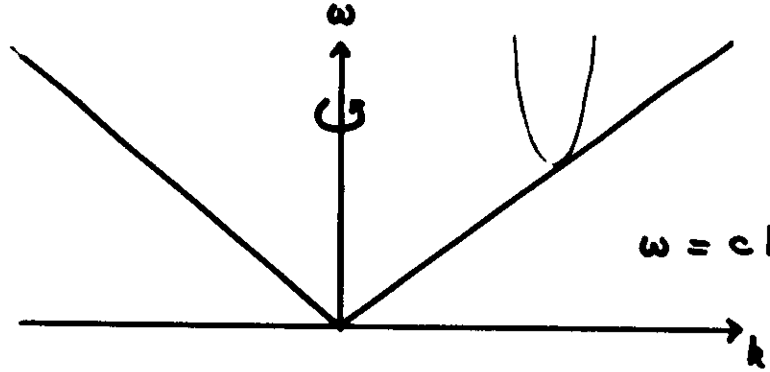
Moreover, $\Delta_+ > 0$ whenever $\boxed{|\lambda_{12}| > |\lambda_{11}\lambda_{22}|^{1/2}}$

even if both waves are stable by themselves!

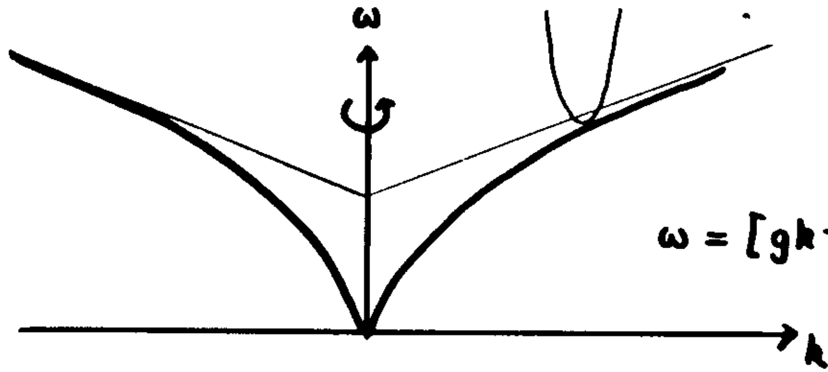
<u>NB</u> independent of sign(λ_{12}) and size of $|A_1|, |A_2|$



$\omega = [\omega_e^2 + c^2k^2]^{1/2}$



$\omega = ck$



$\omega = [gk \tanh(kh)]$

Two Spatial Dimensions

$$\mu \frac{\partial^2}{\partial x^2} \longrightarrow \mu_x \frac{\partial^2}{\partial x^2} + \mu_y \frac{\partial^2}{\partial y^2} \quad ; \quad \mu_x = \left(\frac{\partial^2 \omega}{\partial k_x^2}\right)_0$$

$$\longrightarrow \sigma_x \frac{\partial^2}{\partial x^2} + \sigma_y \frac{\partial^2}{\partial y^2} \quad ; \quad \sigma_x = \mathrm{sign}(\mu_x)$$

