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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE CENTRATOM TRIESTE

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ASPECTS OF HELIOSEISMOLOGY

B. Roberts

University of St. Andrew's
Dept. of Applied Mathematics
Fife. Ky 169Ss
Scotland
U. K.

Lecture 4 B. Roberts
St. Andrews University

ASPECTS OF HELIOSEISMOLOGY

Effects of Stratification

The inclusion of gravity introduces a body force into the eqn. of momentum:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot \text{grad})\,\underline{v}\right) = -\text{grad}\, p + \underline{j}\times\underline{B} + \underline{g}\rho$$

where $\underline{g} = +g\,\hat{\underline{z}}$ for $\hat{\underline{z}}$ pointing DOWNWARDS $\quad \downarrow \underline{g}\rho \quad \downarrow z$

$(g = 274\ \mathrm{m\,s^{-2}})$

Thus we have: a preferred direction (the z-axis)

an additional force

and imposed length and timescales

— these are additional to any the magnetic field may provide.

Absence of $\underline{B}$: Equilibrium

In equilibrium $\left(\frac{\partial}{\partial t} \equiv 0, \underline{v} \equiv 0\right)$ we have $p = p_0$, $T = T_0$, $\rho = \rho_0$:

$$\nabla p_0 = +g\rho_0\hat{\underline{z}}, \text{ i.e. } p_0'(z) = +g\rho_0 \, . \quad ({}' \equiv d/dz)$$

Combined with gas laws:

$$p = \frac{k_B \rho T}{\mu m_H}$$

Boltzmann's constant, k_B

m_H mass of hydrogen atom

μ mean particle mass

Thus,

$$p_0' = +g\left(\frac{\mu m_H}{k_B T_0}\right)p_0 \, .$$

We may integrate to obtain

$$p_0(z) = p_0(0)\, e^{+n(z)}$$

$$\rho_0(z) = \rho_0(0)\, \frac{\Lambda_0(0)}{\Lambda_0(z)}\, e^{+n(z)}$$

where

$$\Lambda_0(z) = \frac{p_0}{p_0'} = \frac{p_0}{g\rho_0} = \frac{k_B T_0(z)}{\mu m_H g}$$ <u>pressure scale-height</u>

and $$n(z) = \int_0^z \frac{dz}{\Lambda_0(z)},$$ the 'number' of scale-hts between $z=0$ and z.

($z=0$ is an <u>arbitrary reference level</u>, usually taken to be at optical depth $\tau_{0.5} = 1$ in the photosphere)

<u>Examples</u> 1. <u>Isothermal atmosphere</u> (Λ_0 = constant, assuming μ constant)

Then ~~the~~ $n(z) = z/\Lambda_0$

and $p_0(z) = p_0(0)\, e^{z/\Lambda_0}$, $\rho_0(z) = \rho_0(0)\, e^{+z/\Lambda_0}$

i.e. pressure and density decline exponentially ~~fast~~ in height $(-z)$, with a scale of Λ_0 (i.e. pressure + density fall by a factor of $e (\simeq 2.718)$ in a heightchange of Λ_0). At the temperature minimum, $\Lambda_0 \simeq 100$ km

2. <u>Linear temperature</u> (constant temperature gradient)

With $\Lambda_0(z) = \Lambda_0(0) + z\Lambda_0'$, Λ_0' constant (and dimensionless)

Then $$n(z) = \frac{1}{\Lambda_0'} \log_e\left(1 + z\frac{\Lambda_0'}{\Lambda_0(0)}\right)$$

Introduce

$$z_0 \equiv \frac{\Lambda_0(0)}{\Lambda_0'} = \frac{c_S^2(0)}{(c_S^2)'}$$ temperature scale-ht. at $z=0$

for <u>sound speed</u> c_S:

$$c_S^2(z) = \frac{\gamma p_0}{\rho_0} = c_S^2(0) + (c_S^2)' z = c_S^2(0)\left(1 + \frac{z}{z_0}\right)$$
$$= \gamma g \Lambda(z)$$

and

$$m = \frac{z_0}{\Lambda_0(0)} - 1 \;; \qquad m+1 = \frac{\gamma g}{(c_S^2)'}$$

Then

$$\frac{p_0(z)}{p_0(0)} = \left(1 + \frac{z}{z_0}\right)^{m+1}, \quad \frac{\rho_0(z)}{\rho_0(0)} = \left(1 + \frac{z}{z_0}\right)^{m},$$

so $p_0(z) \propto \rho_0^{1+\frac{1}{m}}$

m is called the polytropic index.

General Considerations

With no stratification, no g, we have

(isotropic) **sound waves** $c_s = (\gamma p_0/\rho_0)^{1/2}$

If $g \neq 0$, then sound waves must be anisotropic.

Also, because of the new force ($g \neq 0$) another mode of oscillation arises:

gravity waves

These two modes are also called

p- and g- modes

Lengthscales The inclusion of g introduces a lengthscale

$$H = \frac{\rho_0}{\rho_0'} \qquad \text{density scale ht.}$$

Similarly, $\Lambda \equiv \Lambda_0 \equiv p_0/p_0'$ pressure scale ht.

Combined with c_s, we get a

Timescale In terms of frequency,

$$\omega_a = \frac{c_s}{2H} \qquad \text{acoustic cutoff freq.}$$

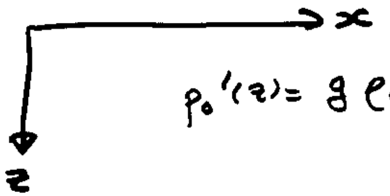
This is the cutoff frequency. Can also construct

$\omega^2 = g - g^2$ (squared) buoyancy freq.

