INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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H4-SMR 393/37

SPRING COLLEGE ON PLASMA PHYSICS

15 May - 9 June 1989

MODULATIONAL INSTABILITIES OF ELECTROSTATIC AND ELECTROMAGNETIC WAVES

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INTRODUCTION:

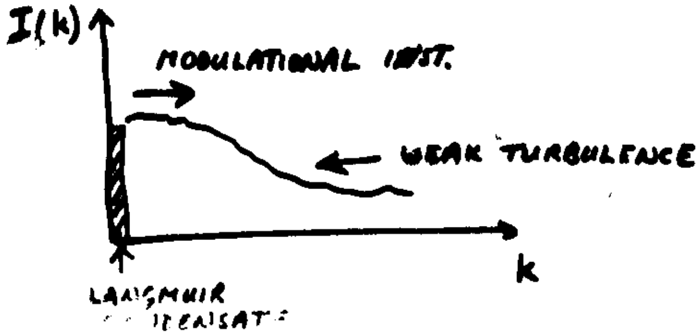
IN A WEAKLY TURBULENT PLASMA WITH LANGMUIR AND ION-ACOUSTIC MODES ($B=0$) THE 3-WAVE INTERACTION

$$k_0 \to k_2 + k_1$$

LEADS TO A CONCENTRATION OF LANGMUIR OSCILLATIONS IN THE LONG WAVELENGTH REGIME ($k_0 \to 0$, $\lambda_0 \to \infty$) THE SO-CALLED LANGMUIR CONDENSATE, SPATIALLY UNIFORM.

THIS IS NOT OBSERVED!

VEDENOV, RUDAKOV & GAMITIS IN (1965) SHOWED THAT SUCH A SPATIALLY UNIFORM SYSTEM IS MODULATIONALLY UNSTABLE.



THE MODULATIONAL ISTABILITY - TAKES A UNIFORM WAVE TRAIN WITH A SLOWLY VARYING MODULATION AND PRODUCES DENSITY DEPLETIONS FILLED WITH LANGMUIR OSCILLATIONS.

A LOCAL LOWERING OF THE DENSITY LEADS TO A DECREASE OF THE LOCAL LANGMUIR FREQUENCY - TRAPPING THE LANGMUIR WAVES WHICH IN TURN THROUGH THE PONDEROMOTIVE FORCE - LEADS TO A FURTHER LOWERING OF THE DENSITY.

THE M.I. LEADS TO DENSITY DEPLETIONS WHICH ARE FILLED WITH PLASMA OSCILLATIONS. THESE STRUCTURES ARE CALLED "CAVITONS" OR LANGMUIR "SOLITONS" (NOT PROPER

OTHER PROCESSES WHICH CAN LOWER THE FREQUENCY ARE

(1) JOULE HEATING (WAVE AMPLITUDE DEPENDENT)

(2) RELATIVISTIC CORRECTIONS (LORENTZ FACTOR)

NOTE
THE FREQUENCY SHIFT MUST BE AMPLITUDE DEPENDENT

$$\omega = \omega(k^2, A)$$

THE MODULATIONAL INSTABILITY TAKES A UNIFORM WAVE TRAIN – AND CREATES A NON-UNIFORM WAVE TRAIN. ⇒ <u>CAUSTONS</u>

(1) THE LANGMUIR MODULATIONAL INSTABILITY
(2) THE OSCILLATING TWO STREAM INSTABILITY
(3) SELF-MODULATION ;- FILAMENTATION OF ELECTROMAGNETIC WAVES.

ARE ALL EXAMPLES

IN ALL CASES THE BASIC MECHANISM IS A FOUR-WAVE INTERACTION. INVOLVING TWO PUMP "QUANTA" AND TWO SIDEBANDS

THE INITIAL PUMP WAVE AT $\underline{k}_0$ GENERATES PAIRS OF SIDEBANDS $\underline{k}_0 \pm n\underline{k}_s$ AND THE ASSOCIATED DENSITY PERTURBATIONS $n\underline{k}_s$

$$\underline{k}_0 \Rightarrow \underline{k}_0 \pm n\underline{k}_s$$

NOTE

4-WAVE PROCESSES INVOLVE CUBIC NONLINEARITIES.

3-WAVE PROCESSES INVOLVE QUADRATIC NONLINEARITIES.