linear group velocity $\vec{v} \parallel \vec{x}$

repeat linear analysis with

$$A_{1,2} \propto e^{\gamma(t - x/v)} e^{i(k_x x + k_y y)}$$

$\vec{k} \perp \vec{v}$ corresponds to filamentation

<u>1-wave</u> $\quad \gamma^2 - \sigma_j k_j^2 (2\Lambda - \sigma_j k_j^2) = 0$

γ depends on <u>direction</u> of $\vec{k}$

<u>2-waves</u>

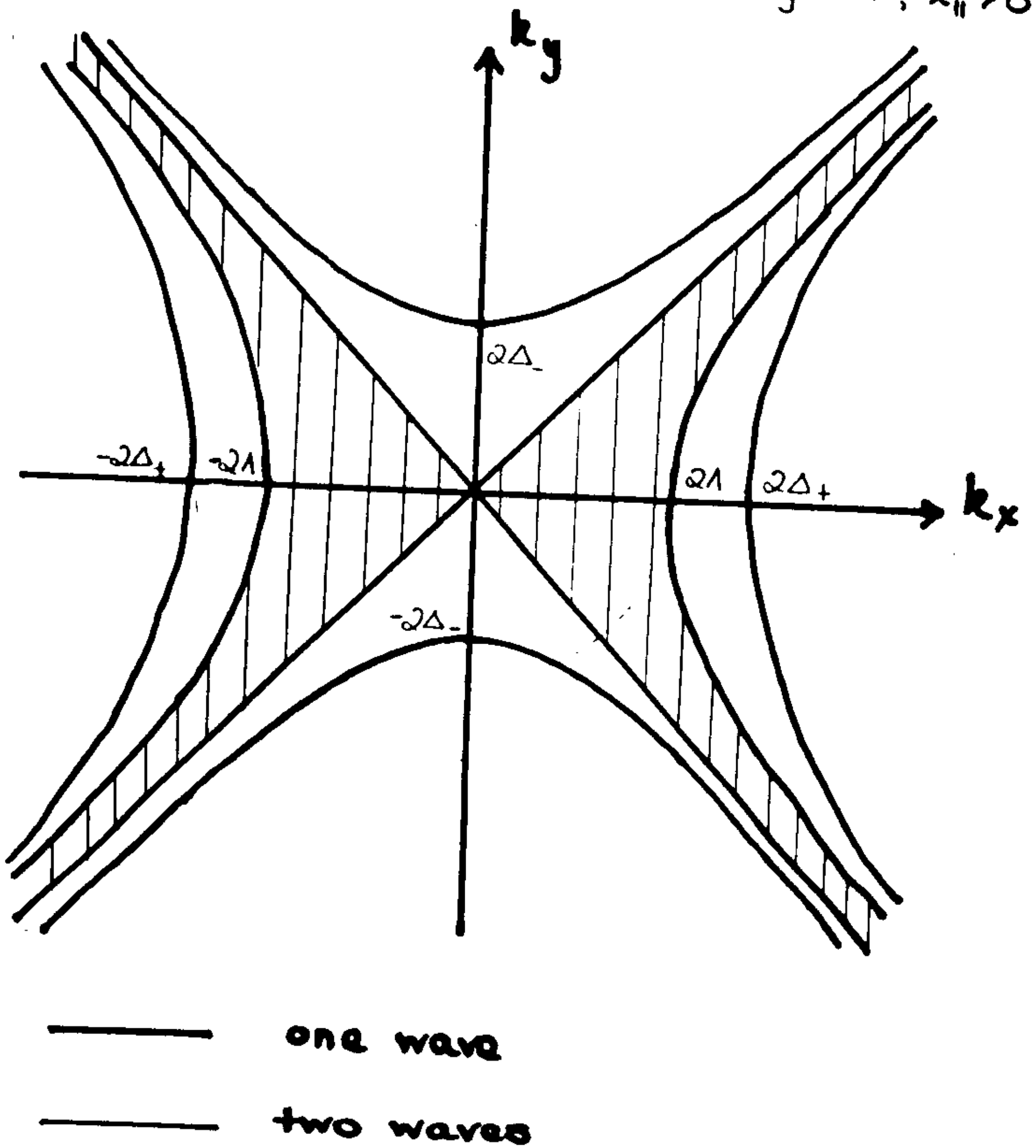
$$\left[\gamma^2 - \sigma_j k_j^2 (2\Delta_+ - \sigma_j k_j^2)\right]\left[\gamma^2 - \sigma_j k_j^2 (2\Delta_- - \sigma_j k_j^2)\right] = 0$$

$\Delta_- < 0 < \Delta_+$ whenever $|\lambda_{12}| > |\lambda_{11} \lambda_{22}|^{1/2}$

