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SPRING COLLEGE ON PLASMA PHYSICS

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PARAMETRIC PROCESSES IN PLASMAS

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Parametric Processes in Plasmas

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1. Intuitive Theory
2. Kinetic Theory for Dipole Pump
3. Finite Wavelength Weak Pump
4. Parametric Processes in Curren Drive

Stimulated Raman Scattering and its Interplay with Stimulated Brillouin Scattering in Laser Produced Plasma.

1. Intuitive theory

Parametric excitation:

$$\frac{d^2X(t)}{dt^2} + \Omega^2 X(t) = 0 \quad \text{(pendulum equation)}$$

$$\underbrace{\Omega^2}_{\text{[parameter]}} = \Omega_0^2 \underbrace{[1+Q(t)]}_{\text{[modulation]}}, \qquad \underbrace{Q(t+T) = Q(t)}_{\text{[periodic]}}$$

For $|Q| \ll 1$ (weak modulation)

$\Rightarrow$ amplification of $X(t)$ when $\underbrace{\boxed{\Omega_0 T \doteqdot l\pi}}_{\text{[matching condition]}}$ $(l = 1, 2, \cdots)$

due to coupling of $\begin{cases} X_+ \sim \exp[-i\Omega_0 t] \\ X_- \sim \exp[+i\Omega_0 t] \end{cases}$ and

[mode coupling]

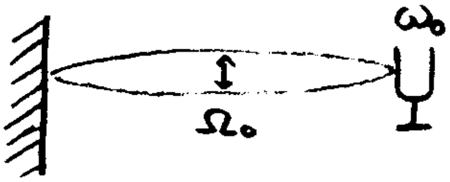
cf. Bragg reflection condition in solids

$$\frac{d^2\psi(x)}{dx^2} + [k^2 + V(x)]\psi(x) = 0, \quad \underbrace{V(x+a) = V(x)}_{\text{[periodic lattice potential]}}$$

$ka \doteqdot l\pi$ ⋯ resonant momentum exchange with lattice

$\Rightarrow$ electron energy forbidden

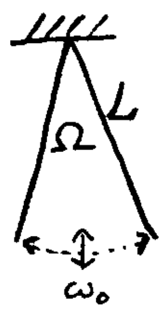
1883: Lord Rayleigh



amplification of string vibration when

$$\omega_0 \doteqdot 2\Omega_0$$

Child's swing



$$\Omega = \sqrt{\frac{g}{L}}$$

$\omega_0 \doteqdot 2\Omega$ modulation of L.

$\Rightarrow$ amplification of swing.

Mathieu equation model

$$\frac{d^2X(t)}{dt^2} + \Omega_0^2\left[1 - 2\varepsilon\cos\omega_0 t\right]X(t) = 0$$

$|\varepsilon| \ll 1 \Rightarrow$ instability when $\omega_0 \doteqdot \frac{2}{\ell}\Omega_0$

Perturbation analysis

$$X(t) = \int d\omega\, e^{-i\omega t} X(\omega)$$

$$D(\omega)X(\omega) = -\varepsilon\Omega_0^2\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$$

where $D(\omega) = \omega^2 - \Omega_0^2$

$D = 0$: linear dispersion relation $\omega = \pm\Omega_0$

$$D(\omega\pm\omega_0)X(\omega\pm\omega_0) = -\varepsilon\Omega_0^2\left[X(\omega) + X(\omega\pm 2\omega_0)\right]$$

. . . .

$$D(\omega\pm m\omega_0)X(\omega\pm m\omega_0) = -\varepsilon\Omega_0^2\Big[X(\omega\pm[m+1]\omega_0) + X(\omega\pm[m-1]\omega_0)\Big]$$

$|\varepsilon| \ll 1$: Resonance condition

$D(\omega\pm m\omega_0) \doteqdot 0$ or $\omega \pm m\omega_0 \doteqdot \pm\Omega_0$

Otherwise $X(\omega\pm m\omega_0) \doteqdot 0$

$m=1$: $\omega_0 \doteqdot 2\Omega_0$ (correspond to $\ell = 1$)

$m=2$: $\omega_0 \doteqdot \Omega_0$

. . . .

. . . .