IF $\underline{k}_s \parallel \underline{k}_0$ WE GET 1-D MODULATION.

IF $\underline{k}_s \perp k_0$ WE GET 2-D MODULATION OR FILAMENTATION.

1-D LANGMUIR MODULATIONAL INSTABILITY.

STARTING FROM FLUID EQUATIONS FOR IONS AND ELECTRONS. (COLLISIONLESS)

$$\frac{\partial n_j}{\partial t} + \underline{\nabla}\cdot(n_j \underline{v}_j) = 0$$

$$\left(\frac{\partial}{\partial t} + \underline{v}_j\cdot\underline{\nabla}\right)\underline{v}_j\gamma_j + \frac{KT_j}{n_j m_j}\underline{\nabla} n_j = \frac{q_j}{m_j}(\underline{E} + \underline{v}_j \times \underline{B})$$

$$\underline{\nabla}\cdot\underline{E} = \frac{1}{\epsilon_0}\sum_j n_j q_j \qquad \text{POISSONS EQU.}$$

$$\gamma_j = (1 - v_j^2/c^2)^{-\frac{1}{2}} \quad ; \quad j = i, e.$$

WE OBTAIN THE 1-D NONLINEAR LANGMUIR WAVE FIELD E IN THE WEAK RELATIVISTIC LIMIT, INCLUDING LOW FREQUENCY DENSITY PERTURBATION δn_e^{ℓ}

GIVEN BY

$$\left\{\frac{\partial^2}{\partial t^2} + \omega_{pe}^2\left(1+\frac{\delta n_e^l}{n_0}\right) + 3v_{Te}^2\frac{\partial^2}{\partial x^2} + \underbrace{\frac{3}{8}\omega_{pe}^2\frac{V_{osc}^2}{c^2}}_{\text{WEAK RELATIVISTIC TERM}}\right\}E = 0 \quad -(1)$$

THE EQUATION FOR THE LOW FREQUENCY DENSITY PERTURBATION IS GIVEN BY

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2\frac{\partial^2}{\partial x^2}\right)\delta n_e^l = +\frac{\epsilon_0}{2m_i}\frac{\omega_0^2}{\omega_{pe}^2}\frac{\partial^2}{\partial x^2}|E|^2 \quad -(2)$$

c_s IS ION SOUND SPEED.

NOTE $V_{osc} = \dfrac{e|E|}{m_e\omega_0}$ IS ELECTRON QUIVER VELOCITY

THE LANGMUIR FREQUENCY IS NOW

$$\omega_0^2 = \omega_{pe}^2\left(1+\frac{\delta n_e^l}{n_0}\right) + 3k_0^2 v_{Te}^2 - \frac{3}{8}\omega_{pe}^2\frac{V_{osc}^2}{c^2} \quad -(3)$$

NOTE

THE RELATIVISTIC EFFECT OPERATES ON ELECTRON PLASMA TIME SCALES (SLIGHTLY LESS)

THE LOW FREQUENCY DENSITY RESPONSE OPERATES ON ION-PLASMA PERIODS WHICH IS ~ 40 TIMES LONGER FOR A HYDROGEN PLASMA

THEREFORE THESE TWO PROCESSES WILL BE DE-COUPLED. (IN TIME)

INTRODUCING A SLOWLY VARYING AMPLITUDE TO DESCRIBE THE NONLINEAR BEHAVIOUR OF THE HIGH FREQUENCY WAVE

$$E(x,t) = Re\left\{E_0(x,t)\exp i(k_0x-\omega_0t)\right\}$$

WHERE E_0 VARIES ON A SLOW TIME SCALE COMPARED WITH LINEAR PHASE.

USING THE W.K.B. APPROX.

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - i\omega_0 \;;\; \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + ik_0$$

WE OBTAIN THE ENVELOPE EQUATION FOR THE SLOWLY VARYING AMPLITUDE.

$$i\left(\frac{\partial}{\partial t} + v_g\frac{\partial}{\partial x}\right)E_0 + \frac{1}{2}v_g'\frac{\partial^2}{\partial x^2}E_0 - \frac{\omega_{pe}^2}{2\omega_0}\left(\frac{\delta n_e^l}{n_0} - \frac{3}{8}\frac{V_{osc}^2}{c^2}\right)E_0 = 0 \quad -(4)$$

EQUATIONS (4) AND (2) ARE THE SO-CALLED ZAKHAROV EQUATIONS

$V_g = 3k_0 V_{Te}^2/\omega_0$ GROUP VELOCITY

$V_g' = 3V_{Te}^2/\omega_0$ GROUP DISPERSION.