Sound Waves in Stratified Atmosphere $B=0$

plane atmosphere

treatment actually includes p and g-modes, f-mode and unstable g-modes (convection).



$$p_0'(z) = g\,\rho_0(z)$$

Eqm $\Rightarrow$ $p_0(z) = p(0)\exp\left(\int_0^z \frac{dz}{\Lambda}\right)$

$$\Lambda(z) = \frac{p_0(z)}{\rho_0(z)\,g} = \frac{k_B T_0(z)}{mg} \qquad \text{pressure scale ht}$$

(Note: $c_s^2(z) = \gamma g \Lambda(z)$)

Perturbations

Lamb (1932), Spiegel + Unno (1964)

Introduce $\Delta = \mathrm{div}\,\mathbf{v}$

Consider [2D]-motions $\mathbf{v} = (v_x, 0, v_z)\exp i(\omega t - k_x x)$

Then

$$\boxed{\begin{aligned}(g^2k_x^2 - \omega^4)v_z &= \omega^2 c_s^2 \frac{d\Delta}{dz} + g(\gamma\omega^2 - k_x^2c_s^2)\Delta \\ \frac{dv_z}{dz} + \left(\frac{g k_x^2}{\omega^2}\right)v_z &= \left(1 - \frac{k_x^2 c_s^2}{\omega^2}\right)\Delta\end{aligned}} \quad (*)$$

f-mode Set $\Delta = 0$. Then (*) are satisfied if $\omega^2 = gk$

(provided b.condns. on v_z are met).

p- and g-modes

Elim. v_z:

$$\boxed{\frac{d^2\Delta}{dz^2} + \left(\frac{c_s^{2\prime}}{c_s^2} + \frac{\gamma g}{c_s^2}\right)\frac{d\Delta}{dz} + \left\{\frac{\omega^2 - k_x^2 c_s^2}{c_s^2} - \frac{g k_x^2}{\omega^2}\left(\frac{c_s^{2\prime}}{c_s^2} - \frac{(\gamma-1)g}{c_s^2}\right)\right\}\Delta = 0} \quad (**)$$

Introduce $Q = \rho_0^{1/2} c_s^2 \Delta$ to give

$$\boxed{\frac{d^2 Q}{dz^2} + \kappa^2(z) Q = 0}$$

where

$$\kappa^2(z) = \frac{\omega^2 - \hat{\omega}_a^2}{c_s^2} + k_x^2\left(\frac{\omega_g^2}{\omega^2} - 1\right)$$

$$\hat{\omega}_a^2 = \frac{c_s^2}{4H^2}(1 + 2H'), \quad H = \frac{\rho_0'}{\rho_0}$$

modified acoustic cutoff, $\hat{\omega}_a$

$$\omega_g^2 = \frac{g}{H} - \frac{g^2}{c_s^2} \quad \text{buoyancy freq.}$$

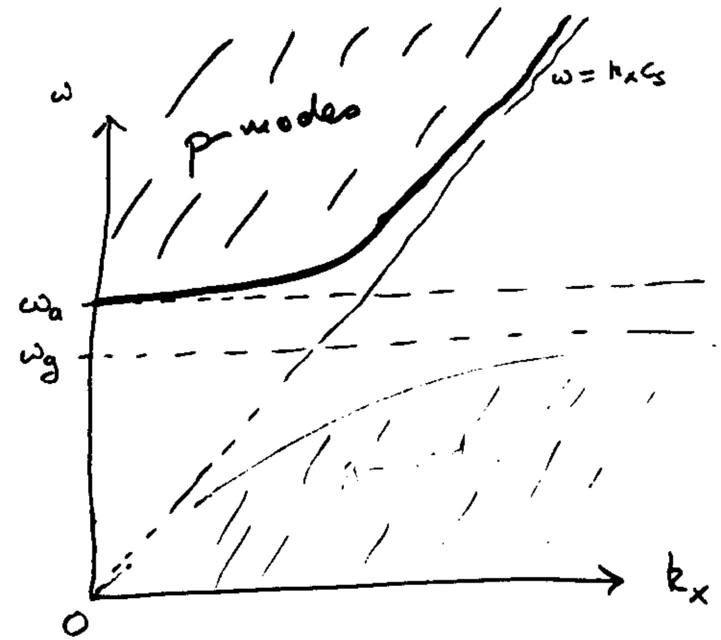
Waves for $\kappa^2 > 0$.

Isothermal atmosphere H, c_s constant.

Dispersion reln. ($Q \propto e^{-ik_z z}$)

$$k_z^2 = \frac{\omega^2 - \omega_a^2}{c_s^2} + k_x^2\left(\frac{\omega_g^2}{\omega^2} - 1\right)$$

waves for $k_z^2 > 0$ $\rightarrow$ diagnostic diagram.