Case I $\omega_0 \doteqdot 2\Omega_0$

$\omega \doteqdot \Omega_0$ [resonant] $\longrightarrow$ $\omega - \omega_0 \doteqdot -\Omega_0$ [resonant]

$\searrow$ $\omega + \omega_0 \doteqdot 3\Omega_0$ [off resonant]

$$\left.\begin{aligned} D(\omega)X(\omega) &= -\varepsilon\Omega_0^2 X(\omega-\omega_0) \\ D(\omega-\omega_0)X(\omega-\omega_0) &= -\varepsilon\Omega_0^2 X(\omega) \end{aligned}\right\} \text{direct coupling of two resonant modes}$$

$\Rightarrow$ Dispersion relation: $\boxed{D(\omega)D(\omega-\omega_0) = \varepsilon^2\Omega_0^4}$

Resonance approximation

$$D(\omega) = \omega^2 - \Omega_0^2 = (\omega-\Omega_0)(\omega+\Omega_0) \doteqdot 2\Omega_0(\omega-\omega_0)$$

$$D(\omega-\omega_0) = (\omega-\omega_0-\Omega_0)(\omega-\omega_0+\Omega_0) \doteqdot -2\Omega_0(\omega-\omega_0+\Omega_0)$$

$$\doteqdot -2\Omega_0(\omega-\Omega_0-\Delta)$$

where $\Delta = \omega_0 - 2\Omega_0$ [frequency mismatch]

Dispersion relation:

$$-4\Omega_0^2(\omega-\Omega_0)(\omega-\Omega_0-\Delta) = \varepsilon^2\Omega_0^4$$

$$\Rightarrow \boxed{\omega = \Omega_0 + \frac{1}{2}\left[\Delta \pm \sqrt{\Delta^2 - \varepsilon^2\Omega_0^2}\,\right]}$$

- $\varepsilon \to 0$: $\omega = \Omega_0$ and $\omega - \omega_0 = -\Omega_0$
- instability when $\varepsilon^2 > \Delta^2/\Omega_0^2$ ($\in 0$ at $\Delta = 0$) [exact matching]
- unstable mode: $\omega = \omega_r + i\gamma$ (γ: growth rate)

$$\omega_r = \Omega_0 + \Delta/2 = \omega_0/2 \quad \text{[frequency locking]}$$

$$\gamma = \frac{1}{2}\sqrt{\varepsilon^2\Omega_0^2 - \Delta^2} \le \frac{|\varepsilon|\Omega_0}{2} \equiv \gamma_{max}$$

- note $\omega - \omega_0 = -\dfrac{\omega_0}{2} + \gamma$ [frequency locking, same growth rate]

Case II $\omega_0 \doteqdot \Omega_0$

$$\omega \doteqdot 0, \quad X(\omega) \begin{cases} X(\omega-\omega_0) & \omega-\omega_0 \doteqdot -\Omega_0 \\ X(\omega+\omega_0) & \omega+\omega_0 \doteqdot +\Omega_0 \end{cases}$$

[nonresonant] [resonant]

Coupling of two resonant modes via nonresonant mode

$$D(\omega)X(\omega) = -\varepsilon\Omega_0^2\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

$$D(\omega\pm\omega_0)X(\omega\pm\omega_0) = -\varepsilon\Omega_0^2 X(\omega)$$

Dispersion relation

$$\boxed{1 = \frac{\varepsilon^2\Omega_0^4}{D(\omega)}\left[\frac{1}{D(\omega-\omega_0)}+\frac{1}{D(\omega+\omega_0)}\right]}$$

Resonance approx. $D(\omega\pm\omega_0) \doteqdot \pm 2\Omega_0(\omega\pm\delta)$

$\delta \equiv \omega_0 - \Omega_0$ (mismatch)

$D(\omega) \doteqdot D(0) = -\Omega_0^2$

$\Rightarrow \quad \omega^2 = \delta\left[\varepsilon^2\Omega_0 + \delta\right]$

- instability when $0 > \delta > -\varepsilon^2\Omega_0$
- unstable modes : $\omega = i\gamma$ (purely growing mode)

$$\omega\pm\omega_0 = \pm\omega_0 + i\gamma$$

[frequency locking] [growth rates are the same for all 3 modes]

- maximum growth at $\delta = -\varepsilon^2\Omega_0/2$

$$\gamma_{max} = \varepsilon^2\Omega_0/2 \propto \varepsilon^2$$

- amplitude ratio : $\dfrac{X(\omega)}{X(\omega\pm\omega_0)} \sim \varepsilon \ll 1$ [resonant modes dominant]