WE CAN ANALYSE EQUATIONS (2) & (6) IN THREE REGIMES (FOR THE COND. $k\lambda_D < \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}$)

(1) QUASI STATIC - HYDRODYNAMIC

(2) RELATIVISTIC REGIME

(3) KINETIC REGIME - TREAT SEPERATELY

CASE (1)

WE ASSUME $\delta n_e^l \sim N_e^l e^{iKx}$

AND $\frac{\partial^2}{\partial t^2} \ll K^2 c_s^2$ STATIC APPROX.

THEN FROM (2)

$$N^l = \frac{\epsilon_0}{2 m_i c_s^2} |E|^2 \qquad -(5)$$

Substituting (5) into equ. (6) we get

$$\left\{ i\left(\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial x}\right) + \frac{1}{2} v_g' \frac{\partial^2}{\partial x^2} + C_K |E_0|^2 \right\} E_0 = 0 \qquad -(6)$$

$$C_K = \frac{1}{8} \frac{\omega_p \epsilon_0}{n_0 K T_e} \qquad \text{HYDRODYNAMIC (FOR STATIC APPROX)}$$

CASE (2)

$$C_K = \frac{3}{16} \frac{e^2}{\omega^3 m^2 c^2} \qquad \text{(FOR REL. MASS VARIATION)}$$

IN A FRAME MOVING WITH THE LINEAR GROUP VELOCITY v_g OF THE WAVE EQU. (6) BECOMES THE NON-LINEAR SCHRÖDINGER EQU.

THE STABILITY OF SUCH A WAVE TO THE MODULATIONAL INSTABILITY IS FOUND BY WRITING

$$E_0 = Re\left\{ \mathcal{E}_0 + \mathcal{E}_1 e^{i(Kx - \Omega t)} + \mathcal{E}_2 e^{-i(Kx - \Omega t)} \right\}$$

THEN (6) becomes

$$(\Omega - K v_g)^2 = \left(\frac{1}{2} K^2 v_g' C_K |E_0|^2 - \frac{1}{4} K^4 v_g'^2 \right) \qquad -(7)$$

$$Re\,\Omega = K v_g \qquad \text{PURELY GROWING IN } v_g \text{ FRAME.}$$

$$Im\,\Omega = \frac{1}{\sqrt{2}} K v_g'^{\frac{1}{2}} \left[C_K |E_0|^2 - \frac{1}{2} K^2 v_g' \right]^{\frac{1}{2}} \qquad -(8)$$

MODULATION OCCURS IN A MEDIUM WHERE THE NONLINEAR FREQUENCY SHIFT AND GROUP DISPERSION HAVE OPPOSITE SIGN.

THERE IS ANOTHER REGIME THE KINETIC REGIME WHERE $\frac{m_e}{m_i}\frac{T_i}{T_e} \simeq k^2\lambda_{De}^2$

THE LOW FREQUENCY PERTURBATION CAN ONLY BE TREATED WITH THE VLASOV DESCRIPTION. - THIS INSTABILITY REGIME CAN BE FASTER THAN AN ION PERIOD.

STARTING FROM VLASOV DESCRIPTION AND POISSONS EQU. WE FIND

$$\delta n_e^\ell = \frac{i\epsilon_0 k_\ell E_\ell}{e}(1+\chi_i) \;;\; \delta n_e^\ell = \int f_{1e}^\ell dv \qquad (9)$$

χ_i IS ION SUSCEPTIBILITY

$$\underline{\nabla}\cdot\underline{E}^\ell = \frac{e}{\epsilon_0}\int f_{1i}\,dv - \frac{e}{\epsilon_0}\int f_{1e}\,dv \qquad (10)$$

$$f_{1i} = -\frac{ie}{m_i}\frac{E^\ell}{(\omega_q^\ell - k_q^\ell\cdot v)}\cdot\frac{\partial f_{0i}}{\partial v} \qquad (11)$$

$$f_{1e} = \frac{ie}{m_e}\frac{E^\ell}{(\omega^\ell - k^\ell\cdot v)}\cdot\frac{\partial f_{0e}}{\partial v} + \frac{ie}{m_e}\frac{E^h}{(\omega^h - k^h\cdot v)}\cdot\frac{\partial f_{0e}}{\partial v} \quad (12)$$

USING (10) AND (12) WE GET

$$E_\ell = \frac{i e k_\ell}{m_e\omega_0^2}\frac{\chi_e}{(1+\chi_e+\chi_i)}|E_0|^2 \qquad (13)$$

WHERE χ_e IS THE ELECTRON SUSCEPTIBILITY.