Diagnostic diagram for Isothermal atmosphere

$$\omega_a^2 = \frac{c_s^2}{4H_0^2}, \quad \omega_g^2 = \frac{g}{H_0}\left(\frac{\gamma-1}{\gamma}\right)$$

solar photosphere at the temp. min.

$$H_0 = 100\text{ km}, \quad \nu_a \equiv \frac{\omega_a}{2\pi} = 5.38\text{ mHz}, \quad \tau_a = 1/\nu_a = 186\text{ s}$$

$$\nu_g \equiv \frac{\omega_g}{2\pi} = 5.27\text{ mHz}, \quad \tau_g = \nu_g^{-1} = 190\text{ s}$$

Linear Profile $\quad c_s^2(z) = c_0^2\left(1+\frac{z}{z_0}\right), \quad z \geq 0$

So $\rho_0(z) \propto (1+z/z_0)^{m+1}$ - $p_0(z) \propto (1+z/z_0)^{m}$

where $\quad m+1 = \frac{z_0}{\Lambda(0)} = \frac{\gamma g}{(c_s^2)'}$ $\quad$ [Note: $m \downarrow$ if temp. grad grows $\uparrow$]

Then $\quad \omega_a^2 = \frac{m(m+2)\,c_0^2}{4 z_0 (z+z_0)}, \quad \omega_g^2 = \left[m - \frac{1}{\gamma}(m+1)\right]\frac{g}{(z+z_0)}$

Eqn. (**) has exact soln. (Lamb)

$$\boxed{\Delta = e^{-k_x(z+z_0)}\left\{C\,M(-a, m+2, 2k_x z + 2k_x z_0) + D\,U(-)\right\}}$$

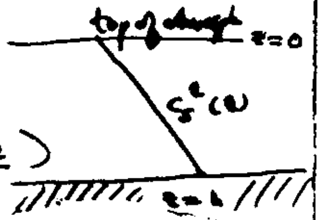
M, U confluent hypergeometric fns

C, D arb. constants

$$2a = \left(\frac{m+1}{\gamma}\right)\Omega^2 + \left(\left(\frac{\gamma-1}{\gamma}\right)m - \frac{1}{\gamma}\right)\frac{1}{\Omega^2} \quad -(m+2)$$

$$\Omega^2 \equiv \omega^2/g k_x$$

Polytrope $\quad$ Take $\left.\begin{matrix} c_0^2 = 0 \\ z_0 = 0 \end{matrix}\right\}$ but $c_0^2/z_0 \neq 0$



Then if $v_z = 0$ at $z = h$ $\quad$ (base)

and v_z finite at $z=0$ $\quad$ (top)

we get $D=0$ and the dispersion reln. (Spiegel + Unno, '64)

$$\boxed{2\Omega^2\frac{M'}{M} = 1+\Omega^2 - \frac{(m+1)\Omega^2}{k_x h}}$$

When $k_x h \gg 1$ (deep atmosphere) the polytrope dispn reln. yields

$$a \approx n-1$$

i.e. $\quad \boxed{(m+1)\Omega^4 - \gamma(m+2n)\Omega^2 + [(\gamma-1)m - 1] = 0}$

Two roots : $\quad$ p- modes $\longrightarrow$ Sturmian sequence of $\omega^2(\Omega^2)$

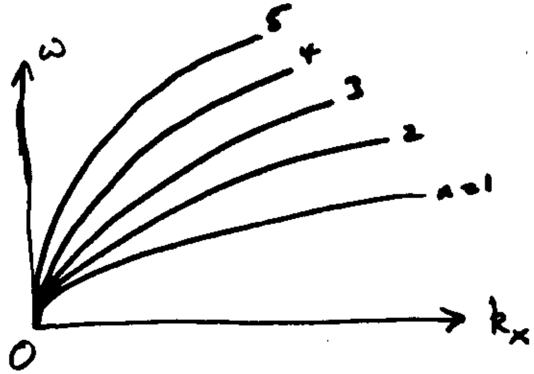
$\qquad\qquad\quad$ g- modes $\longrightarrow$ anti-Sturmian " " frequencies accumulating at $\omega^2=0$ ...

If $(\gamma-1)m < 1$ (i.e. if $\omega_g^2 < 0$), then the g-modes are UNSTA[BLE]

For adiabatic stratification (until $\omega_g = 0$), g-modes give $\omega^2 = 0$

and p-modes are

$$\Omega^2 = \Omega_n^2 \equiv 1 + \frac{2n}{m}, \quad n = 1, 2, 3, \ldots$$



parabolas
$\omega_n^2 \sim k_x$

Such modes are observed!

Observations. Parabolas first obtained by Deubner('75), when no theoretical consensus existed (to explain 5^m oscilln. reported by Leighton et al. in 1962); Deubner's measurements showed that the models proposed (indep.) by Ulrich('70) and Leibacher + Stein ('71) were the correct ones.

$$\boxed{(m+1)\Omega^4 - \gamma(m+2n)\Omega^2 + [(\gamma-1)m - 1] = 0}$$

(Spiegel + Unno '62)

– quadratic for $\Omega^2 \equiv \omega^2/gk_x$

– 2 roots, $+ \rightarrow$ p-modes, $- \rightarrow$ g-modes

Large n: $\omega^2 \sim \omega_n^2$ where

$$\omega_n^2 = \begin{cases} 2\gamma g k_x (n + \frac{1}{2}m)/(m+1) & \text{p-modes} \\ 2\gamma g k_x (m - \frac{m+1}{\gamma})/(m+2n) & \text{g-modes} \end{cases}$$

p-modes are Sturmian i.e.

$$\omega_1^2 < \omega_2^2 < \omega_3^2 < \dots < \omega_n^2 < \dots$$

with $\omega_n^2 \rightarrow \infty$ as $n \rightarrow \infty$

g-modes are anti-Sturmian, i.e. (with $(\gamma-1)m > 1$)

$$\omega_1^2 > \omega_2^2 > \omega_3^2 > \dots > \omega_n^2 > \dots$$

with $\omega_n^2 \rightarrow 0$ as $n \rightarrow \infty$

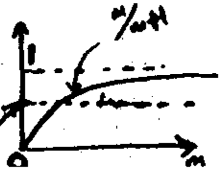
g-modes become unstable if $(\gamma-1)m < 1$

i.e. if $\omega_g^2 < 0$

Schwarzschild (1906) criterion

– this corresponds to convection.