Effect of damping

$$\frac{d^2}{dt^2}X(t) + 2\Gamma \frac{d}{dt}X(t) + [\Omega_0^2 + \Gamma^2] X(t) = 0$$

$$X(t) \sim e^{-i\omega t}$$

$$\Rightarrow (\omega - \Omega_0 + i\Gamma)(\omega + \Omega_0 + i\Gamma) = 0$$

$$\omega = \pm\Omega_0 - i\Gamma \quad \text{(damped oscillation)}$$

Parameter modulation

$$\Omega_0^2 \rightarrow \Omega^2 = \Omega_0^2 (1 - 2\varepsilon \cos\omega_0 t)$$

Let $\tilde{X}(t) \equiv X(t)\, e^{\Gamma t}$

$$\Rightarrow \frac{d^2}{dt^2}\tilde{X}(t) + \Omega^2 \tilde{X}(t) = 0$$

If $\omega = \omega_r + i\gamma$ for $\tilde{X}(t)$

then $\omega = \omega_r + i(\gamma - \Gamma)$ for $X(t)$

Minimum threshold for instability

$$\gamma_{max} > \Gamma$$

Case I $(\omega_0 \doteq 2\Omega_0)$ $\quad \varepsilon^2 > \dfrac{4\Gamma^2}{\Omega_0^2}$ or $\varepsilon > \dfrac{2\Gamma}{\Omega_0}$

Case II $(\omega_0 \doteq \Omega_0)$ $\quad \varepsilon^2 > \dfrac{2\Gamma}{\Omega_0}$ or $\varepsilon > \sqrt{\dfrac{2\Gamma}{\Omega_0}}$

Note: Γ is the damping rate of the resonant modes.

Coupled mode parametric excitation

Two different modes

X_L and X_H couple via Z

[low freq. mode] [high freq. mode] [large amplitude pump]

Examples in unmagnetized plasma

electromagnetic wave $\omega^2 = \omega_{pe}^2 + c^2k^2$ (photon)

electron plasma wave $\omega^2 = \omega_{pe}^2 + 3k^2V_{Te}^2$ (plasmon)

ion acoustic wave $\omega^2 = k^2C_s^2/[1+k^2\lambda_D^2]$ (phonon)

where $\omega_{pe}^2 = ne^2/\varepsilon_0 m_e$, $V_{Te}^2 = T_e/m_e$, $C_s^2 = T_e/m_i$, $\lambda_D^2 = \varepsilon_0 T_e/ne^2$

- photon → photon + plasmon (stimulated Raman scattering)
- photon → photon + phonon (stimulated Brillouin scattering)
- photon → plasmon + phonon (parametric decay into plasmon and phonon)
- photon → photon + electron or ion (Compton scattering)
- photon → plasmon + electron or ion (stimulated mode conversion)
- plasmon → plasmon + phonon (resonant decay of plasmon)
- plasmon → photon + phonon (parametric back conversion into photon)
- plasmon → photon + electron or ion (stimulated mode back conversion)
- plasmon → plasmon + ion (induced scattering of plasmon on ions)

[
- plasmon → phonon + electron (stimulated mode conversion
- photon → plasmon + plasmon (two plasmon decay)

]

Manley Rowe relation for three coupled modes

Coupling condition: $\omega_1 + \omega_2 + \omega_3 = 0$
$\mathbb{k}_1 + \mathbb{k}_2 + \mathbb{k}_3 = 0$

Wave energy and momentum

$\omega_j N_j$, $\mathbb{k}_j N_j$ (N_j : action, $j = 1, 2, 3$)

Energy and momentum conservation

$$\omega_1 \Delta N_1 + \omega_2 \Delta N_2 + \omega_3 \Delta N_3 = 0$$

$$\mathbb{k}_1 \Delta N_1 + \mathbb{k}_2 \Delta N_2 + \mathbb{k}_3 \Delta N_3 = 0$$

$$\Rightarrow \quad \Delta N_3 = -\frac{1}{\omega_3}\left[\omega_1 \Delta N_1 + \omega_2 \Delta N_2\right]$$

$$(\mathbb{k}_1 \omega_3 - \mathbb{k}_3 \omega_1) \Delta N_1 + (\mathbb{k}_2 \omega_3 - \mathbb{k}_3 \omega_2) \Delta N_2 = 0$$

$$\mathbb{k}_1 \omega_3 - \mathbb{k}_3 \omega_1 = -(\mathbb{k}_2 + \mathbb{k}_3)\omega_3 + \mathbb{k}_3(\omega_2 + \omega_3)$$