SUBSTITUTING EQU. (13) IN (9) YIELDS

$$\frac{\delta n_e^\ell}{n_0} = -\frac{\epsilon_0 k^2}{m_e\omega_{pe}^2 n_0}\frac{\chi_e(1+\chi_i)}{(1+\chi_e+\chi_i)}|E_0|^2 \qquad -(14)$$

SUBSTITUTING THIS IN THE EQU. FOR THE HIGH FREQUENCY WAVE FIELD EQUATION E_0 YIELDS

$$i\left(\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial x}\right)E_0 + \frac{1}{2}v_g'\frac{\partial^2}{\partial x^2}E_0 + C_k|E_0|^2E_0 = 0 \qquad -(15)$$

WHERE

$$C_K = \frac{\epsilon_0\omega_{pe}^4}{2n_0 KT_e\omega_0^2}\frac{v_{Te}^2}{c^2}\frac{k^2\lambda_e^2\;\chi_e(1+\chi_e)}{(1+\chi_e+\chi_i)} \qquad -(16)$$

$\lambda_e = c/\omega_{pe}$ IS THE COLLISIONLESS SKIN DEPTH.

$$\gamma_{GROWTH} \cong 1.2\,K v_{Te}\left(\frac{\epsilon_0|E_0|^2}{n_0 KT_e}\right) \gg \gamma_{REL.}$$

BENJAMIN FEIR STABILITY DIAGRAM

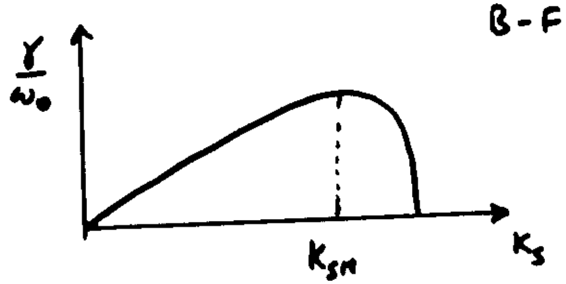
THE GROWTH RATE γ CAN IN GENERAL BE EXPRESSED AS

$$\frac{\gamma}{\omega_0} = K_s \left(\Gamma - K_s^2 \right)^{\frac{1}{2}}$$

K_s IS NORMALIZED MODULATION WAVENUMBER

$$K_s = \frac{k_s v_{Te}}{\omega_0} \qquad \text{FOR LANGMUIR WAVES}$$

B-F diagram



$$K_{sm} = \left(\tfrac{1}{2}\Gamma\right)^{\frac{1}{2}}$$

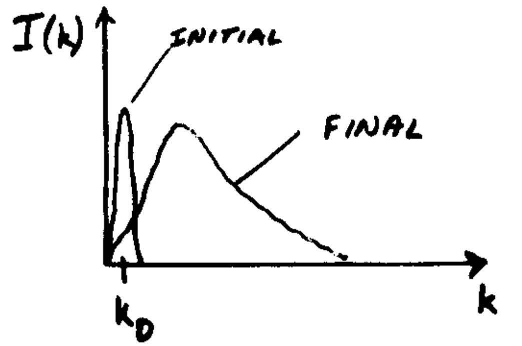
RANGE OF UNSTABLE K_s MODES

$$\frac{\Delta K_s}{K_{sm}} = \frac{\sqrt{2}}{2}$$

THERE CAN BE A LARGE NUMBER OF MODES EXCITED

$$k \rightarrow k_0 \pm \Delta k_s$$

GIVING RISE TO A TURBULENT SPECTRUM



DOES OUR SIMPLE DETERMINISTIC MODEL BREAK DOWN?

DO WE NEED TO DEVELOP FOR WEAKLY DRIVEN CASES A STATISTICAL DESCRIPTION?

