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SPRING COLLEGE ON PLASMA PHYSICS

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SATURATION AND CROSS FIELD COUPLING OF BEAT-DRIVEN 3-D PLASMA WAVES

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SATURATION AND CROSS FIELD COUPLING OF BEAT-DRIVEN 3-D PLASMA WAVES

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Spring College on Plasma Physics:
" LARGE AMPLITUDE WAVES and FIELDS in PLASMAS "
22-26 May 1989
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COLLABORATION BETWEEN:

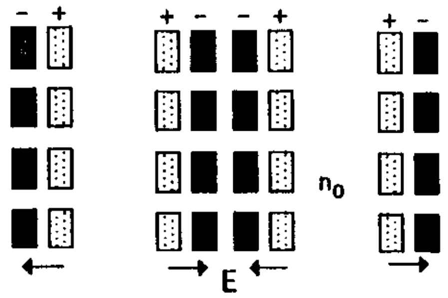
ELECTRICAL ENGINEERING DEPARTMENT, UNIVERSITY of NAPLES: Luciano De Menna, Giovanni Miano;

DEPARTMENT of PHYSICAL SCIENCE, UNIVERSITY of NAPLES: Umberto de Angelis;

RUTHERFORD APPLETON LABORATORY, OXFORD: Robert Bingham,

on:

Three Dimensional Analysis of Plasma Wave Dynamics



$$\omega_p = \left(\frac{4\pi e^2 n_0}{m_e}\right)^{1/2}$$

FREE PLASMA OSCILLATIONS:

L. Tonks, I. Langmuir, Phys. Rev., (1929);

A.I. Akhiezer, V. Polovin, Soviet Phys. JETP, (1956)

J.M. Dawson, Phys. Rev., (1959);

FORCED PLASMA OSCILLATIONS:

BEAT WAVE-DRIVEN PLASMA WAVES (PBW) :

M.N. Rosenbluth, C.S. Liu, Phys. Rev. Lett., 1972;

PLASMA WAKE FIELD (PWF) :

P. Chen , J.M. Dawson, R.W. Huff, T. Katsouleas, Phys. Rev. Lett., 1985;

ELECTROMAGNETIC WAKE FIELD IN PLASMA(EWF):
(or Laser Wake Field (LWF))

P. Sprangle, E. Esarey, A. Ting, G. Joyce, Appl. Phys. Lett.,1988;

E. Esarey, A. Ting, P. Sprangle, G. Joyce, Comments on Plasma Physics and Controlled Fusion, 1989.

"External Electromagnetic Field"

$$(\rho_{ext}, J_{ext}) \rightarrow (E_{ext}, B_{ext})$$

may generate charge separation in plasma, then plasma oscillations and plasma waves.

TWO FLUID MODEL :

- cold and collisionless plasma;
- infinitely massive ions;
- unmagnetized plasma;
- relativistic electrons;

- electron momentum equation:

$$\left(\frac{\partial}{\partial t} + v\cdot\nabla\right)p = -e\,(E + E_{ext}) - e\frac{v}{c}\times(B + B_{ext}),$$

$$p = m_e\gamma v \quad , \quad \gamma = (1 - v^2/c^2)^{-1/2} ,$$

- electron continuity equation:

$$\frac{\partial n}{\partial t} + \operatorname{div}(nv) = 0 ,$$

- Maxwell equations:

$$\operatorname{rot} E = -\frac{1}{c}\frac{\partial B}{\partial t} , \qquad \operatorname{div} B = 0$$

initial conditions for the electromagnetic field

$$\operatorname{rot} B = \frac{1}{c}\frac{\partial E}{\partial t} - \frac{4\pi}{c} e\,nv . \qquad \operatorname{div} E = 4\pi e(n_0 - n)$$

WEAK RELATIVISTIC LIMIT:

$$\frac{\partial v}{\partial t} + \frac{e}{m_e}E = -\frac{1}{2}\operatorname{grad}(v^2) - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{v^2}{c^2}v\right) + \frac{1}{m_e}F_{ext} ,$$

$$\operatorname{rot} E + \frac{1}{c}\frac{\partial B}{\partial t} = 0 ,$$

$$\operatorname{rot} B - \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} e\, n_0 v = -\frac{4\pi}{c} e\,\delta n\, v ,$$

$$\operatorname{div} E + 4\pi e\,\delta n = 0 ,$$

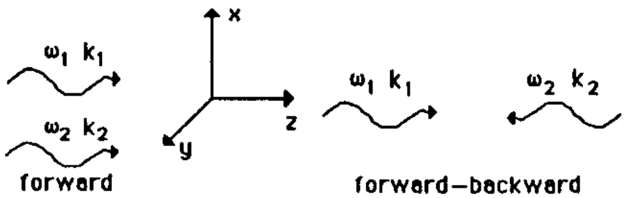
where $\delta n \equiv n - n_0$.

By using momentum and Maxwell equations we obtain:

$$B + B_{ext} = \operatorname{rot}\left(\frac{e}{c}p\right).$$

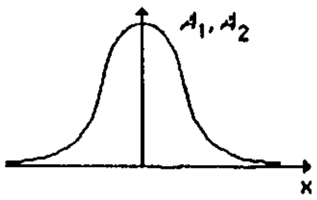
The external force F_{ext} depends on the particular kind of excitation: F_{ext} is assumed known.