$$= -(\mathbb{k}_2 \omega_3 - \mathbb{k}_3 \omega_2)$$

$$\Rightarrow \quad \boxed{\Delta N_1 = \Delta N_2 = \Delta N_3}$$

Suppose $|\omega_3| > |\omega_2| > |\omega_1|$

then $|\omega_3| = |\omega_1| + |\omega_2|$

$$\mathrm{sign}\{\omega_3 \Delta N_3\} \neq \mathrm{sign}\{\omega_1 \Delta N_1\} = \mathrm{sign}\{\omega_2 \Delta N_2\}$$

$\Rightarrow$ Energy flow: $3 \leftrightarrow 1$, $3 \leftrightarrow 2$

$\Rightarrow$ only the highest frequency mode can act as the PUM

Energy partition: $\dfrac{\omega_1 \Delta N_1}{\omega_2 \Delta N_2} = \dfrac{\omega_1}{\omega_2}$

energy predominantly goes to high freq. mode.

Two basic problems :

I. Find general instability characteristics,

such as threshold for instability, maximum growth rate, frequency at threshold or maximum growth etc.

Can use MODEL EQUATIONS.

II. Derive coupled-mode equations or coupling constant

Need case by case analysis.

I. Model analysis

high freq. mode denoted by $X_H(\mathbb{k}_H, \omega)$

low freq. mode denoted by $X_L(\mathbb{k}_L, \omega)$

pump or modulator : $Z(t) = 2Z_0 \cos(\mathbb{k}_0 \cdot \mathbb{r} - \omega_0 t)$.

$(\mathbb{k}_H \pm \mathbb{k}_L = \mathbb{k}_0)$ [Z_0 : constant]

Linear dispersion relation :

$$D_H(\mathbb{k}, \omega) X_H(\mathbb{k}, \omega) = 0$$

$$D_L(\mathbb{k}, \omega) X_L(\mathbb{k}, \omega) = 0$$

where

$$D_H(\mathbb{k}, \omega) = \omega^2 - \omega_H^2(\mathbb{k})$$

$$D_L(\mathbb{k}, \omega) = \omega^2 - \omega_L^2(\mathbb{k})$$

[natural frequencies]

— can include damping by $\omega \to \omega + i\Gamma_{H,L}$

Restrict ourselves to the case

$$\omega_0 \doteqdot \omega_H \gg \omega_L \quad \text{and} \quad Z_0^2 \ll 1$$

[low freq. excitation] (A) [weak pump]

Coupling of $X_H(\mathbb{k}\pm\mathbb{k}_0,\ \omega\pm\omega_0)$ and $X_L(\mathbb{k},\ \omega)$ $\Big\rangle$ via Z

$$D_L(\mathbb{k},\omega)X_L(\mathbb{k},\omega) = Z_0\{\lambda_+ X_H(\mathbb{k}+\mathbb{k}_0,\omega+\omega_0) + \lambda_- X_H(\mathbb{k}-\mathbb{k}_0,\omega-\omega_0)\}$$

$$D_H(\mathbb{k}\pm\mathbb{k}_0,\omega\pm\omega_0)X_H(\mathbb{k}\pm\mathbb{k}_0,\omega\pm\omega_0) = Z_0\mu_\pm X_L(\mathbb{k},\omega)$$

$\Rightarrow$ Dispersion relation

$$\boxed{1 = \frac{Z_0^2}{D_L(\mathbb{k},\omega)}\left\{\frac{\lambda_+\mu_+}{D_H(\mathbb{k}+\mathbb{k}_0,\omega+\omega_0)} + \frac{\lambda_-\mu_-}{D_H(\mathbb{k}-\mathbb{k}_0,\omega-\omega_0)}\right\}}$$

Two cases of instabilities:

i) $D_L \fallingdotseq 0$ and $D_H(\omega+\omega_0)\fallingdotseq 0$ or $D_H(\omega-\omega_0)\fallingdotseq 0$

— resonant-type instability (cf case I)

ii) $D_H(\omega+\omega_0)\fallingdotseq 0$ and $D_H(\omega-\omega_0)\fallingdotseq 0$

— nonresonant-type (cf case II)