PROBLEMS

1. MOST THEORIES TEND TO TREAT THE MONOCHROMATIC WAVE PROBLEM.

2. SELF - GENERATED MAGNETIC FIELDS, WILL CAUSE PROBLEMS IN THE FULLY NONLINEAR STAGE. (TSYTOVICH, see NAAR (1981)).
3. PLASMA HEATING - PLASMA IS USUALLY TREATED AS A FLUID, NO ELECTRON, ION HEATING.

THE FIRST PROBLEM COULD BE TACKLED USING THE WAVE - KINETIC DESCRIPTION FOR PLASMONS, PHOTON WAVE-PACKETS. TAPPERT (1971) SOLVED THE WAVE KINETIC NUMERICALLY TO STUDY SELF TRAPPING OF TRANSVERSE ELECTROMAGNETIC PULSES.

$$\frac{\partial N_k}{\partial t} + \frac{\partial \omega}{\partial \underline{k}} \cdot \nabla N_k - \frac{\partial \omega}{\partial \underline{x}} \cdot \nabla_k N_k = 0 \qquad -(17)$$

$N_k = \dfrac{|E|^2}{\omega}$ WAVE ACTION; PLASMON NUMBER DENSITY.

THIS EQUATION PLUS VLASOV AND MAXWELL'S EQUATIONS INCLUDING POISSON'S EQU. DESCRIBES THE BEHAVIOUR OF PARTICLES + PLASMONS INTERACTING.

IT CAN BE SHOWN THAT THE DIELECTRIC FUNCTION CAN BE WRITTEN AS

$$\epsilon = 1 + \chi_e + \chi_i + \chi_{NL} = 0 \qquad -(18)$$

WHERE $$\chi_{NL} = \alpha \int \frac{\underline{k}_s \cdot \frac{\partial N_0}{\partial \underline{k}}}{\left(\omega_s - \underline{k}_s \cdot \frac{\partial \omega}{\partial \underline{k}}\right)} \, dk$$

SOLVING (18) LEADS TO THE MODULATIONAL INSTABILITY ALSO.

NOTE χ_{NL} HAS REAL AND IMAGINARY PARTS.

$\Rightarrow$ DAMPING (GROWTH) OF WAVES IN PLASMA.

THE ANALYSIS WE HAVE CARRIED OUT SO FAR IS TO DESCRIBE LANGMUIR WAVES. SIMILAR ANALYSIS CAN BE CARRIED OUT FOR TRANSVERSE WAVES IN PLASMAS.

E.G. LASER LIGHT PASSING THROUGH NON-LINEAR MEDIA - SELF MODULATION INCREASES THE BAND WIDTH, FILAMENTATION TENDS TO BREAK THE BEAM UP INTO FILAMENTS.

THE SAME TYPE OF NONLINEARITIES ALSO ACT.

ie. PONDEROMOTIVE.

JOULE HEATING.

RELATIVISTIC.

WE GET THE NONLINEAR SCHRÖDINGER EQUATION. IN V_g FRAME.

$$\left[i\frac{\partial}{\partial t} + \frac{1}{2}\frac{c^2}{\omega}\underline{\nabla} + \lambda|E|^2 \right] E = 0 \qquad -19$$

$$\lambda = \frac{1}{2}\ \frac{n_0 e^2 \mu_0 c^2}{\omega^3 m_e^2\, KT_e}$$

{ FROM PONDEROMOTIVE FORCE

EQU. 19 HAS AN EQUILIBRIUM SOLUTION

$$E(x,t) = E \exp\left[i\lambda|E|^2 t \right]$$

ie
$$\mathcal{E} = E(x,t)\, e^{-i\omega_0 t}$$
$$= E\, e^{-i(\omega_0 - \lambda|E|^2)}$$

FREQUENCY SHIFT $-\lambda|E|^2$

CAN THIS BE THE REASON FOR THE SPREAD IN FREQUENCY.

PROBLEMS NOT TACKLED.

ROLE OF DENSITY GRADIENTS

ROLE OF B FIELD - NEW MODES.

ROLE OF TRAPPED PARTICLES ($\Delta\omega \propto \dot{\phi}^{\frac{1}{2}}$)

ROLE OF INCREASING THE BAND-WIDTH

MULTI-MODE PICTURE.

ROLE OF PARTICLE DISTRIBUTION FUNCTION.

PROBLEMS NOT TACKLED.

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