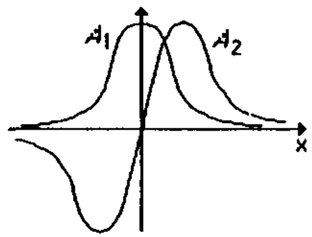
BEAT-WAVE DRIVEN PLASMA WAVES:



$$A_0(r;t) = \sum_i A_{0i}\, \mathcal{A}_i(r,t)\, [\cos(k_i z - \omega_i t + \phi_{0i})\ e_x + \sin(k_i z - \omega_i t + \phi_{0i})\ e_y]$$

$\mathcal{A}_i = \mathcal{A}_i(r,t)$ describes the spatial-time envelope of the i^{th}-electromagnetic wave





$$F_{ext} = -e\ \mathrm{grad}\Psi_{NL},$$

$$\Psi_{NL} = \frac{m_e c^2}{e} v^2(r;t) \cos(k_p z - \Delta\omega t + \Delta\Phi_0),\ \Delta\omega \equiv \omega_p \text{ (resonant interaction)},$$

where:

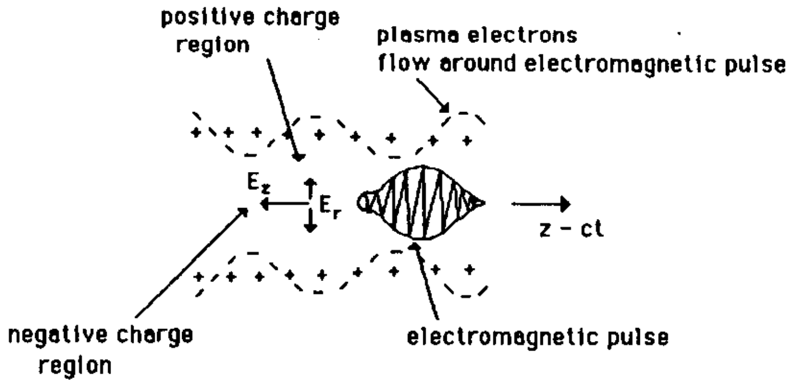
$$a_i = \frac{eA_{0i}}{m_e c^2}\ ,\quad v_0^2 = a_1 a_2\ ,\quad v^2(r,t) \equiv \mathcal{A}_1(r,t)\mathcal{A}_2(r,t) v_0^2,$$

$$\Delta\omega = \omega_1 - \omega_2,\quad \Delta\Phi_0 = \Phi_{01} - \Phi_{02}\ .$$

Forward propagation $\quad k_p = k_1 - k_2$.

Forward-Backward propagation $\quad k_p = k_1 + k_2$.

PLASMA ELECTROMAGNETIC WAKE FIELD



$$A_0(r;t) = A_0 \mathcal{A}(r,t)\, [\cos(kz - \omega t + \phi_0)\ e_x + \sin(kz - \omega t + \phi_0)\ e_y]$$

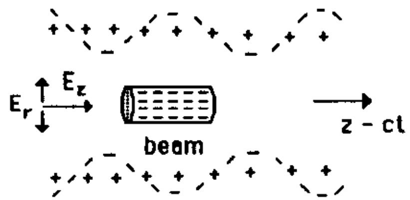
$$F_{ext} = -e\ \mathrm{grad}\Psi_{NL},$$

$$\Psi_{NL} = \frac{m_e c^2}{e}\frac{v^2}{2}(r;t) \text{ (no-resonant interaction)},$$

where:

$$a = \frac{eA_0}{m_e c^2}\ ,\quad v_0^2 = a^2\ ,\quad v^2(r,t) \equiv \mathcal{A}^2(r,t) v_0^2.$$

PLASMA WAKE FIELD



$$\rho_{ext}(r,t) = q n_{beam}(r,t) \ \rightarrow E_{ext}$$

$$F_{ext} = -e\, E_{ext}$$

$$\mathrm{div}F_{ext} = -4\pi e q\, n_{beam}(r,t),\quad \mathrm{rot}F_{ext} \neq 0.$$

LINEAR THEORY

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{e}{m_e}\mathbf{E}=\frac{1}{m_e}\mathbf{F}_{NL},$$

$$\mathrm{rot}\,\mathbf{E}+\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}=0,$$

$$\mathrm{rot}\,\mathbf{B}-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}+\frac{4\pi}{c}e\,n_0\mathbf{v}=0,$$

$$\mathrm{div}\mathbf{E}+4\pi e\,\delta n=0.$$

with the initial conditions:

$$\mathbf{v}(t=0)=0, \quad \mathbf{B}(t=0)=0, \quad \mathbf{E}(t=0)=0.$$

We may decompose the field quantities in the following manner (Helmholtz H.):

$$\mathbf{v}=\mathbf{v}_{irr}+\mathbf{v}_{sol}, \quad \mathbf{B}=\mathbf{B}_{sol}, \quad \mathbf{E}=\mathbf{E}_{irr}+\mathbf{E}_{sol},$$

$\mathbf{v}_{irr}$, $\mathbf{E}_{irr}$, $(\mathbf{F}_{NL})_{irr}$: curl-free components; $\mathbf{v}_{sol}$, $\mathbf{B}_{sol}$, $\mathbf{E}_{sol}$, $(\mathbf{F}_{NL})_{irr}$: div-free components.