Resonance approx. for D_H

$$D_H(\mathbb{k}\pm\mathbb{k}_0,\omega\pm\omega_0) \fallingdotseq \pm 2\omega_0\left[\omega\pm\omega_0\mp\omega_H(\mathbb{k}\pm\mathbb{k}_0)\right]$$

$$= \pm 2\omega_0\left[(\omega-\alpha)\pm\delta\right]$$

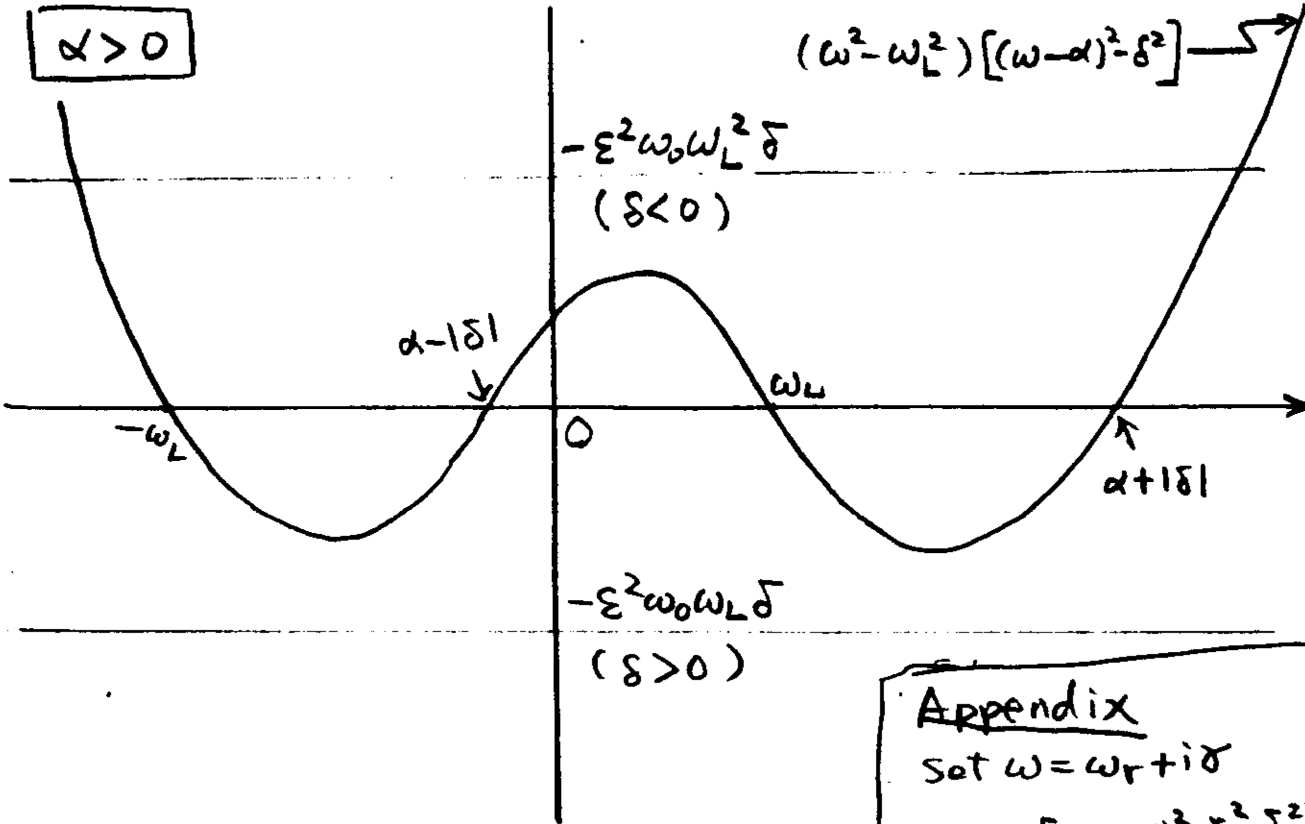
where $\alpha = [\omega_H(\mathbb{k}+\mathbb{k}_0) - \omega_H(\mathbb{k}-\mathbb{k}_0)]/2$

$\delta = \omega_0 - [\omega_H(\mathbb{k}+\mathbb{k}_0) + \omega_H(\mathbb{k}-\mathbb{k}_0)]/2$

Assume $Z_0^2 \lambda_+ \mu_+ = Z_0^2 \lambda_- \mu_- \equiv \varepsilon^2 \omega_0^2 \omega_L^2(\mathbb{k})$