Convection occurs if $\frac{m}{m+1} < \frac{m_{adiab}}{1+m_{adiab}}$, $m_{adiab} \equiv \frac{1}{\gamma-1}$



$(\gamma = 5/3:$ $m < \frac{3}{2})$

Libbrecht + Kaufman (1988, Ap. J. 324, 1172)

Low and intermediate degree modes

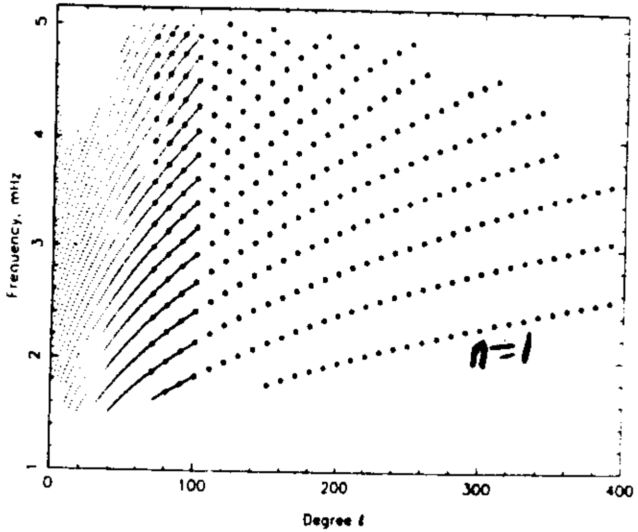


FIG. 5.—Schematic ℓ-ν diagram of the p-mode frequencies added by the intermediate-resolution data set (open circles) along with the points from DHL,PP and the ... resolution analysis. The lowest-n ridge has n = 1. The intermediate-resolution points are listed in Table 2.

High degree modes

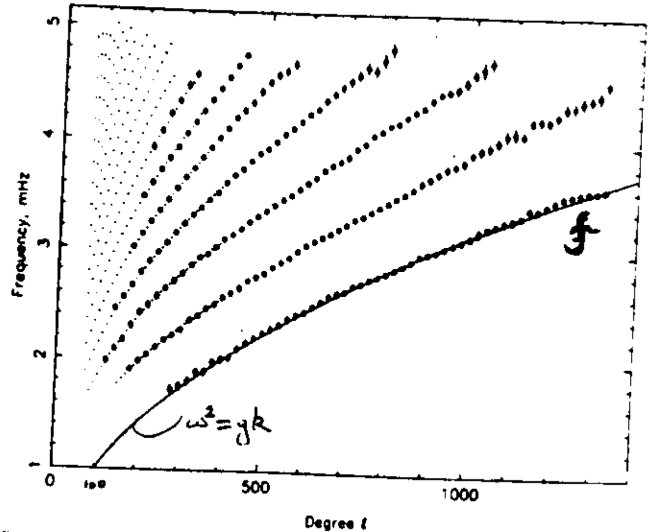
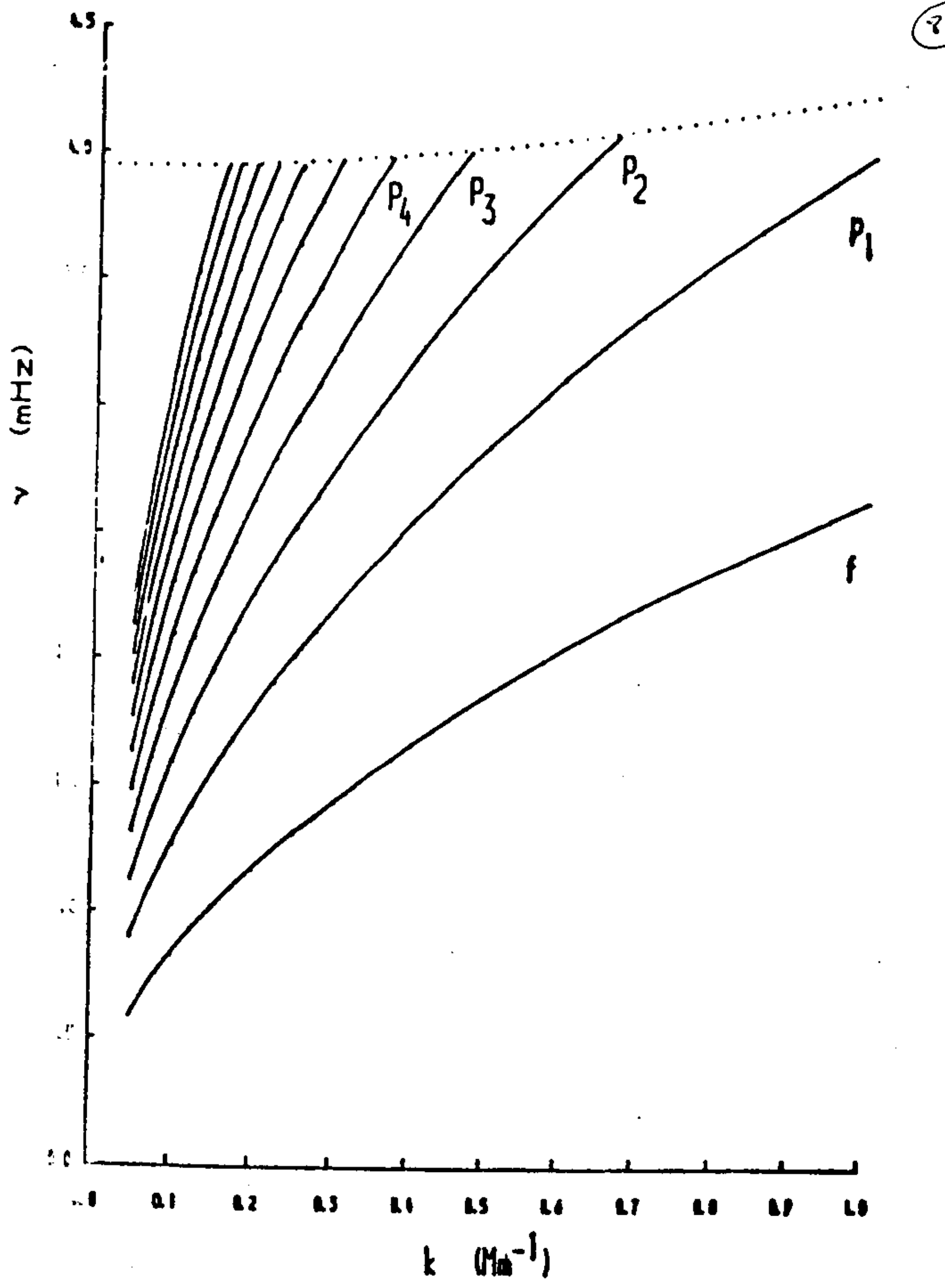