Equations for the curl-free components:

$$\frac{\partial \mathbf{v}_{irr}}{\partial t}+\frac{e}{m_e}\mathbf{E}_{irr}=\frac{1}{m_e}(\mathbf{F}_{NL})_{irr},$$

$$\frac{1}{c}\frac{\partial \mathbf{E}_{irr}}{\partial t}-\frac{4\pi}{c}e\,n_0\mathbf{v}_{irr}=0,$$

Equations for the div-free components:

$$\frac{\partial \mathbf{v}_{sol}}{\partial t}+\frac{e}{m_e}\mathbf{E}_{sol}=\frac{1}{m_e}(\mathbf{F}_{NL})_{sol},$$

$$\mathrm{rot}\,\mathbf{E}_{sol}=-\frac{1}{c}\frac{\partial \mathbf{B}_{sol}}{\partial t},$$

$$\mathrm{rot}\,\mathbf{B}_{sol}=\frac{1}{c}\frac{\partial \mathbf{E}_{sol}}{\partial t}-\frac{4\pi}{c}e\,n_0\mathbf{v}_{sol}.$$

The curl-free component of the electric field is solution of the ordinary differential equation:

$$\frac{\partial^2 \mathbf{E}_{irr}}{\partial t^2}+\omega_p{}^2\mathbf{E}_{irr}=\frac{4\pi e n_0}{m_e}(\mathbf{F}_{NL})_{irr},$$

with the initial conditions: $\mathbf{E}_{irr}=0$, $\frac{\partial \mathbf{E}_{irr}}{\partial t}=0$.

The div-free component of the electric field is solution of the wave differential equation:

$$\nabla^2\mathbf{E}_{sol}-\frac{1}{c^2}\frac{\partial^2 \mathbf{E}_{sol}}{\partial t^2}-\frac{\omega_p{}^2}{c^2}\mathbf{E}_{sol}=-\frac{\omega_p{}^2}{c^2}\frac{(\mathbf{F}_{ext})_{sol}}{e}$$

with the initial conditions: $\mathbf{E}_{sol}=0$, $\frac{\partial \mathbf{E}_{sol}}{\partial t}=0$ and with suitable boundary conditions.

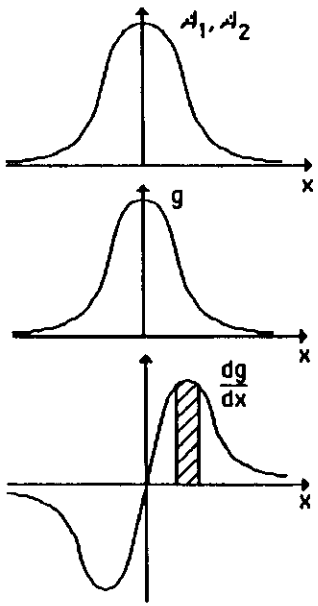
BEAT-WAVE DRIVEN PLASMA WAVE

$$E = - E_D \cdot (\omega_p t) \frac{1}{\beta_f^2 k_p} \mathrm{grad}\left[\frac{v^2(r,t)}{2} \sin(k_p z - \omega_p t + \Delta\Phi_0) \right],$$

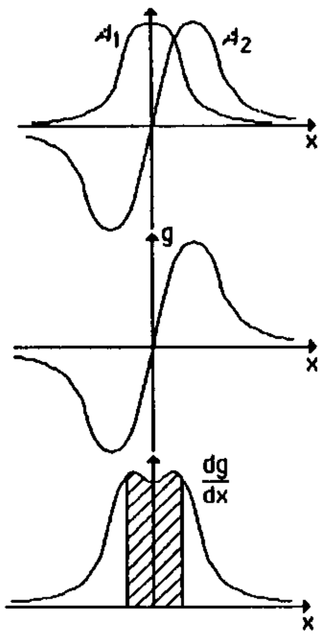
- E_D Dawson limit electric field: $E_D \equiv 4\pi e n_0/k_p$;
- $\beta_f \equiv \omega_p/(k_p c)$.

If $v^2(r,z)$ does not depend on z coordinate (in the waist of the external electromagnetic waves) then we obtain for $v^2 = v_0^2 g(x)$:

$$E = \begin{cases} -E_D \dfrac{v_0^2}{2\beta_f^2 k_p} \dfrac{dg}{dx} \cdot (\omega_p t) \sin(k_p z - \omega_p t + \Delta\Phi_0)\, e_x , \\ -E_D \dfrac{v_0^2}{2\beta_f^2} g(x) \cdot (\omega_p t) \cos(k_p z - \omega_p t + \Delta\Phi_0)\, e_z . \end{cases}$$



Fedele R. et al., Phys. Rew. (1986).



Miano G., Ph.D. dissertation, (1988).

AN APPROXIMATE NONLINEAR MODEL

(Miano G., de Angelis U., Bingham R., Plasma Phys. and Controlled Fusion, accepted for publication, 1989)

$$\frac{\partial v}{\partial t} + \frac{e}{m_e} E = -\frac{1}{2} \frac{\partial}{\partial t} \left(\frac{v^2}{c^2} v \right) + \frac{1}{m_e} F_{NL} ,$$

$$\frac{1}{c} \frac{\partial E}{\partial t} - \frac{4\pi}{c} e n_0 v = 0 ,$$

then we obtain

$$\frac{d^2 E}{dt^2} + \frac{1}{8(4\pi e n_0 c)^2} \frac{d}{dt} \left[\left(\frac{dE}{dt} \cdot \frac{dE}{dt} \right) \frac{dE}{dt} \right] + \omega_p^2 E = \frac{4\pi e n_0}{m_e} F_{NL} , \quad (A)$$

with the initial conditions:

$E(r,z;t=0) = 0,$

$\frac{dE}{dt}\Big|_{t=0} = 0 .$

Gaussian transverse profile

$\varepsilon \equiv \beta_f^{-2/3} \nu^{4/3}$, $T \equiv \Delta\omega\, t$, $f \equiv \frac{\omega_p}{\Delta\omega} = 1 + \varepsilon\sigma$, $R \equiv k_p r$, $Z \equiv k_p z$,

$E_r \equiv E_D\, \beta_f^{-4/3} \nu^{2/3}\, v(R,Z;T)$, $E_z \equiv E_D\, \beta_f^{-4/3} \nu^{2/3}\, u(R,Z;T)$.

