INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

H4-SMR 393/4

## SPRING COLLEGE ON PLASMA PHYSICS

15 May - 9 June 1989

# MHD SOLITIONS IN MAGNETIC FLUX TUBES

B. Roberts

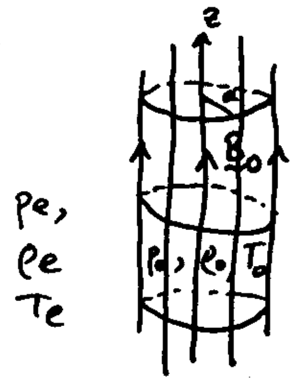
University of St. Andrew's
Dept. of Applied Mathematics
Fife. Ky 169Ss
Scotland
U. K.

Lecture 3 B. Roberts
St. Andrews University

MHD SOLITIONS IN MAGNETIC FLUX TUBES

# Waves in Magnetic Flux Tubes

## Equilibrium




$$p_e = p_0 + \frac{B_0^2}{2\mu_0}$$

(so $p_0 < p_e$)

**Linear perturbations** lead to (for general $\rho_0(r)$, $B_0(r)$, $p_0(r)$ subject to $\frac{d}{dr}\left(p_0 + \frac{B_0^2}{2\mu_0}\right) = 0$)

$$\boxed{\begin{aligned}
\rho_0(r)\left(\frac{\partial^2}{\partial t^2} - v_A^2\frac{\partial^2}{\partial z^2}\right)v_r &= -\frac{\partial^2 p_T}{\partial r\,\partial t}\\
\rho_0(r)\left(\frac{\partial^2}{\partial t^2} - v_A^2\frac{\partial^2}{\partial z^2}\right)v_\theta &= -\frac{1}{r}\frac{\partial^2 p_T}{\partial\theta\,\partial t}\\
\rho_0(r)\left(\frac{\partial^2}{\partial t^2} - c_T^2\frac{\partial^2}{\partial z^2}\right)v_z &= -\left(\frac{c_s^2}{c_f^2}\right)\frac{\partial^2 p_T}{\partial z\,\partial t}
\end{aligned}}$$

where

$$p_T = p + \frac{B_0 B_z}{\mu_0}$$

(perturbation in total pressure)

and

$$\boxed{\frac{\partial p_T}{\partial t} = \rho_0 v_A^2\frac{\partial v_z}{\partial z} - \rho_0 c_f^2 \,\mathrm{div}\,\underline{v}}$$

Fourier components $v_r = v_r(r)\, e^{i(\omega t + n\vartheta - kz)}$ etc.

$\Longrightarrow$

$$\left.\begin{aligned} &\rho_0(k^2v_A^2-\omega^2)\frac{1}{r}\frac{d}{dr}(rv_r) + \left(m_0^2+\frac{n^2}{r^2}\right) i\omega\, p_T = 0 \\ &i\omega\frac{dp_T}{dr} + \rho_0(k^2v_A^2-\omega^2)v_r = 0 \end{aligned}\right\}$$

$$m_0^2 = \frac{(k^2c_s^2-\omega^2)(k^2v_A^2-\omega^2)}{(c_s^2+v_A^2)(k^2c_T^2-\omega^2)}$$

$\Longrightarrow$

$$\boxed{\begin{aligned} &\frac{d}{dr}\left\{\frac{\rho_0(k^2v_A^2-\omega^2)}{\left(m_0^2+\frac{n^2}{r^2}\right)}\frac{1}{r}\frac{d}{dr}(rv_r)\right\} - \rho_0(k^2v_A^2-\omega^2)v_r = 0 \qquad \text{twist-free Hain-Lüst '58} \\ &\rho_0(k^2v_A^2-\omega^2)\frac{1}{r}\frac{d}{dr}\left\{\frac{1}{\rho_0(k^2v_A^2-\omega^2)}\, r\frac{dp_T}{dr}\right\} - \left(m_0^2+\frac{n^2}{r^2}\right)p_T = 0 \end{aligned}}$$

Note: one soln. is $v_r = p_T = 0$, and then $v_z = 0$, $\operatorname{div}\underline{v} = 0$ and

$$\frac{\partial^2 v_\vartheta}{\partial t^2} = v_A^2(r)\frac{\partial^2 v_\vartheta}{\partial z^2} \qquad \text{torsional Alfvén waves}$$