FIG. 6.—Schematic ℓ-ν diagram of the p-mode frequencies added by the high-resolution data set (open circles), along with the intermediate-resolution points. The lowest-n ridge has n = 0, consisting of the f-modes. The smooth curve is the analytic expression for the mode frequencies, $\omega^2 = g k$. After considering residual systematic errors in this data set, the points are not significantly displaced from the line (see text).

Observations



Theory

(k=1:2=7cc)

WKB Aspects

Take $\kappa^2 = \frac{\omega^2}{c_s^2(z)} - k_x^2$

(setting $\omega_g^2=0$ and neglecting $\hat{\omega}_a^2$)

Bohr-Sommerfeld reln. gives

$$\boxed{\int_{cavity} \kappa \, dz \sim (n+\alpha)\pi}$$ (XX) $(n \to \infty)$.

phase α (= ½ in 2 point cavity)

Polytrope
E.g. Take top of cavity as $z=0$ and bottom at $\kappa^2(z_c)=0$, viz

$$z = z_c = (m+1)\frac{\omega^2}{\gamma g k_x^2}$$

$$\therefore \int_0^{z_c} \left[\frac{\omega^2(m+1)}{\gamma g z} - k_x^2\right]^{1/2} dz \sim (n+\alpha)\pi$$

$$\Rightarrow \frac{\omega^2}{g k_x} = \frac{2}{m}(\alpha+n) = 1 + \frac{2n}{m}$$ (as before)

provided $\alpha = m/2$

and then

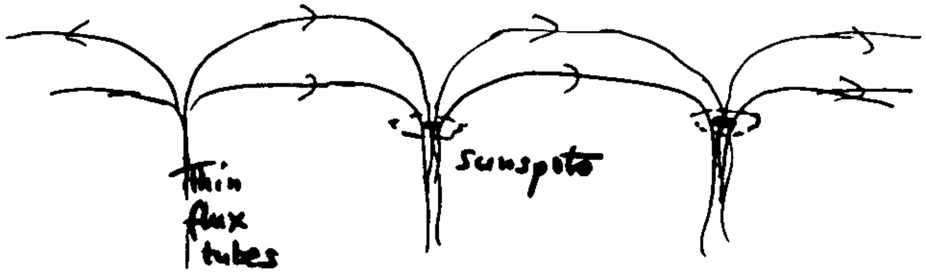
cavity depth $z_c = \frac{2(n+\alpha)}{k_x} \propto \lambda_x$ wavelength

Eqn. (XX) may be cast in the form of Abel's integral eqn. and inverted [with $k_x^2 = \frac{l(l+1)}{r^2}$ for a sphere] to get the sound speed. This has been done for the Sun (Christensen-Dalsgaard et al. '85).

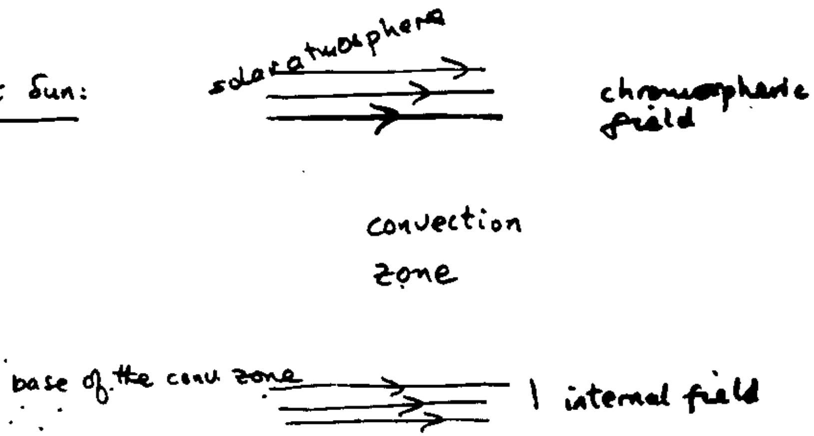
Thus, helioseismology has given us the run of sound speed in the Sun