In the limit $\varepsilon \ll 1$ eq.(A) becomes:

$$\ddot{v} + [\,1 + \varepsilon(\sigma - \tfrac{3}{2}\dot{v}^2 - \tfrac{1}{2}\dot{u}^2\,)]v - \varepsilon\, \dot{u}\cdot\dot{v}\, v = \frac{\varepsilon\alpha}{2}\cos(\,Z_0 - T + \Delta\Phi_0)\,,$$

(B)

$$\ddot{u} + [\,1 + \varepsilon(\sigma - \tfrac{3}{2}\dot{u}^2 - \tfrac{1}{2}\dot{v}^2\,)]u - \varepsilon\, \dot{u}\cdot\dot{v}\, u = \frac{\varepsilon}{2}\sin(\,Z_0 - T + \Delta\Phi_0)\quad,$$

where $\alpha \equiv -\frac{d\nu^2}{dR}\frac{1}{\nu^2}$.

Initial conditions:

$v(T{=}0) = 0,\quad \dot{v}(T{=}0) = 0,$

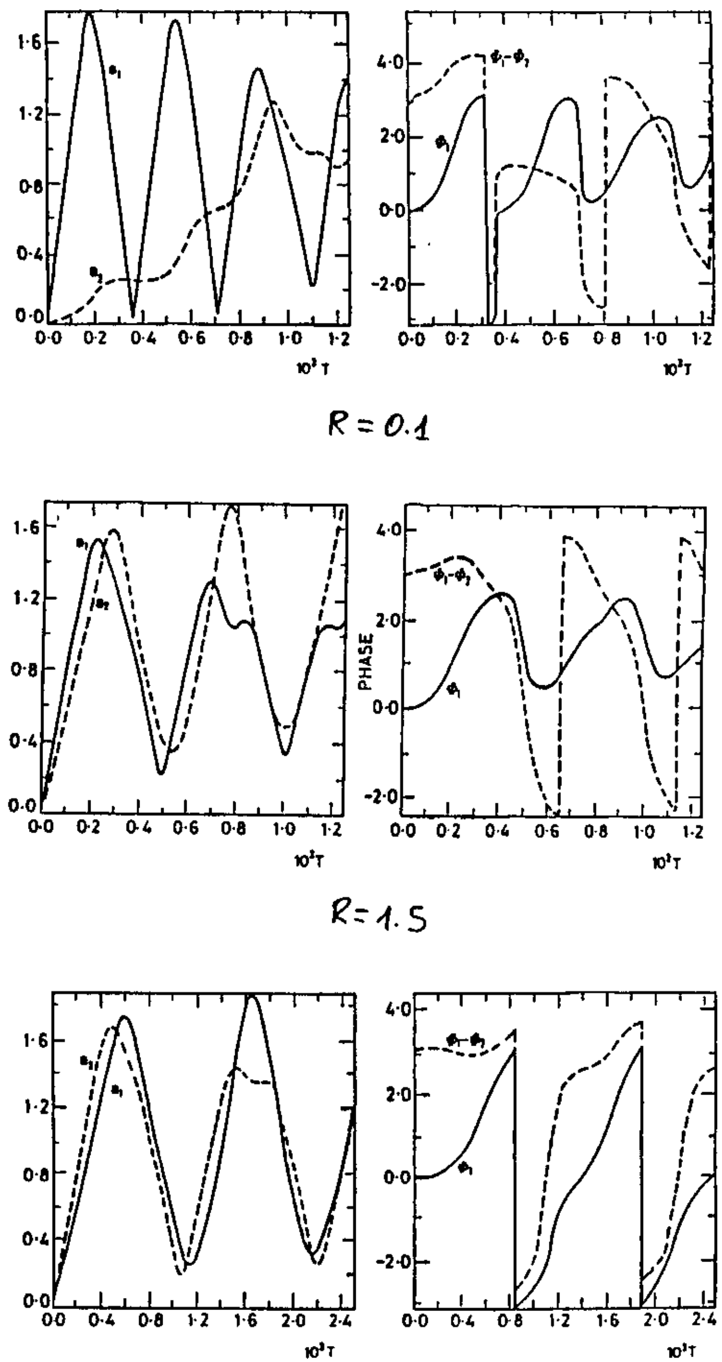
$u(T{=}0) = 0,\quad \dot{u}(T{=}0) = 0.$

We try a solution in the form (variation parameter method):

$$u(R,Z;T) = a_1(R;T)\cos[\,Z - T + \Phi_1(R;T) + \Delta\Phi_0]\,,$$

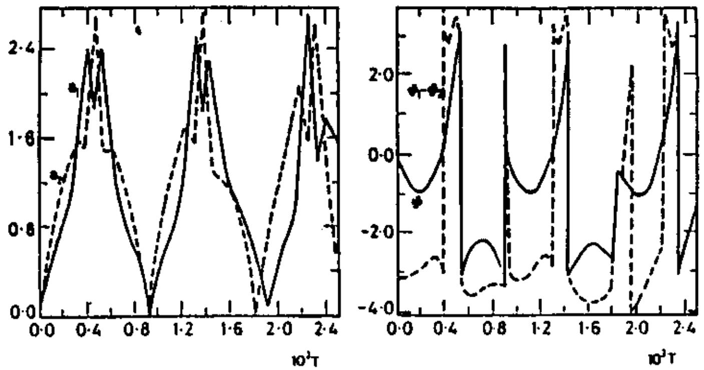
$$v(R,Z;T) = a_2(R;T)\sin[\,Z - T + \Phi_2(R;T) + \Delta\Phi_0]\,.$$