In a uniform medium we get

$$\boxed{\frac{d^2p_T}{dr^2} + \frac{1}{r}\frac{dp_T}{dr} - \left(m_0^2+\frac{n^2}{r^2}\right)p_T = 0} \qquad \text{Bessel's eqn}$$



Solution, subject to $p_T$ being bounded at $r=0$:

$$r<a:\; p_T = \begin{cases} A_0 I_n(m_0 r), & m_0^2>0 \\ A_0 J_n(n_0 r), \quad n_0^2 = -m_0^2, & m_0^2<0 \end{cases}$$

Outside tube, in $r>a$, we choose

$p_T$ to be evanescent in $r$, $p_T \downarrow 0$ as $r\to\infty$

corresponding to

$$p_T = A_e K_n(m_e r),\; r>a, \qquad m_e^2 = \frac{k^2c_e^2-\omega^2}{c_e^2} > 0$$

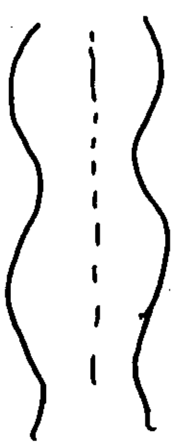
Matching ($p_T$ and $v_r$ etc) yields the dispersion reln. (Wilson 80, Spruit 82; Edwin + R 83)

$$\boxed{\rho_e\,\omega^2 m_0\frac{I_n'(m_0a)}{I_n(m_0a)} + \rho_0(k^2v_A^2-\omega^2)m_e\frac{K_n'(m_ea)}{K_n(m_ea)} = 0}$$

transcendental

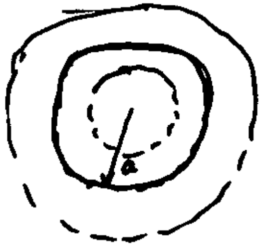
for case $m_0^2>0$ (and $m_e^2>0$).

## Modes



Sausage mode
$n=0$

Kink mode
$n=1$

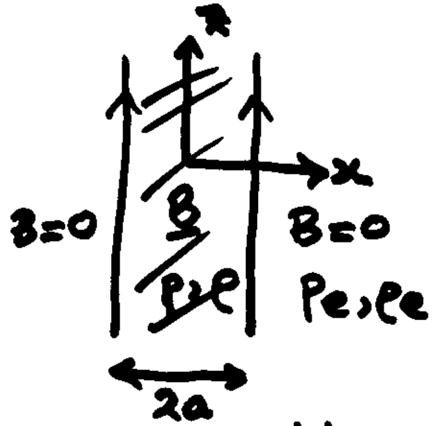



Fluting modes ($n \geq 2$)

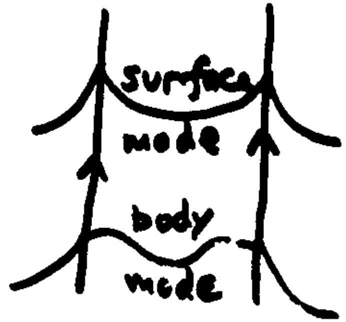
Structured Fields : $\delta = 0$, $a \neq 0$.

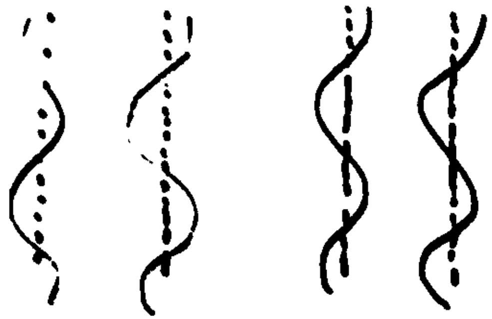
# Waves in an isolated fluxtube

($g=0$) **Linear**




Magnetic slab

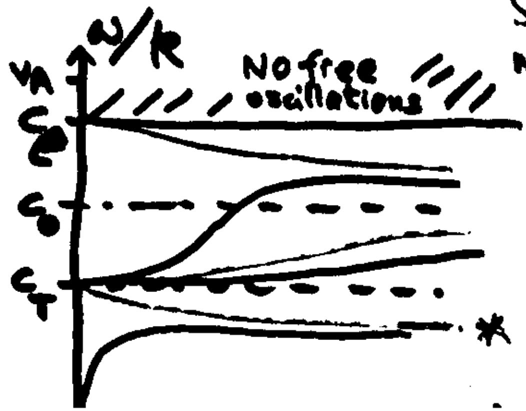


Sausage mode.