MAGNETIC EFFECTS

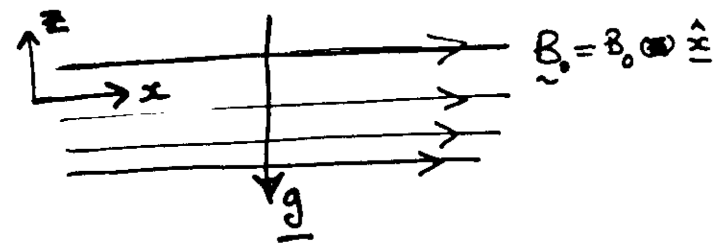
Magnetic Canopy



A flat Sun:



Effect of a horizontal $\underset{\sim}{B}$



arbitrary stratification in $c_s(z)$.

EQM. $\left(p_0(z) + \dfrac{B_0^2(z)}{2\mu_0}\right)' = -\rho_0(z)\,g$

Linear perturbations: consider [2D] motions,

$\underline{v} = (v_x, 0, v_z)\, e^{i(\omega t - kx)}$

Find (from ideal mhd eqns)

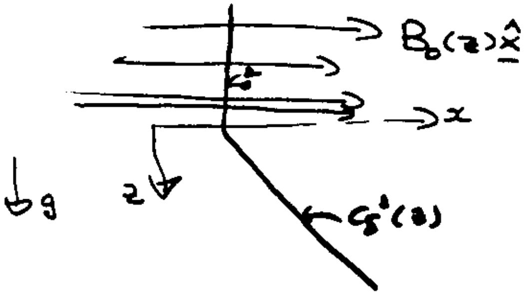
$$\boxed{0 = \frac{d}{dz}\left\{\frac{\rho_0(c_s^2+v_A^2)(\omega^2-k^2c_T^2)}{(\omega^2-k^2c_s^2)}\frac{dv_z}{dz}\right\} + \left\{\rho(\omega^2-k^2v_A^2) - \frac{g^2k^2\rho}{(\omega^2-k^2c_s^2)} - gk^2\left(\frac{\rho c_s^2}{\omega^2-k^2c_s^2}\right)'\right\}v_z} \quad (*)$$

Complicated eqn! No exact solns. except in very special cases (e.g. constant c_s, v_A)

Continuous spectra – associated with slow wave

includes: fast waves, slow waves, gravity waves, magnetic buoyancy, compressible magnetoconvection in horiz. $\underset{\sim}{B}$, etc. !

Atmosphere



$\rho_0(0-)$

$= (\ldots + \frac{\gamma}{2\beta}) \ldots$

$\beta = \frac{c_0^2}{v_A^2/2}$

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Atmosphere ($z<0$) is taken to be isothermal.

Mhd eqn. is

$$\frac{d}{dz}\left\{ \frac{\rho_0(c_s^2+v_A^2)(\omega^2-k^2c_T^2)}{(\omega^2-k^2c_s^2)} \frac{du_z}{dz}\right\} = \left\{ \frac{Ag^2k^2}{\omega^2-k^2c_s^2} - \rho_0(\omega^2-k^2c_T^2) - gk^2\left(\frac{A c^2}{\omega^2-k^2c_s^2}\right)'\right\} u_z$$

Take $v_A =$ constant. Then

$$u_z = u_0 e^{z/2H_B\left(-1 \pm \sqrt{1-4AH_B^2}\right)}, \quad z<0$$

$$H_B^{-1} = \frac{\rho_0'}{\rho_0} = \frac{\Gamma g}{c_0^2}, \quad \Gamma = \frac{\gamma}{(1+\frac{\gamma}{2\beta})}$$

$[\Gamma = \gamma,\ H_B = H_0 \text{ if } B_0 = 0]$

Suppose $4AH_B^2 < 1$, and choose + sign so $e^{z/2H_B}u_z \to 0$ as $z \to -\infty$.

Across $z=0$: u_z cts

$$\frac{\rho_0(c_s^2+v_A^2)(\omega^2-k^2c_T^2)}{(\omega^2-k^2c_s^2)}\frac{du_z}{dz} + \left(\frac{gk^2\rho_0 c_s^2}{\omega^2-k^2c_s^2}\right)u_z \quad \text{cts.}$$

$\rho_0 u_z^2$ bounded as $z \to \infty$

In non-magn. region: $\Delta = e^{-k(z+z_0)}U(-a, m+2, 2kz+2kz_0)$, $z>0$

satisfying $\rho_0 u_z^2$ bounded as $z \to \infty$

Dispersion Reln

Campbell + R. ('89) Ap. J.

$$\boxed{2k\omega^2c_0^2\frac{U'}{U} + \gamma g\omega^2 - k\omega^2c_0^2 - gk^2c_0^2 = \frac{(\omega^2-k^2c_0^2)(g^2k^2-\omega^4)(c_0^2+\frac{1}{2}\gamma v_A^2)}{gk^2c_0^2 + (c_0^2+v_A^2)(\omega^2-k^2c_T^2)\lambda}}$$

$$\lambda = \frac{1}{2H_B}\left[-1 + (1-4AH_B^2)^{1/2}\right]$$

complicated!