$\sigma = 0\,,\ \gamma_0^2 = 0.01$

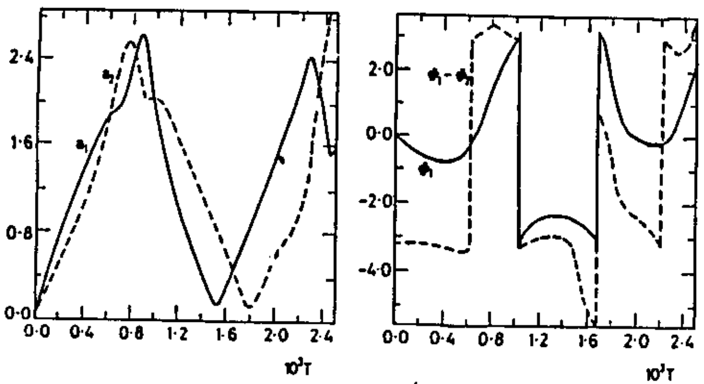


D – 2

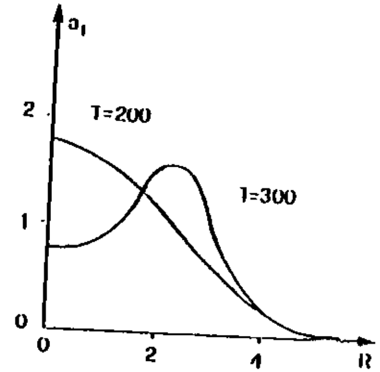
$\sigma = 1 \qquad \gamma_0^2 = 0.01$



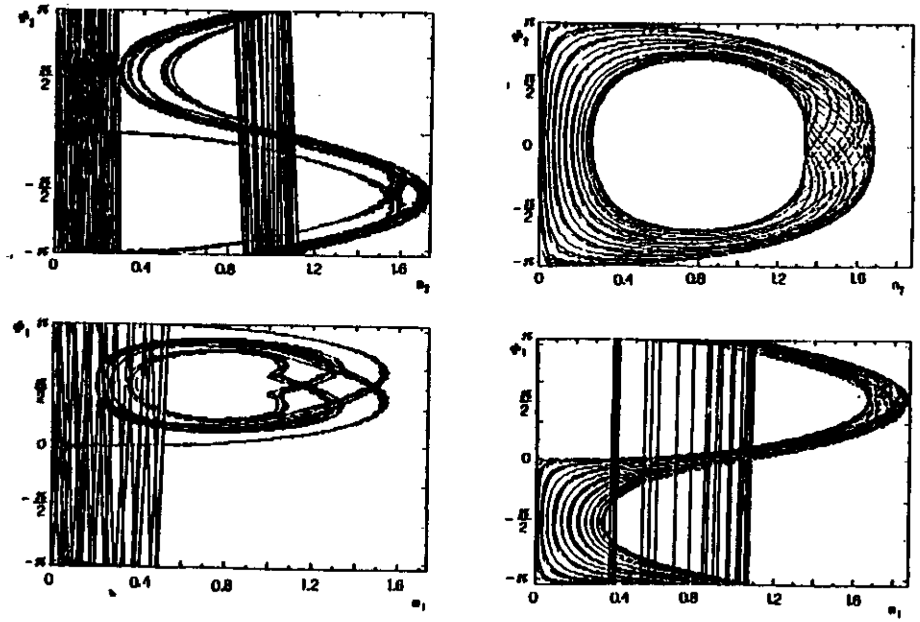
$R = 1.5$



$R = 3$



$\sigma = 0 \ , \ \gamma_0^2 = 0.01$



$\gamma_0^2 = 0.01 \ , \sigma = 0, \ R = 1.5 \qquad \gamma_0^2 = 0.01 \ , \sigma = 0 , \quad R = 3$

COMPLETE NONLINEAR MODEL

$\varepsilon \equiv (v_0^2 \beta_f^{-1})^{1/3}$, $t \equiv \Delta\omega^{-1} T$, $f \equiv \omega_p / \Delta\omega$, $r \equiv (k_p \beta_f)^{-1} R$, $f = 1 + \frac{\varepsilon^2 \sigma}{2}$, $\sigma \sim O(\varepsilon^0)$

$\delta n \equiv \varepsilon n_0 N$, $v \equiv \varepsilon c \beta$, $E \equiv \varepsilon E_D \beta_f^{-1} e$, $B \equiv \varepsilon E_D \beta_f^{-1} b$, $F_{NL} \equiv v_0^2 (m_e \omega_p c) f_{NL}$.

$$\frac{\partial \beta}{\partial T} + e = -\frac{\varepsilon}{2}\,\mathrm{grad}(\beta^2) - \frac{\varepsilon^2}{2}\left[\sigma e + \frac{\partial}{\partial T}(\beta^2 \beta) - f_{NL}\right],$$

$$\mathrm{rot}\, e + \frac{\partial b}{\partial T} = 0,$$

$$\mathrm{rot}\, b - \frac{\partial e}{\partial T} + \beta = -\varepsilon N \beta - \frac{\varepsilon^2 \sigma}{2}\beta - \frac{\varepsilon^3 \sigma}{2} N \beta,$$

$$\mathrm{div}\, e + N = -\frac{\varepsilon^2 \sigma}{2} N,$$

where:

$$f_{NL} = \beta_f\, \mathrm{grad}_R [A_1(R) A_2(R) \cos(\beta_f^{-1} Z - T + \Delta\Phi_0)].$$