Dispersion relation

$$(\omega^2 - \omega_L^2)\left[(\omega-\alpha)^2 - \delta^2\right] = -\varepsilon^2 \omega_0 \omega_L^2 \delta$$



$\delta > 0$: two growing modes

(see Appendix) ① $\omega_r \geq \alpha > 0$

② $0 \geq \omega_r$

$\delta < 0$: one growing mode

③ $\alpha > \omega_r \geq 0$

Appendix

Set $\omega = \omega_r + i\gamma$

$$2\gamma\omega_r[(\omega_r-\alpha)^2-\gamma^2-\delta^2] + 2\gamma[\omega_r-\alpha][\omega_r^2-\gamma^2-\omega_L^2] = 0$$

$$[\omega_r^2-\gamma^2-\omega_L^2][(\omega_r-\alpha)^2-\gamma^2-\delta^2] - 2\gamma\omega_r 2\gamma[\omega_r-\alpha] = -\varepsilon^2\omega_0\omega_L\delta$$

$$\frac{\omega_r-\alpha}{\omega_r}[\omega_r^2-\gamma^2-\omega_L^2] + 4\gamma^2\omega_r[\omega_r-\alpha] = \varepsilon^2\omega_0\omega_L\delta$$

$$\Rightarrow \omega_r[\omega_r-\alpha]\underbrace{\left\{4\gamma^2 + \frac{(\omega_r^2-\gamma^2-\omega_L^2)^2}{\omega_r^2}\right\}}_{\gamma} = \varepsilon^2\omega_0\omega_L\delta$$

Dipole pump $\mathbf{k}_0 = 0$

$\alpha = 0, \quad \delta = \omega_0 - \omega_H(\mathbf{k})$

i) Resonant-type $\quad \delta \doteq \omega_L > 0$

$\omega = \omega_r + i\gamma, \quad \omega_r \doteq \pm \omega_L \cdots$ modes ①, ②

$\gamma \doteq \sqrt{\gamma_{max}^2 - \Delta^2/4}$ $\quad$ [resonant decay instability]

$\gamma_{max}^2 = \varepsilon^2 \omega_0 \omega_L / 4$ $\quad (\Delta = \omega_0 - \omega_H - \omega_L)$

ii) Nonresonant-type $\quad -\varepsilon^2\omega_0 < \delta < 0$ $\quad$ mode ③

$\omega = i\gamma$ $\quad$ (purely growing mode)

$\gamma_{max} = \varepsilon^2\omega_0/2$ at $\delta = -\varepsilon^2\omega_0/2$

[oscillating two-stream instability OTSI]

Effect of damping

$$D_{H,L}(\mathbf{k},\omega) = [\omega - \omega_{H,L} - i\Gamma_{H,L}][\omega + \omega_{H,L} - i\Gamma_{H,L}]$$

Threshold : resonant-type $\quad \gamma_{max}^2 > \Gamma_H \Gamma_L$

or $\quad \varepsilon^2 > 4\Gamma_H\Gamma_L/\omega_0\omega_L$

nonresonant-type $\quad \gamma_{max} > \Gamma_H$

or $\quad \varepsilon^2 > 2\Gamma_H/\omega_0$

Well above threshold

$\varepsilon^2 \gg \omega_L/\omega_0$ $\quad$ Maximum growth mode :

$$\omega = \begin{cases} \dfrac{1}{2}\left[\dfrac{\varepsilon^2\omega_0\omega_L^2}{4}\right]^{1/3}[\pm\sqrt{5} + \sqrt{3}\,i] & (\delta > 0) \\ i\left[\dfrac{\varepsilon^2\omega_0\omega_L^2}{2}\right]^{1/3} & (\delta < 0) \end{cases}$$

[Quasi-reactive mode instability]

Stimulated scattering

$\omega_0 = \omega_H(k_0)$,

$\Rightarrow \quad \delta = \omega_H(\mathbb{k}_0) - \dfrac{\omega_H(\mathbb{k}_0+\mathbb{k}) + \omega_H(\mathbb{k}_0-\mathbb{k})}{2}$

sign of δ determined by dispersion characteristics

1D propagation: $\quad \delta > 0$ if $\dfrac{d^2\omega_H}{dk^2} < 0$ mode ① and ②