Instability for <u>arbitrary</u> direction of $\vec{k}$

Mixed Dispersion

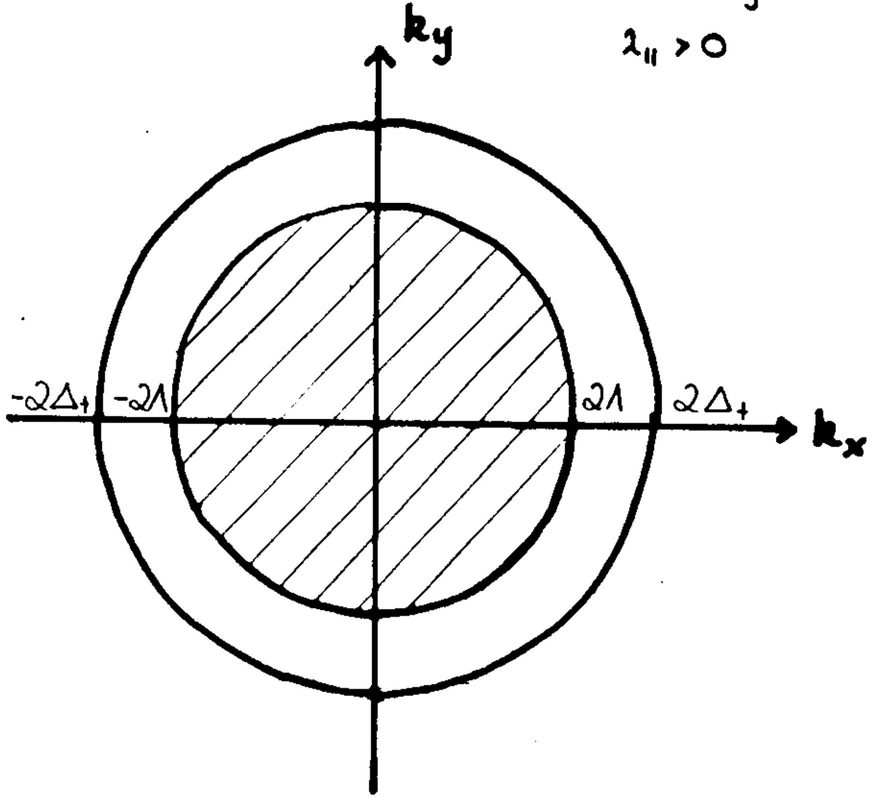
$\sigma_x = 1 ; \ \sigma_y = -1 ; \ \lambda_{11} > 0$