By using momentum and Maxwell equations we obtain also:

$$b - \mathrm{rot}\beta = \frac{\varepsilon^2}{2}\,\mathrm{rot}(\beta^2 \beta - \sigma\, \mathrm{rot}\beta).$$

MULTIPLE TIME SCALE METHOD

$$a = a_0(T_0, T_1, T_2, \ldots; R) + \varepsilon\, a_1(T_0, T_1, T_2, \ldots; R) + \varepsilon^2 a_2(T_0, T_1, T_2, \ldots; R) + \ldots$$

$$T_0 \equiv T, \quad T_1 \equiv \varepsilon T, \quad T_2 \equiv \varepsilon^2 T, \ldots$$

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \ldots$$

The expansion contains arbitrary functions of the time scales $T_1, T_2, \ldots$. In order to determine these functions, additional conditions need to be imposed. If the expansion is to be valid for times as large as $\varepsilon^{-M} T$, $\varepsilon^M a_M$ should be a small correction to $\varepsilon^{M-1} a_{M-1}$, which in turn should be a small correction to $\varepsilon^{M-2} a_{M-2}$. Thus we require the so-called:

Lighthill conditions (Lighthill M.J.)

$$\left|\frac{a_m}{a_{m-1}}\right| < \infty \quad m = 1, M,$$

for all $T_0, T_1, \ldots, T_M$.

i) ε^0:

$$\frac{\partial \beta_0}{\partial T_0} + e_0 = 0 \quad ,$$

$$\text{rot}\, e_0 + \frac{\partial b_0}{\partial T_0} = 0\,,$$

$$\text{rot}\, b_0 - \frac{\partial e_0}{\partial T_0} + \beta_0 = 0\,,$$

$$\text{div} e_0 + N_0 = 0 \quad ,$$

$$b_0 - \text{rot}\beta_0 = 0\,,$$

and with zero-initial conditions for β_0, e_0, b_0;

ii) ε^1:

$$\frac{\partial \beta_1}{\partial T_0} + e_1 = -\frac{\partial \beta_0}{\partial T_1} - \frac{1}{2}\text{grad}(\beta_0^2)$$

$$\text{rot}\, e_1 + \frac{\partial b_1}{\partial T_0} = -\frac{\partial b_0}{\partial T_1}\,,$$

$$\text{rot}\, b_1 - \frac{\partial e_1}{\partial T_0} + \beta_1 = \frac{\partial e_0}{\partial T_1} - N_0\beta_0\,.$$

$$\text{div} e_1 + N_1 = 0\,,$$

$$b_1 - \text{rot}\beta_1 = 0\,,$$

and with zero-initial conditions for β_1, e_1, b_1;

iii) ε^2:

$$\frac{\partial \beta_2}{\partial T_0} + e_2 = -\frac{\partial \beta_0}{\partial T_2} - \frac{\partial \beta_1}{\partial T_1} - \text{grad}(\beta_0 \cdot \beta_1) - \frac{1}{2}\sigma e_0 - \frac{1}{2}\frac{\partial}{\partial T_0}(\beta_0^2 \beta_0) + \frac{1}{2} f_{NL}\,.$$

$$\text{rot}\, e_2 + \frac{\partial b_2}{\partial T_0} = -\frac{\partial b_0}{\partial T_2} - \frac{\partial b_1}{\partial T_1}\,,$$

$$\text{rot}\, b_2 - \frac{\partial e_2}{\partial T_0} + \beta_2 = \frac{\partial e_0}{\partial T_2} + \frac{\partial e_1}{\partial T_1} - N_0\beta_1 - N_1\beta_0 - \frac{\sigma}{2}\beta_0\,,$$

$$\text{div} e_2 + N_2 = -\frac{\sigma}{2}N_2,$$

$$b_2 - \left(1 - \frac{\sigma}{2}\right)\text{rot}\beta_2 = \frac{1}{2}\,\text{rot}\beta_0^2\,\beta_0\,,$$

and with zero-initial conditions for β_2, e_2, b_2.

SOLUTION TO 0^{th} ORDER

$$\frac{\partial^2 N_0}{\partial T_0^2} + N_0 = 0.$$

$$N_0 = \tilde{N}_0(T_1, T_2, ...; \tilde{R})\, e^{i\theta_0} + \text{c.c.}\,,$$

where $\theta_0 = \beta_f^{-1}Z - T_0 + \Delta\Phi_0$.

Initial conditions for $\tilde{N}_0$:

$$\tilde{N}_0(T_1{=}0, T_2{=}0, ...) = 0.$$

$$\frac{\partial}{\partial T_0}(\text{rot} b_0) = \frac{\partial^2 e_0}{\partial T_0^2} + e_0\,,$$

$$e_0 \sim e^{-iT_0} \;\rightarrow\; \frac{\partial}{\partial T_0}(\text{rot} b_0) = 0,$$

$$b_0 = b_0(T_1, T_2, ...; \hat{R}) \;\rightarrow\; \text{rot} e_0 = 0.$$

$$\text{rot}\, e_0 = 0\,,$$

$$\text{div} e_0 = -N_0\,.$$

SOLUTION TO 1th ORDER

$$\left|\frac{N_1}{N_0}\right| < \infty \text{ for all } T_0, T_1, \ldots, T_M ,$$