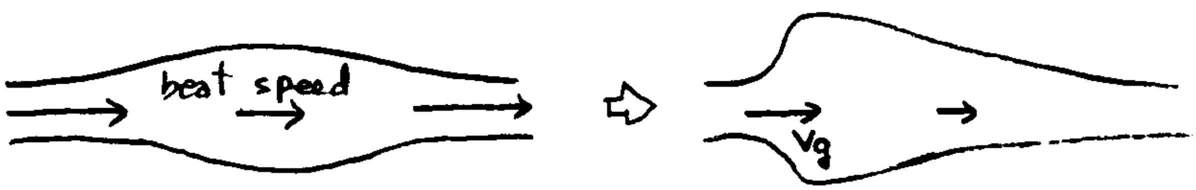
$\delta < 0$ if $\dfrac{d^2\omega_H}{dk^2} > 0$ mode ③

(photon, plasmon)

Physical mechanism (consider mode ③ with $\alpha > \omega_I > 0$)

Note α/k : beat propagation speed in linear approx.

$0 > \dfrac{\omega_r - \alpha}{k}$: nonlinear shift of beat propagation speed



modulation of amplitude $\Rightarrow$ antisteapening

$\Rightarrow$ results in amplification of modulation.

Maximum growth

at $\omega_H(\mathbb{k}_0) = \omega_H(\mathbb{k}_0 - \mathbb{k}) + \omega_L(\mathbb{k})$

(resonance condition) $(\Delta = 0)$

Isotropic case:

γ_{max} at $\mathbb{k} = 2\mathbb{k}_0$ (back scattering)

$(\mathbb{k}_H = -\mathbb{k}_0)$

$\mathbb{k} = \mathbb{k}_0 - \mathbb{k}_H = 2\mathbb{k}_0$

Nonlinear wave-particle interactions

$$\omega_0 = \omega_H + \mathbf{k}\cdot\mathbf{v}$$

[pump] [plasma wave] [particle]

$$\omega_H(\mathbf{k}) = \omega_H(\mathbf{k}') + (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}$$

[nonlinear Landau damping]

Energy flows following Manley-Rowe relation

Absolute vs Convective instabilities

$\omega_0 \to \omega_1 \quad \mathbf{v}_{g1}$

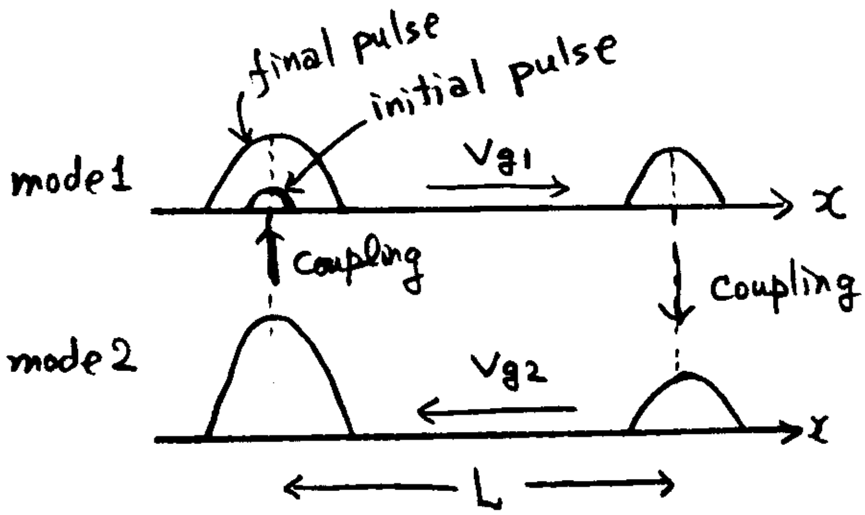
$\omega_0 \to \omega_2 \quad \mathbf{v}_{g2}$

[group vel.]

If $\mathbf{v}_{g1}\cdot\mathbf{v}_{g2} > 0$ always convective

If $\mathbf{v}_{g1}\cdot\mathbf{v}_{g2} < 0$ can have absolute insta. via feedback loop

1D uniform plasma with uniform pump



spatial amplification rate of the pulse

$$g = \frac{\gamma}{\sqrt{|V_{g1}V_{g2}|}}$$

net amplification at original position

$$\exp\left\{2gL - \left[\frac{\Gamma_1}{|V_{g1}|} + \frac{\Gamma_2}{|V_{g2}|}\right]L\right\}$$

Threshold for abs. insta.