Kink mode

SLAB

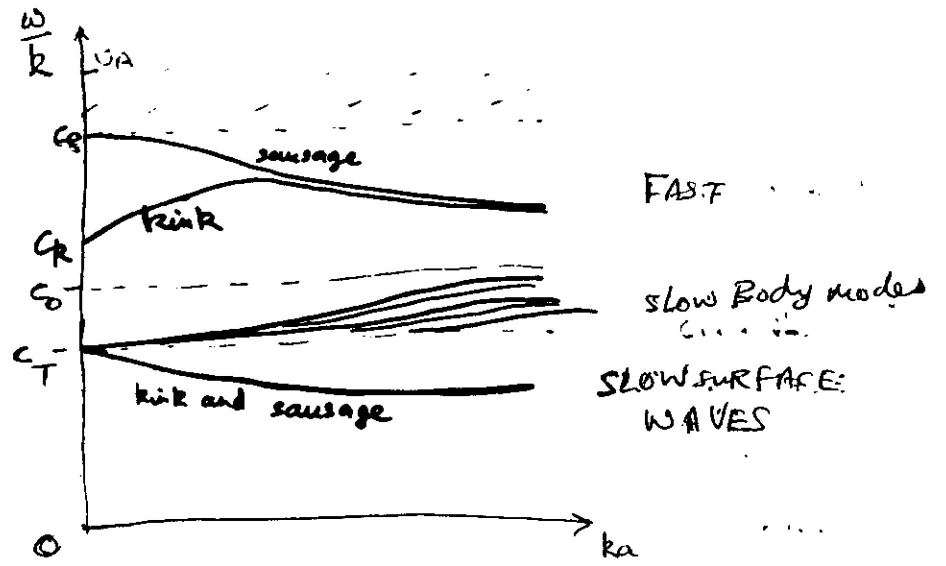



$c_e$ = sound speed in external mediu

$c_0, v_A$ : sound + Alfv speeds in slab

Solutions

## CYLINDER




Case $v_A > c_e > c_0$

[For $c_e \sim c_0 \sim v_A \sim 10\ \mathrm{km\,s^{-1}}$
find
$c_T \sim 7\ \mathrm{km\,s^{-1}}$
$c_k \sim 6\ \mathrm{km\,s^{-1}}$]

—— : kink (n=1) modes
—— : sausage (n=0) modes

SURFACE WAVES if $m_0^2 > 0$ ?
Body waves if $m_0^2 < 0$.

$$c_k^2 = \frac{\rho_0 v_A^2 + \rho_e v_{Ae}^2}{\rho_0 + \rho_e} \qquad \text{in general}$$

$$= \left(\frac{\rho_0}{\rho_0 + \rho_e}\right) v_A^2 \qquad \text{in an } \underline{\text{isolated tube}}$$

[cf. $c_k$ is speed of propn. of surface wave on a single interface between two incompressible media (Kruskal + Schwarzschild '54)]

## Approximations

**Wide structure ($|k|a \gg 1$)**

Suppose $m_e a, m_0 a \to \infty$ as $ka \to \infty$

Then $\tanh m_0 a \to 1$ (slab)
$\coth m_0 a \to 1$

giving

$$\boxed{(k^2 v_A^2 - \omega^2)\, m_e = (\rho_e/\rho_0)\, \omega^2 m_0}$$

which is the surface wave dispersion reln. for a single interface. So fast and slow surface modes may exist, depending upon circumstances (the slow always exists).

**Thin structure ($|k|a \ll 1$)** Long wavelength limit
Set $c = \omega/k$ (phase speed)

$$c \sim c_T \left\{ 1 - \frac{1}{4}\left(\frac{\rho_e}{\rho_0}\right)\left(\frac{c_T}{v_A}\right)^4 k^2 a^2 K_0(\lambda |k| a) \right\},$$

sausage mode

$$c \sim c_k \left\{ 1 + \frac{1}{2}\left(\frac{\rho_e}{\rho_e + \rho_0}\right) \nu^2 k^2 a^2 K_0(\nu |k| a) \right\}$$

kink mode

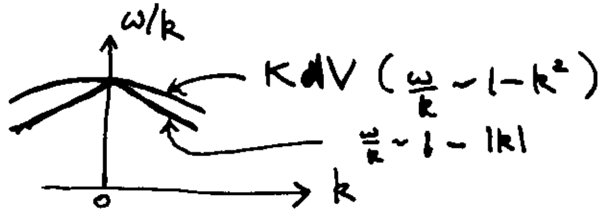
$$\lambda = \frac{(c_e^2 - c_T^2)^{1/2}}{c_e} \qquad (> 0 \text{ assumed})$$

$$\nu = \frac{(c_e^2 - c_k^2)^{1/2}}{c_e} \qquad (> 0 \text{ assumed})$$

## Differential Equations

$$\omega \sim k c_T (1 - \alpha |k|)$$

What type of eqn. will describe the Fourier operation $k|k|$ ?