U' eval. at $2kz_0$, $\quad z_0 = \frac{c_s^2(0)}{(c_s^2)'}$

Dispn. reln. includes p-modes (modified by $\underline{B}$)
f-mode (modif. by $\underline{B}$)
MHD surface waves

MAGNETOACOUSTIC CUTOFF

For $kz_0 \ll 1$ may show that

$$\nu_n = \nu_0 + C_n(c_s, v_A)\,k^{m+5/2}$$

p-modes

freq. in absence of an atmosphere

correction due to an atmosphere

Similar for f-mode:

$$\nu_n = \bar{\nu}_0\left[1 + \frac{(2c_0^2+v_A^2)v_A^2}{(c_0^2+v_A^2)c_0^2}\frac{2^{m-1}(m+1)^{m+2}}{m\,\Gamma(m+2)}(kH_0)^{m+2}\right]$$

Here atmosph. correction is due entirely to the atmosphere being magnetic. $\nu \uparrow$ when $B \uparrow$

(13')

Atmospheric corrections

(analytical expansions)

$\left(\gamma = \frac{5}{3}, \quad m = 3/2\right)$

$B_0 = 0$

p_n-modes

l	$n=1$	$n=2$	$n=3$	$n=10$
100	−0.01	−0.009	−0.008	−0.008
500	−8.2	−5.4	−4.8	−5.1
900	−86	−57	−50.5	−54
1300	−375	−248	−220	−236

Atmospheric corrections (to mode p_n) in μHz

$B_0 = 10G$

'quiet' sun

l	$n=1$	2	3	10
100	−0.01	−0.009	−0.008	−0.008
500	−8.2	−5.4	−4.8	−5.2
900	−86	−57	−51	−54
1300	−374	−249	−221	−236

Note: negligible dif. from $B_0=0$ case

$B_0 = 10^2 G$

'Active' Sun

l	$n=1$	2	3	10
100	−0.01	−0.01	−0.01	−0.01
500	−6.7	−7.2	−7.3	−8.7
900	−70	−76	−76	−92
1300	−306	−329	−331	−399

$B_0 = 200G$

l	$n=1$	2	3	10
100	−.01	−0.02	−02	−0.03
500	−4.1	−13.1	−14.9	−19.7
900	−43	−137	−156	−207
	−187	−597	−677	−900

Evolution of B_0

(Campbell + R. '89)

$$\Delta\nu \equiv \nu(B_0) - \nu(B_0=0)$$
$$= \text{atmosph. corr.}(B_0) - \text{atmos. corr}(B_0=0)$$

Low to moderate l: effect negligible.

p-modes

$\Delta\nu$ (in μHz)

	l	$B_0=10G$	$100G$	$200G$
p_1:	500	0.02	1.5	4.1
	900	0.2	15.7	43.
	1300	0.8	68.	187.
p_2:	500	−0.02	−1.8	−7.6
	900	−0.2	−25.	−80.
	1300	−0.8	−83.	−349.
p_{10}:	500	−0.03	−3.6	−14.6
	900	−0.4	−38.	−153.0
	1300	−1.6	−164.	−665.7

Note sign change in $\Delta\nu$ for $n=1$ compared with $n \geq 2$

f-mode

$$\Delta\nu \equiv \nu(B_0) - \nu(0)$$

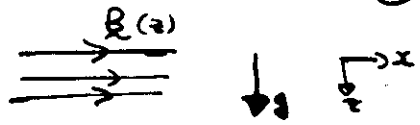
f-mode

$$\nu(0) = \frac{(gk)^{1/2}}{2\pi}$$

l	$B_0 = 10G$	$10^2 G$	$200G$
500	0.01	0.85	2.79
900	0.10	8.90	29.2
1300	0.44	38.7	127.1

Note: $\Delta\nu_f > 0$ whereas $\Delta\nu_{p_n} < 0$ $(n=2,3,\ldots)$

A Magnetic Lamb's Eqn.



Horizontal magnetic field, $B(z)\,\underline{\hat{x}}$

Magnetostatic equil^{m}.:

$$\left.\begin{aligned} &\frac{d}{dz}\left(p + \frac{B^2}{2\mu_0}\right) = g\rho \\ &p = \frac{k_B}{\bar{m}}\rho T \end{aligned}\right\}$$

arb. $T(z)$

Perturbations

$$\underline{v} = (v_x(z), 0, v_z(z))\, e^{i(\omega t - kx)}$$

Eqns. of ideal MHD yield

$$\left.\begin{aligned} &\left(1 - \frac{k^2 c_s^2}{\omega^2}\right)\Delta = \frac{dv_z}{dz} + \left(\frac{gk^2}{\omega^2}\right) v_z \\ &c_s^2 \frac{d\Delta}{dz} + (\gamma - 1) g\Delta - \gamma\left(\frac{BB'}{\mu_0\rho}\right)\Delta = -g\frac{dv_z}{dz} + (k^2 v_A^2 - \omega^2) v_z \\ &\qquad\qquad - \frac{1}{\rho}\frac{d}{dz}\left\{\frac{B^2}{\mu_0}\frac{dv_z}{dz}\right\} \end{aligned}\right\}$$

$[\Delta \equiv \operatorname{div} \underline{v}]$

Elim. of $\Delta \Rightarrow$

$$\frac{d}{dz}\left\{\frac{\rho(c_s^2 + v_A^2)(\omega^2 - k^2 c_T^2)}{(\omega^2 - k^2 c_s^2)}\frac{dv_z}{dz}\right\}$$

$$= \left\{\frac{\rho g^2 k^2}{\omega^2 - k^2 c_s^2} - \rho(\omega^2 - k^2 v_A^2) - gk^2\left(\frac{\rho c_s^2}{\omega^2 - k^2 c_s^2}\right)'\right\} v_z$$