$$\downarrow$$

$$N_1 = N_{10}(T_2, \ldots; \vec{R}) + [\tilde{N}_1(T_2, \ldots; \vec{R})\, e^{i2\theta_0} + c.c.] \quad ,$$

where:

$$N_{10} = \nabla^2(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^*), \; \tilde{N}_1\, e^{i2\theta_0} = -\frac{1}{3}[2\mathrm{div}(\tilde{N}_0 \tilde{\mathbf{E}}_0 e^{i2\theta_0}) - \frac{1}{2}\nabla^2(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0 e^{i2\theta_0})].$$

$$\mathbf{e}_1 = \mathbf{E}_{10}(T_2, \ldots; \vec{R}) + [\tilde{\mathbf{E}}_1(T_2, \ldots; \vec{R})\, e^{i2\theta_0} + c.c.],$$

$$\beta_1 = \beta_{10}(T_2, \ldots; \vec{R}) + [\tilde{\beta}_1(T_2, \ldots; \vec{R})\, e^{i2\theta_0} + c.c.],$$

$$\mathbf{b}_1 = \mathbf{b}_{10}(T_2, \ldots; \vec{R}) + [\tilde{\mathbf{b}}_1(T_2, \ldots; \vec{R})\, e^{i2\theta_0} + c.c.].$$

$$\mathrm{rot\ rot}\ \beta_{10} + \beta_{10} = -i\, \tilde{N}_0 \tilde{\mathbf{E}}_0 + c.c.,$$

$$\mathrm{rot\ rot}\ (\tilde{\beta}_1\, e^{i2\theta_0}) - 3\tilde{\beta}_1\, e^{i2\theta_0} = -i\, \mathrm{grad}(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0 e^{i2\theta_0}) + i\tilde{N}_0 \tilde{\mathbf{E}}_0\, e^{i2\theta_0},$$

$$\mathbf{E}_{10} = -\mathrm{grad}(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^*),$$

$$\tilde{\mathbf{E}}_1\, e^{i2\theta_0} = \frac{1}{2}\mathrm{grad}(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0 e^{i2\theta_0}) + 2i\tilde{\beta}_1\, e^{i2\theta_0},$$

$$\mathbf{b}_{10} = \mathrm{rot}\ \beta_{10},$$

$$\tilde{\mathbf{b}}_1\, e^{i2\theta_0} = \mathrm{rot}\ (\tilde{\beta}_1\, e^{i2\theta_0}).$$

In the monodimensional limit we obtain:

$$N_1 = 2\tilde{N}_0^2\, e^{i2\theta_0} + c.c.,$$

$$\beta_{10} = -2\,|\tilde{N}_0|^2\, \mathbf{e}_z, \qquad \tilde{\beta}_1 = 2\tilde{N}_0^2\, \mathbf{e}_z,$$

$$\mathbf{E}_{10} = 0, \qquad \tilde{\mathbf{E}}_1 = 2i\tilde{\beta}_1,$$

$$\mathbf{b}_{10} = 0, \qquad \tilde{\mathbf{b}}_1 = 0.$$

SOLUTION TO 2th ORDER

$$\left|\frac{N_2}{N_1}\right| < \infty \text{ for all } T_0, T_1, \ldots, T_M ,$$

$$\downarrow$$

$$\begin{aligned}
&(2i\frac{\partial \tilde{N}_0}{\partial T_2} - \sigma \tilde{N}_0)\, e^{i\theta_0} + \\
&\quad -\mathrm{div}[\frac{1}{2}(2\tilde{\mathbf{E}}_0(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^*) + \tilde{\mathbf{E}}_0^*(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0))e^{i\theta_0}] + \qquad \text{(A)} \\
&\quad \mathrm{div}[i(\tilde{N}_0 \beta_{10} + \tilde{N}_0^* \tilde{\beta}_1 - i N_{10} \tilde{\mathbf{E}}_0 + \tilde{N}_1 \tilde{\mathbf{E}}_0^*)e^{i\theta_0}] + \\
&\quad \nabla^2[i(\tilde{\mathbf{E}}_0^* \cdot \tilde{\beta}_1 - \tilde{\mathbf{E}}_0 \cdot \beta_{10})e^{i\theta_0}] + \mathrm{div}(\tilde{\mathbf{f}}_{NL} e^{i\theta_0}) = 0,
\end{aligned}$$

where:

$$\mathrm{div}(\tilde{\mathbf{E}}_0 e^{i\theta_0}) = -\tilde{N}_0\, e^{i\theta_0},$$

$$\mathrm{rot}(\tilde{\mathbf{E}}_0 e^{i\theta_0}) = 0,$$

$$\mathrm{rot\ rot}\ \beta_{10} + \beta_{10} = -i\, \tilde{N}_0 \tilde{\mathbf{E}}_0 + c.c.,$$

$$\mathrm{rot\ rot}\ (\tilde{\beta}_1\, e^{i2\theta_0}) - 3\tilde{\beta}_1\, e^{i2\theta_0} = -i\, \mathrm{grad}(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0 e^{i2\theta_0}) + i\tilde{N}_0 \tilde{\mathbf{E}}_0\, e^{i2\theta_0},$$

$$N_{10} = \nabla^2(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^*),$$

$$\tilde{N}_1\, e^{i2\theta_0} = -\frac{1}{3}[2\mathrm{div}(\tilde{N}_0 \tilde{\mathbf{E}}_0 e^{i2\theta_0}) - \frac{1}{2}\nabla^2(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0 e^{i2\theta_0})],$$

with initial and suitable boundary conditions.