Whitham's eqn.

Suppose $c(k) \equiv \frac{\omega}{k}$ is given (e.g. $c(k) = 1 - k^2$ or $c(k) = 1 - |k|$)

Then Whitham ('67) suggested the integrodifferential eqn. to describe long wave behaviour :-

$$\boxed{\frac{\partial v}{\partial t} + [\text{N.L. term}] + \int_{-\infty}^{\infty} K(z-s) \frac{\partial v}{\partial s}(s,t)\, ds = 0}$$

where
$$K(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{ikz} dk$$
$$c(k) = \int_{-\infty}^{\infty} K(s) e^{-iks} ds$$
Fourier transform pair



## Application

(a) KdV eqn. – straightforward (see W.'s book), gives $\delta, \delta''$ for $K(z)$.

(b) $c(k) = c_T - \alpha|k|$     SLOW [illegible]

$$K(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (c_T - \alpha|k|) e^{ikz} dk$$
$$= c_T \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikz} dk - \alpha \frac{1}{2\pi} \int_{-\infty}^{\infty} k \operatorname{sgn}(k) e^{ikz} d$$
$$= c_T \delta(z) + \frac{\alpha}{\pi} \left(\frac{1}{z^2}\right).$$

Whitham's eqn. becomes

$$\frac{\partial v}{\partial t} + \beta v \frac{\partial v}{\partial z} + c_T \frac{\partial v}{\partial z} = \alpha\, \mathcal{J}\left(\frac{\partial v}{\partial z}(z,t)\right),$$

where
$$\mathcal{J}(u(z,t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |k| e^{ikz} \left\{ \int_{-\infty}^{\infty} u(s,t) e^{-iks} ds \right\} d$$
$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{z-s} \frac{\partial u(s,t)}{\partial s} ds$$
$$= -H\left(\frac{\partial u}{\partial z}\right), \text{ Hilbert transf.}$$

$$\boxed{H(u(z,t)) = \frac{1}{\pi} \int_{-\infty} \frac{u(s,}{s-}}$$

Hence, Benjamin–Ono Eqn.:

# Weakly nonlinear, weakly dispersive ($ka \ll 1$) waves

Parker '74
Defouw '76
Roberts + Webb '78

## Slender flux tube equations: SAUSAGE MODE

Continuity: $\frac{\partial}{\partial t}\rho A + \frac{\partial}{\partial z}\rho v A = 0$

Momentum: $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$

Energy: $\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial z} = \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial z}\right)$

Magnetic flux: $BA = \text{constant}$

Transverse pressure balance: $p + \frac{B^2}{2\mu_0} = \pi_e(z,t)$ pressure on external bdy of tube

(A)

– valid for longitudinal motions $v \equiv v_z \gg v_{\text{transverse}}$.

External medium: (B)

$$\frac{\partial \rho^e}{\partial t} + \operatorname{div}\rho^e \underline{u} = 0, \quad \underline{u} = \text{velocity field}$$

$$\rho^e\left\{\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}\right\} = -\nabla p^e, \text{ plus adiab. } (g=0)$$

with boundary condns.

(C)

normal compt. of vel. cts. across bdy of flux tube }
pressure (gas + magn.) cts. " " " "
$p^e \to 0$ as $r \to \infty$



## Scaling:

Slab. Bearing in mind $\omega = kc_T - \alpha k|k|$, introduce slow time and spatial (moving with the wave) scales by

$$\tau = \epsilon^2 t, \quad \xi = \epsilon(z - c_T t)$$

tube variables

and then expand eqns. for interior of tube thus:

$$\rho = \rho_0 + \epsilon\rho_1 + \cdots, \quad A = A_0 + \epsilon A_1 + \cdots$$
$$v = \epsilon v_1 + \epsilon^2 v_2 + \cdots$$

The choice of scales, etc. allows us to eliminate at 2nd order ($\epsilon^2$) $v_2, A_2$, etc., to give

$$\boxed{\frac{\partial v_1}{\partial \tau} + \beta_0 v_1\frac{\partial v_1}{\partial \xi} + \frac{c_T^2}{2\rho_0 v_A^2}\frac{\partial}{\partial \xi}\pi_e^{(2)} = 0} \quad (*)$$

$$\beta_0 = \frac{v_A^2(6v_A^2 + 3c_0^2)}{2(c_0^2 + v_A^2)^2}$$

(depends only on tube param.)

cf. Adam ('78)

NB $\beta_0 \to 0$ in the incompressible limit.
Significance?