Normal Dispersion

$\sigma_x = \sigma_y = 1$

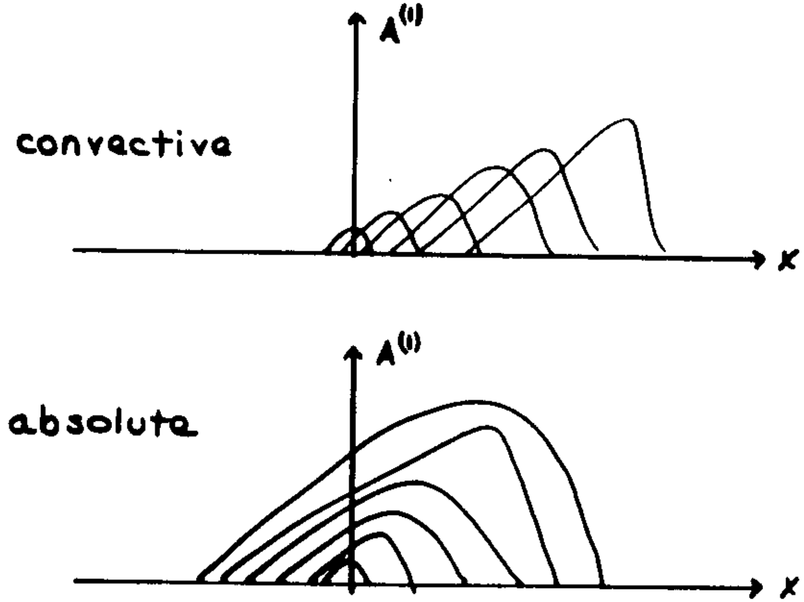
$\lambda_{\parallel} > 0$



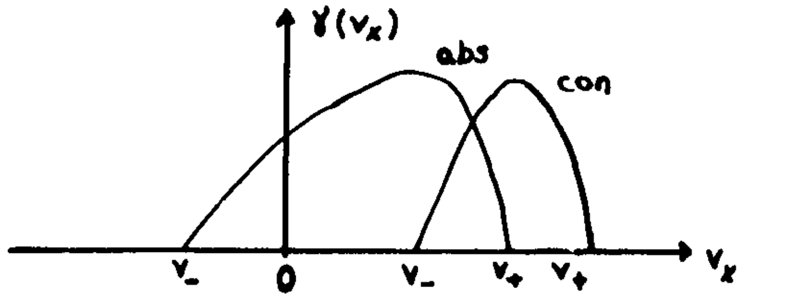
—— boundary of unstable region for one wave

—— boundary of unstable region for two waves

Absolute & Convective Instabilities



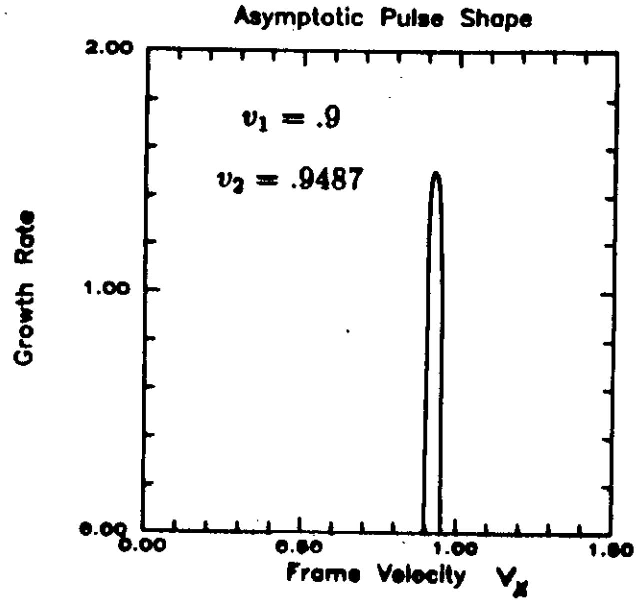
There is a convenient way to represent the unstable pulse shape

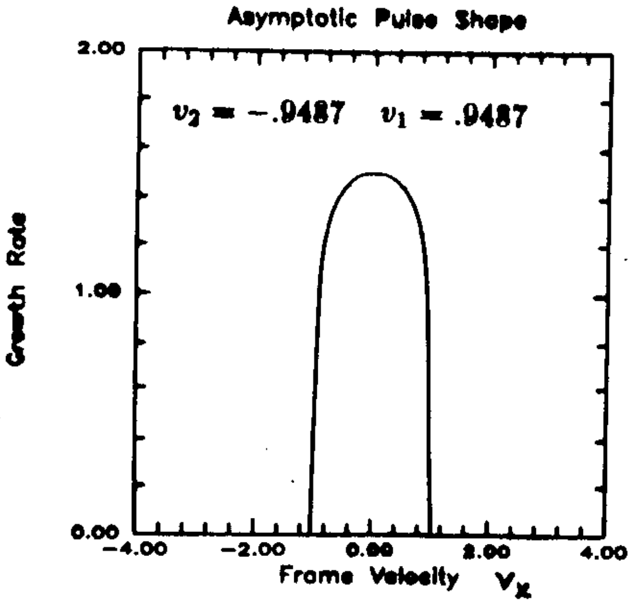


$\gamma(v_x)$ is growth rate in moving frame

$R = 1$

Filamentation





Coupled Modulational Instabilities have been discovered independently by many authors:

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A related paper on nonlinear focusing of coupled waves is McKinstrie & Russell, Phys. Rev. Lett. 61, 2929 (1988).