NONLINEAR FREQUENCY SHIFT:

$$S_R\, e^{i\theta_0} \equiv -\mathrm{div}[\frac{1}{2}(2\tilde{\mathbf{E}}_0(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0^*) + \tilde{\mathbf{E}}_0^*(\tilde{\mathbf{E}}_0 \cdot \tilde{\mathbf{E}}_0))e^{i\theta_0}]$$

$$S_{NC}\, e^{i\theta_0} \equiv \mathrm{div}[i(\tilde{N}_0 \beta_{10} + \tilde{N}_0^* \tilde{\beta}_1 - i N_{10} \tilde{\mathbf{E}}_0 + \tilde{N}_1 \tilde{\mathbf{E}}_0^*)e^{i\theta_0}]$$

$$S_{CONV}\, e^{i\theta_0} \equiv \nabla^2[i(\tilde{\mathbf{E}}_0^* \cdot \tilde{\beta}_1 - \tilde{\mathbf{E}}_0 \cdot \beta_{10})e^{i\theta_0}]$$

[Bell A.R., Gibbon P.,(1988)]

In the monodimensional limit we obtain:

$$S_R = \frac{3}{2}|\tilde{N}_0|^2 \tilde{N}_0, \; S_{NC} = -|\tilde{N}_0|^2 \tilde{N}_0, \quad S_{CONV} = |\tilde{N}_0|^2 \tilde{N}_0,$$

then eq.(A) becomes:

$$\frac{\partial \tilde{N}_0}{\partial T_2} + i(\frac{1}{2}\sigma - \frac{3}{4}|\tilde{N}_0|^2)\tilde{N}_0 = -\frac{1}{4}.$$

By using the variation parameter method (Lagrange G.L.), we obtain:

$$\tilde{E}_0 = \mathcal{L}_1(\tilde{N}_0),$$
$$\tilde{\beta}_{10} = \mathcal{L}_2(\tilde{N}_0 \tilde{E}_0),$$
$$\tilde{\beta}_1 = \mathcal{L}_3(\tilde{N}_0 \tilde{E}_0, \tilde{E}_0 \tilde{E}_0),$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are known linear operators.

Then equation (A) becomes:

$$\left(2i\frac{\partial \tilde{N}_0}{\partial T_2} - \sigma \tilde{N}_0\right) = \mathcal{N}\{\tilde{N}_0, \frac{\partial \tilde{N}_0}{\partial R}, \frac{\partial^2 \tilde{N}_0}{\partial R^2}\} + \tilde{f}\,, \qquad \text{(B)}$$

where $\mathcal{N}$ is a nonlinear integro-differential spatial operator representing $S_R + S_{NC} + S_{CONV}$ and $\tilde{f} = \tilde{f}(R)$ is known function depending on the transverse profile of the ponderomotive force (we are assuming an axisymmetric profile).

Equation (B) may be solved numerically (explicit method or forward method):

$$\tilde{N}_0^{(n+1)} \cong \tilde{N}_0^{(n)} + \frac{\Delta T_2}{2i}\{\sigma \tilde{N}_0^{(n)} + \mathcal{N}^{(n)} + \tilde{f}\}\,,$$

where $\mathcal{N}^{(n)}$ e $\tilde{N}_0^{(n)}$ are respectively $\mathcal{N}$ and $\tilde{N}_0$ values at time $T_2 = n\Delta T_2$, with $\Delta T_2 << 1$, for every $0 \le R \le R_\infty$, where R_∞ is the value of R in which we can assume $\tilde{N}_0 \cong 0$, $\frac{\partial \tilde{N}_0}{\partial R} \cong 0$, $\frac{\partial^2 \tilde{N}_0}{\partial R^2} \cong 0$ $\tilde{N}_0^{(0)}(R_\infty) = 0$. Equation (B) must be solved with:

- -initial conditions : $\tilde{N}_0(T_2 = 0, R) = 0$;
- -boundary conditions : $\frac{\partial \tilde{N}_0}{\partial R}\big|_{R=0} = 0$, $\tilde{N}_0(R_\infty) = 0$.

$\tilde{N}_0^{(n)}(R)$ may be represented as function of R by using cubic-splines:

$\tilde{N}_0^{(n)}(R)$ is a C^2 function in $(0, R_\infty)$.

$$N_0(R,Z,T) = |\tilde{N}(R,\epsilon^2 T)|\, e^{i\theta(R,\epsilon^2 T)}\, e^{i(z-T)} + c.c.$$

$$\text{rot}\, \bar{E}_0 = 0$$
$$\text{div}\, \bar{E}_0 = -N_0$$

$$\bar{E}_0 = -\text{grad}\, V \quad , \quad V = V(R,Z,T)$$

$$\nabla^2 V = N_0$$

$$V(R,Z,T) = |\tilde{V}(R,\epsilon^2 T)|\, e^{i\varphi(R,\epsilon^2 T)}\, e^{i(z-T)} + c.c.$$

$$E_R = -\left[\frac{\partial}{\partial R}|\tilde{V}|\, e^{i(z-T+\varphi)} + i|\tilde{V}|\frac{\partial \varphi}{\partial R} e^{i(z-T+\varphi)} + c.c.\right]$$

$$E_z = -i\, \tilde{V}\, e^{i(z-T+\varphi)} + c.c.$$

$$\Delta\varphi(R,\epsilon^2 T) = \frac{\pi}{2} - tg^{-1}\left\{\frac{\partial \varphi}{\partial R} \Big/ \left(\frac{\partial}{\partial R} \ln|\tilde{V}|\right)\right\}$$

phase slipping on the slow time scale between transverse and longitudinal components