Scaling : External medium

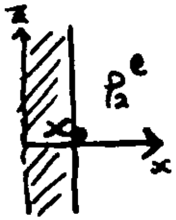
$$\tau = \epsilon^2 t, \quad X = x, \quad Z = z - c_T t$$

Find

$$\left[\frac{\partial^2}{\partial X^2} + \left(1 - \frac{c_T^2}{c_e^2}\right)\frac{\partial^2}{\partial Z^2}\right] p_2^e = 0,$$

Laplace's eqn. for $c_e > c_T$

$$p^e = \underbrace{p_0^e}_{\text{equilm.}} + \epsilon^2 p_2^e(x,z,t)$$




Require soln. of Laplace's eqn. subject to

$$\left.\begin{aligned} &p_2^e \to 0 \text{ as } (x^2+z^2)^{1/2} \to \infty \\ &p_2^e(x,z,t) = \pi_e^{(2)}(z,t) \quad \text{on } x = x_0 \end{aligned}\right\}$$

Soln. $$p_2^e(x,Z,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\pi_e^{(2)}(s,t)(X - X_0)\,ds}{(x-x_0)^2 + (s-\hat{Z})^2},$$

$$\hat{Z} = \left(1 - c_T^2/c_e^2\right)^{1/2} Z.$$

Substituting in ⊛, and using conty. of normal velocity, gives (in original variables):

Roberts & Mangeney (1982)

$$\boxed{\frac{\partial v}{\partial t} + c_T\frac{\partial v}{\partial z} + \beta v\frac{\partial v}{\partial z} + \alpha H\left(\frac{\partial^2 v}{\partial z^2}\right) = 0}$$

Benjamin-Ono Eqn.

(agrees

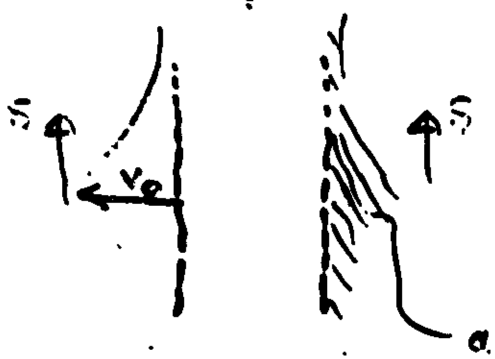


**Soliton** The B.-O. eqn. has a soliton soln. (Benjamin '67), viz.

$$\boxed{v(z,t) = \frac{v_0}{1 + \lambda_*^2 (z - st)^2}}$$

$$s = c_T + \frac{1}{4}\beta v_0, \quad \lambda_* \propto \frac{\beta v_0}{\alpha}$$

soliton speed — geometrical scale




In intense tube $s \approx 7$ km $s^{-1}$ (in photosphere

α κn is constant (indep. of

**Applications** :

Spicules → biggest difficulty: $g=0$

→ otherwise, looks attractive

Other solitons →? in coronal loops ($g \neq 0$) seems likely

(13)

## Cylinder (radius $a$)

Dispersion reln. is:— **symmetric pulsations**

$$\boxed{(k^2 v_A^2 - \omega^2) m_e = \left(\frac{\rho_e}{\rho_0}\right)\omega^2 m_0 \, \frac{I_1(m_0 a)}{I_0(m_0 a)} \cdot \frac{K_0(m_e a)}{K_1(m_e a)}}$$

(kink modes may be treated similarly.)