Compare with Lamb's eqn. when $\underline{v_A = 0}$:

$$\boxed{\frac{d^2 Q}{dz^2} + \kappa^2(z) Q = 0,} \qquad Q = \rho^{1/2} c_s^2 \Delta$$

where

$$\boxed{\kappa^2 = \frac{\omega^2 - \omega_a^2}{c_s^2} + k^2\left(\frac{\omega_g^2}{\omega^2} - 1\right)}$$

Note simplicity!

$$\omega_g^2 = \frac{g}{H} - \frac{g^2}{c_s^2}$$ buoyancy freq.

$$\omega_a^2 = \frac{c_s^2}{4H^2}(1 + 2H')$$ cutoff freq.

$$H = \rho/\rho'$$ density scale-height.

What happens if $\underline{v_A \neq 0}$?

In general, κ is a mess!

But for $\omega \gg k v_A$ we may obtain (Roberts + Campbell '86)

(constant B)

$$\boxed{\kappa^2 = -k^2 + \frac{\omega^2}{c_s^2 + v_A^2} - \frac{\omega_a^2}{c_s^2} + \frac{k^2\omega_g^2}{\omega^2}\left(\frac{c_s^2}{c_s^2 + v_A^2}\right) + \frac{k^2 c_s^2 v_A^2}{(c_s^2 + v_A^2)^2}} + O\left(\frac{k^2 v}{\omega^2}\right.$$

This is the desired expression for a "magnetic Lamb's eqn."

Magnetic effects

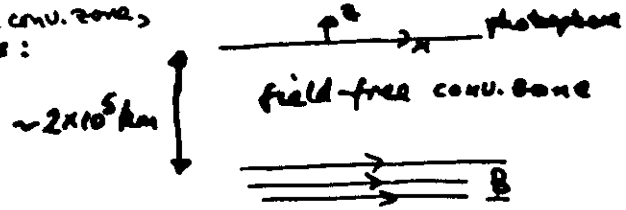
g-modes – cts. spectrum (of slow waves) destroys anti-sturmian g-mode prop. at low ω_n^2 (high n)

p-modes

p-modes scatter off
- isolated fluxtubes (Bogdan + Zweibel '85, ZW + Bogdan '86)
- sunspots (which form a sink for p-modes; Braun et al. '87 preprint, Abdelatif and Thomas '87 ")

what is $\underline{B}$ at base of convection zone?

Also, p(and g-modes) interact with $\underline{B}$ in chromosphere, and B at base of convection zone. Roberts + Campbell '86, in an attempt to deduce $\underline{B}$ at base conv. zone, have argued as follows:



The magnetic equivalent of (XX) may be derived – but is very complicated in algebra. For high frequency limit appropriate for p-modes, R+C get (for $\omega \gg$ large)

$$\boxed{K^2(r) \equiv -k^2 + \frac{\omega^2}{c_s^2+v_A^2} - \frac{\omega_a^2}{c_s^2} + \frac{k^2 c_T^2}{c_s^2+v_A^2} + \frac{k^2 v_A^2}{\omega^2}\left(\frac{c_s^2}{c_s^2+v_A^2}\right)}$$

– result is exact for $g=0$, $k=0$ or $v_A=0$.

Applying standard WKB theory then gives

$$\boxed{\nu_B = \left(n+\tfrac{1}{2}l+\underbrace{\varepsilon}_{\text{phase factor}}\right)(\Delta\nu_0)\left(1+\frac{v_A^2}{2c_s^2}\right)} \qquad \text{(XX)}$$

for cyclic freq. $\nu_B \equiv \omega/2\pi$, and $k^2 \equiv \dfrac{l(l+1)}{R_\odot^2}$, $\Delta\nu_0 = \left(2\int_0^{R_\odot}\frac{dr}{c_s}\right)^{-1}$

Consider now the change in freq. due to a change in $\underline{B}$ over solar cycle. If ν_B is freq. in a field of strength $|\underline{B}_0|$ and $\nu_{B'}$ is freq. in field of strength $|\underline{B}_0'|$ then (from (XX))

$$\boxed{B^2 - B'^2 = 8\pi \rho_0 c_s^2 \frac{1}{\nu_0}(\nu_B - \nu_{B'})}$$

where ν_0 = freq. in absence of B ($\nu_0 \simeq 3.1$ mHz)

Now SMM ACRIM OBSERVATIONS by Woodard and Noyes (1985) give

a freq. change (decrease) of $\simeq 0.4\ \mu$Hz (in p-modes low l)

from 1980 (near solar max.) to 1984 (approaching min)

Applied to the above formula (with $\rho_0 = 0.2$ g cm^{-3}, $c_s = 2\times10^7$ cm s^{-1}) gives

B must be at least 5×10^5 G (This is strong; latest dynamo calc. anti-chase $B = 10^4$ G)

if a changing B is the cause of a change in freq.

But note:
- observs. are at limit of instrument so uncertain
- Elsworth et al ('87) do not confirm W+Noyes result (ground-based network)
- but very recently, Fossat et al. '87 confirm W+N result on analysing 704 hrs. of South Pole data, and Woodard has given ACRIM results for '87 – showing turn up.
- only a longer chain of obs. (say to 1990) will confirm or refute the claim that p-modes change over the solar cycle.