$$\gamma_{max} > \sqrt{|V_{g1}V_{g2}|}\left[\frac{\Gamma_1}{|V_{g1}|} + \frac{\Gamma_2}{|V_{g2}|}\right]\frac{1}{2} \geq \sqrt{\Gamma_1\Gamma_2}$$

[convective threshold

Effects of spatial nonuniformity (1D

Local dispersion relations

$k_0(x,\omega_0)$, $k_1(x,\omega_1)$, $k_2(x,\omega_2)$ with $\omega_0 = \omega_1 +$

Wavenumber mismatch

Let $k_0 = k_1 + k_2$ at $x = 0$

then at $x \neq 0$: $\kappa(x) = k_0(x,\omega_0) - k_1(x,\omega_1) - k_2$

Interaction region

$\kappa(x) \Rightarrow$ frequency mismatch $\Delta^2 = \kappa^2 |V_{g1} V$

instability when $\gamma_{max}^2 > \Delta^2/4$

$\Rightarrow$ interaction region $\kappa^2(x) < \dfrac{4\gamma_{max}^2}{|V_{g1}V_{g2}|}$

For $\kappa = \kappa' x \Rightarrow L_{int} = \dfrac{2\gamma_{max}}{|\kappa'|\sqrt{|V_{g1}V_{g2}|}}$

Absolute vs Convective instabilities

initial pulse X_1 at $x=0$ with $k_1 = k_1(x=0)$

propagation of X_1 to $x=L \Rightarrow k_1(x=L)$

driven mode X_2 at $x=L$ has $k_2^d = k_0(x=L) - k_1$

propagation of X_2 to $x=0 \Rightarrow k_2^d + k_2(x=0) - k_2($

driven mode X_1 at $x=0$ has $k_0(x=0) - [k_2^d + k_2(x=0)$

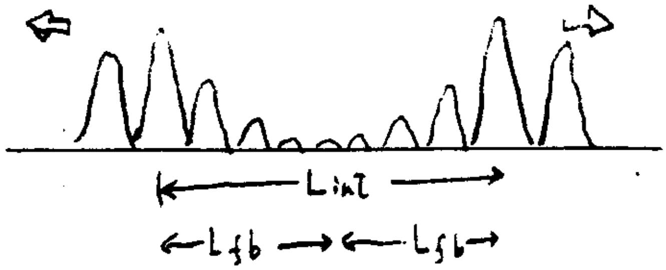
$= k_1(x=0) + \kappa(L)$

Feedback possible only when

$$\kappa^2(L) < \frac{4\gamma_{max}}{|V_{g1}V_{g2}|} \Rightarrow L < L_{fb}$$

For $\kappa(x) = \kappa' x$ $\qquad L_{int}/2 = L_{fb}$

$\Rightarrow$ convection loss dominates



$\rightarrow$ $\boxed{\text{no absolute instability}}$

Maximum amplification factor for convective instability

$\kappa(x) = \kappa' x$

$$\left[\frac{X_1(x = L_{int}/2)}{X_1(x=0)}\right]_{max} = \exp\left\{\int_0^{L_{int}/2} dx\, q(x)\right\}$$

$$= \exp\left\{\int_0^{L_{int}/2} dX \left[\frac{\gamma_{max}^2 - \Delta^2/4}{|V_{g1}V_{g2}|}\right]^{1/2}\right\} = \exp\left\{\int_0^{L_{int}/2} dx \left[\frac{\gamma^2_{max}}{|V_{g1}V_{g2}|} - \frac{\kappa'^2 x^2}{4}\right]^{1/2}\right\}$$

$$= \exp\left\{\frac{\pi \gamma^2_{max}}{|\kappa' V_{g1} V_{g2}|}\right\}$$ (Rosenbluth amplification factor)

Effective threshold

$$\gamma^2_{max} > |\kappa' V_{g1} V_{g2}| = \left|\frac{V_{g1}}{\ell}\right| \cdot \left|\frac{V_{g2}}{\ell}\right| \qquad (\kappa' \equiv \ell^{-2})$$

(effective damping. $\Gamma_{eff} = V_g/\ell$)

For $\kappa(x) = \kappa'' x^2/2$:

feedback possible between two symmetric points

$$\left(\frac{L_{int}}{2}\right)^4 = 16\,\gamma^2_{max} \Big/ \kappa''^2 |V_{g1} V_{g2}|$$

Effective threshold

$$\gamma_{max} > 4^{-1/3} |\kappa''|^{1/3} \sqrt{|V_{g1} V_{g2}|}$$