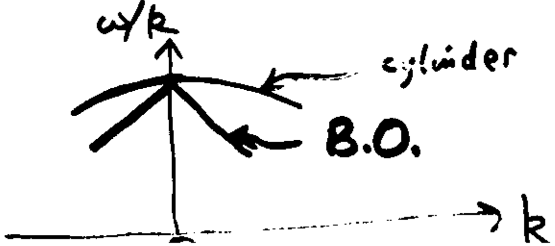
$kr_0 << 1$:

$$\boxed{\omega \sim kc_T - 2\alpha' k^3 K_0(\lambda |k| a)}$$ (cylinder) **tube**

$\lambda = \frac{c_T}{v_A}$, $\alpha' = \frac{1}{8}(\rho_e/\rho_0)\left(\frac{c_T}{v_A}\right)^4 c_T a^2$

(cf. slab): $\omega \sim kc_T - \alpha k|k|$ (slab)

$$\alpha = \tfrac{1}{2}(\rho_e/\rho_0)\left(\frac{c_T}{v_A}\right)^3 c_T x_0$$




(14)

$$c(k) = c_T - 2\alpha' k^2 K_0(\lambda |k| a)$$

$$\therefore K(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_T e^{ikz} dk - \frac{\alpha'}{\pi}\int_{-\infty}^{\infty} k^2 K_0(\lambda|k|a) e^{ikz} dk$$

$$= c_T \delta(z) + \frac{\alpha'}{\pi}\frac{d^2}{dz^2}\left\{\int_{-\infty}^{\infty} K_0(\lambda|k|a) e^{ikz} dk\right\}$$

i.e. $K(z) = c_T \delta(z) + \alpha' \frac{d^2}{dz^2}\left\{\frac{1}{(a^2\lambda^2 + z^2)^{1/2}}\right\}$,

on using the Fourier inverse of Bassets' integral:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} K_0(\lambda|k|a) e^{ikz} dk = \frac{1}{(a^2\lambda^2 + z^2)^{1/2}}.$$

Whitham's eqn. [viz. $v_t + \beta v v_z + \int_{-\infty}^{\infty} K(z-s) v_s(s,t) ds = 0$] then yields (Roberts 85, Phys. Fluids 28, 3280)

$$\boxed{\frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \beta v \frac{\partial v}{\partial z} + \alpha' \frac{\partial^3}{\partial z^3}\int_{-\infty}^{\infty} \frac{v(s,t)\, ds}{\{a^2\lambda^2 + (z-s)^2\}^{1/2}} = 0}$$

After detailed multiple-scale analysis, similar to slab case, we obtain (Roberts 1985)

$$\boxed{0 = \frac{\partial v}{\partial t} + c_T \frac{\partial v}{\partial z} + \beta_0\, v \frac{\partial v}{\partial z} + \alpha' \frac{\partial^3}{\partial z^3} \int_{-\infty}^{\infty} \frac{v(s,t)\,ds}{\{\lambda^2 a^2 + (z-s)^2\}^{1/2}}}$$

$\beta_0$ = same coeff. as in slab case, $\lambda = \frac{c_T}{v_A}$

Notes (i) same eqn. arises in rotating fluids (Leibovich 1970; see also Pritchard '70 – both in JFM 42)

(ii) eqn. has been termed the L-R eqn.

(iii) Both the slab and cylinder eqns. for the sausage mode have been derived without using thin tube eqns. (Merzljakov + Ruderman '85, '86)

Body modes also discussed – no solitary waves can exist (M.+R. '86)

(iv) Fast surface (sausage) waves in a slab satisfy the Khokhlov-Zabolotskaya eqn., viz.

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(v_t + v_z + v v_z) + v_{xx} = 0 \\ &v_x = \alpha v_{zz} \text{ at } x = 0 \\ &v \to 0 \text{ as } x \to \infty \end{aligned}\right\}$$

KdV ... speculated by R.+M. '82



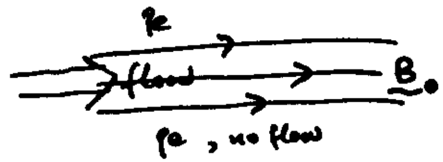
(v) No known analytical soln. for L-R. eqn.

Bogdan + Lerche ('88) have found asymptotic behaviour (large distances) of the eqn.

Molotovshchikov and Ruderman ('87) have found numerically solns. of cylinder eqn. retaining $K_1$-term (their method didn't work on the L-R. eqn.)

Weisshaar ('89) has solved numerically the L-R e (see 17' etc.)

