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SPRING COLLEGE ON PLASMA PHYSICS

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PARAMETRIC PROCESSES IN PLASMA (II)

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Parametric Processes in Plasma

II. Kinetic Theory for Dipole Pump Exciting Electrostatic Waves

Klimontovich eq. for particle distribution:

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{q}{m}\left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]\cdot\frac{\partial}{\partial \mathbf{v}} \right\} f(\mathbf{r},\mathbf{v},t) = 0$$

$$\mathbf{B} = \mathbf{B}_0 = B_0\hat{z} \quad (\text{constant})$$

$$\mathbf{E} = 2\mathbf{E}_0\cos\omega_0 t + \mathbf{E}'(\mathbf{r},t)$$

$$\mathbf{E}_0 = E_{0\perp}\hat{x} + E_{0\parallel}\hat{z} \qquad (\text{pump wave})$$

$$\mathbf{E}' = -\nabla\varphi(\mathbf{r},t) \qquad (\text{excited wave})$$

Oscillating frame:

$$\mathbf{r} - \mathbf{r}_0(t) = \tilde{\mathbf{r}}, \qquad \mathbf{v} - \mathbf{v}_0(t) = \tilde{\mathbf{v}}$$

$$\frac{d\mathbf{r}_0(t)}{dt} = \mathbf{v}_0(t), \qquad \frac{d\mathbf{v}_0}{dt} = \frac{q}{m}\left[2\mathbf{E}_0\cos\omega_0 t + \mathbf{v}_0\times\mathbf{B}_0\right]$$

$$A(\mathbf{r},\mathbf{v},t) = \tilde{A}(\tilde{\mathbf{r}},\tilde{\mathbf{v}},t)$$

$$\left\{ \frac{\partial}{\partial t} + \tilde{\mathbf{v}}\cdot\frac{\partial}{\partial \tilde{\mathbf{r}}} + \frac{q}{m}\left[\tilde{\mathbf{E}}' + \tilde{\mathbf{v}}\times\mathbf{B}_0\right]\cdot\frac{\partial}{\partial \tilde{\mathbf{v}}} \right\} \tilde{f}(\tilde{\mathbf{r}},\tilde{\mathbf{v}},t) = 0$$

$$\mathbf{v}_0(t) = \begin{pmatrix} \frac{2q}{m} E_{0\perp} \frac{\omega_0}{\omega_0^2 - \omega_c^2} \sin\omega_0 t \\ \frac{2q}{m} E_{0\perp} \frac{\omega_c}{\omega_0^2 - \omega_c^2} \cos\omega_0 t \\ \frac{2q}{m\omega_0} E_{0\parallel} \sin\omega_0 t \end{pmatrix}$$

$$\mathbf{r}_0(t) = \begin{pmatrix} -\frac{2q}{m} E_{0\perp} \frac{1}{\omega_0^2 - \omega_c^2} \cos\omega_0 t \\ \frac{2q}{m} E_{0\perp} \frac{\omega_c}{\omega_0} \frac{1}{\omega_0^2 - \omega_c^2} \sin\omega_0 t \\ -\frac{2q}{m\omega_0^2} E_{0\parallel} \cos\omega_0 t \end{pmatrix} \equiv \begin{pmatrix} x_0\cos\omega_0 t \\ y_0 \sin\omega_0 t \\ z_0 \cos\omega_0 t \end{pmatrix}$$

Fourier transform: (D. Arnush et. al)

$$\tilde{A}(\mathbf{k},\omega) = \int dt\, e^{i\omega t} \int d^3\tilde{\mathbf{r}}\, e^{-i\mathbf{k}\cdot\tilde{\mathbf{r}}}\, \tilde{A}(\tilde{\mathbf{r}},t)$$

$$= \int dt\, e^{i\omega t + i\mathbf{k}\cdot\mathbf{r}_0(t)} A(\mathbf{k},t)$$

$$= \sum_{m} e^{i\ell\theta} J_\ell(a)\, A(\mathbf{k}, \omega + \ell\omega_0)$$

$$= \int d\omega' \Delta(\mathbf{k},\omega-\omega')\, A(\mathbf{k},\omega') \equiv \Delta A$$

$$A(\mathbf{k},\omega) = \sum_{m} e^{-i\ell\theta} J_\ell(a)\, \tilde{A}(\mathbf{k}, \omega - \ell\omega_0)$$

$$= \int d\omega' \Delta^{-1}(\mathbf{k},\omega-\omega')\, A(\mathbf{k},\omega') \equiv \Delta^{-1} A$$

where $a^2 = (k_x x_0 + k_z z_0)^2 + (k_y y_0)^2$,

$\theta = \tan^{-1}\left[(k_x x_0 + k_z z_0)/(k_y y_0)\right]$

$$\Delta(\mathbf{k},\omega) = \int dt\, e^{i\omega t} e^{i\mathbf{k}\cdot\mathbf{r}_0(t)} = \sum_{\ell} e^{i\ell\theta} J_\ell(a)\, \delta[\omega + \ell\omega_0$$

$$\ldots \int dt\, e^{i\omega t} e^{-i\mathbf{k}\cdot\mathbf{r}_0(t)} = \ldots e^{-i\ell\theta} \ldots$$