(vi) L.-R. eqn. also occurs in jets, provided flow in tube $< c_e - c_T$ (Roberts '87)




[Viewed in frame of the jet (for uniform flow), we have an isolated tube with no flow but an environment with a flow. Hence dispersion is modified but nonlinear term is unchanged. Find (R'87)

$$0 = \frac{\partial v}{\partial t} + (c_T + U)\frac{\partial v}{\partial z} + \beta_0 v \frac{\partial v}{\partial z} + \alpha' \frac{\partial^3}{\partial z^3} \int_{-\infty}^{\infty} \frac{v(s,t)\,ds}{\{\lambda^2 a^2 + (z-s)^2\}^{1/2}}$$

$$\alpha' = \frac{1}{8}(c_T + U)^2 \left(\frac{c_T}{v_A}\right)^3 \left(\rho_e/\rho_0\right) \frac{a^2}{v_A}$$

$$\lambda = \left[1 - \frac{(U + c_T)^2}{c_e^2}\right]^{1/2} \quad (>0 \text{ assumed})$$

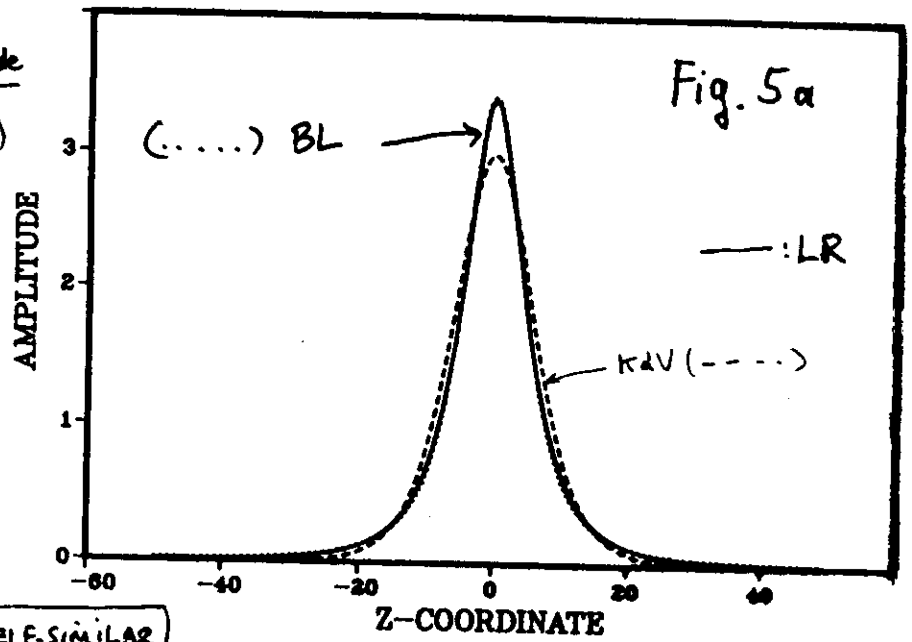
Thus get solitons provided $\lambda > 0$, i.e. $U < c_e - c_T$

WEISSHAAR (1989) - Phys. Fluids - in press (17)

SOLITARY WAVE PROFILES

High Amplitude

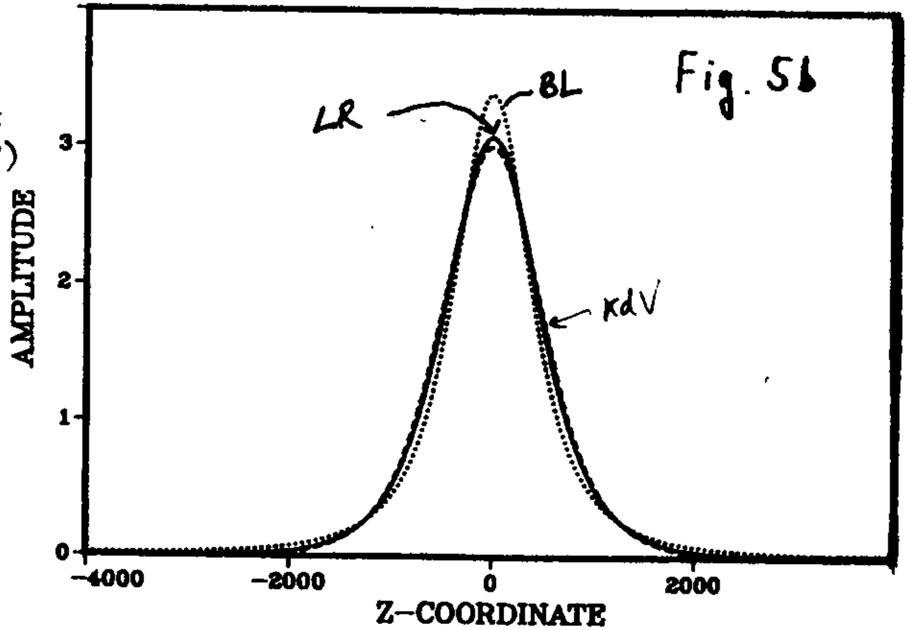
LR ≈ BL)




BL:
NOT SELF-SIMILAR

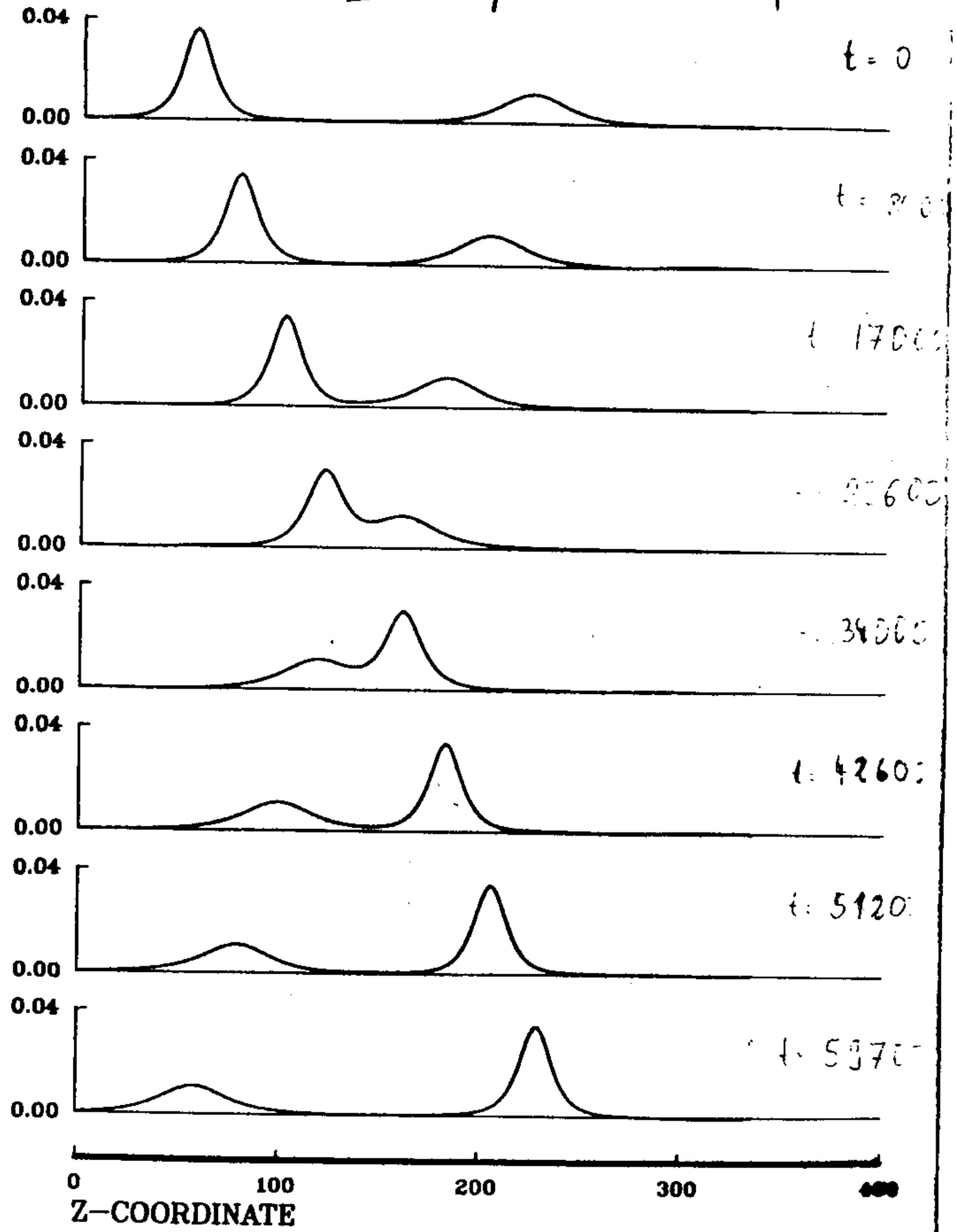
Low Amplitude

LR ≈ KdV)




Low amplitude

moving frame plots (18)



Medium (to high) amplitude