Linear response to $\tilde{\mathbf{E}}'(\mathbf{k},\omega) = -i\mathbf{k}\tilde{\varphi}(\mathbf{k},\omega)$

Let $\tilde{f}(\tilde{\mathbf{r}},\tilde{\mathbf{v}},t) = \tilde{F}(\tilde{\mathbf{v}}) + \tilde{f}'(\tilde{\mathbf{r}},\tilde{\mathbf{v}},t)$

$$\tilde{n}(\tilde{\mathbf{r}},t) = n_0 + \tilde{n}'(\tilde{\mathbf{r}},t)$$

$$\tilde{n}'(\tilde{\mathbf{r}},t) \equiv \int d^3\tilde{v}\, \tilde{f}'(\tilde{\mathbf{r}},\tilde{\mathbf{v}},t)$$

$$\tilde{n}(\mathbf{k},\omega) = \int dt\, e^{i\omega t}\int d^3\tilde{r}\, e^{-i\mathbf{k}\cdot\tilde{\mathbf{r}}}\tilde{n}'(\tilde{\mathbf{r}},t)$$

then

$$q\tilde{n}(\mathbf{k},\omega) = -\varepsilon_0 k^2 \chi(\mathbf{k},\omega)\tilde{\varphi}(\mathbf{k},\omega)$$

$$\chi(\mathbf{k},\omega) = \frac{\omega_p^2}{k^2 n_0}\sum_{\ell=-\infty}^{+\infty}\int d^3\tilde{v}\, \frac{J_m^2(k_\perp\tilde{v}_\perp/\omega_c)}{\omega - k_z\tilde{v}_z - \ell\omega_c}\left\{\frac{\ell\omega_c}{\tilde{v}_\perp}\frac{\partial}{\partial\tilde{v}_\perp} + k_z\frac{\partial}{\partial\tilde{v}_z}\right\}\tilde{F}(\tilde{\mathbf{v}})$$

where $\omega_p^2 = \varepsilon_0 n_0/mq^2$,

If $\tilde{F}(\tilde{\mathbf{v}}) = n_0\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left[-\frac{m|\tilde{\mathbf{v}}|^2}{2T}\right]$

then $\chi(\mathbf{k},\omega) = \frac{k_D^2}{k^2}\left\{1 + \frac{\omega}{\sqrt{2}k_z v_T}\sum_{\ell=-\infty}^{\infty} Z(\zeta_\ell)\Lambda_\ell(k_\perp^2\rho^2)\right\}$

where $k_D^2 = \lambda_D^{-2} = n_0 q^2/\varepsilon_0 T$, $\quad v_T = \sqrt{T/m}$

$$Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx\, e^{-x^2}\Big/(x-\zeta-i\eta) \qquad (\eta>0)$$

$$\zeta_\ell = (\omega+\ell\omega_c)/\sqrt{2}\,k v_T$$

$$\Lambda_\ell(\beta) = e^{-\beta} I_\ell(\beta), \qquad \rho = v_T/\omega_c$$

Linear response of electron-ion system:

electron oscillating frame:

$$-e\tilde{n}_e(\mathbf{k},\omega) = -\chi_e(\mathbf{k},\omega)\varepsilon_0 k^2\tilde{\varphi}(\mathbf{k},\omega)$$

ion oscillating frame:

$$Ze\tilde{n}_i(\mathbf{k},\omega) = -\chi_i(\mathbf{k},\omega)\varepsilon_0 k^2\tilde{\varphi}(\mathbf{k},\omega)$$

Poisson eq.

$$\varepsilon_0 k^2\varphi(\mathbf{k},\omega) = e\{Z n_i(\mathbf{k},\omega) - n_e(\mathbf{k},\omega)\}$$

Suppress $(\mathbf{k},\omega)$, for simplicity:

$$-e\Delta_e n_e = -e\chi_e\{Z\Delta_e n_i - \Delta_e n_e\}$$

$$Ze\Delta_i n_i = -e\chi_i\{Z\Delta_i n_i - \Delta_i n_e\}$$

$$n_e = \Delta_e^{-1}(1+\chi_e)^{-1}\chi_e Z\Delta_e n_i$$

$$= \Delta_i^{-1}\chi_i^{-1}(1+\chi_i)Z\Delta_i n_i$$

Note $(1+\chi_e)^{-1}\chi_e = 1-(1+\chi_e)^{-1}$

$$\chi_i^{-1}(1+\chi_i) = 1+\chi_i^{-1}$$

$$\boxed{\left\{1+\chi_i\Delta_i\Delta_e^{-1}(1+\chi_e)^{-1}\Delta_e\Delta_i^{-1}\right\}\Delta_i n_i = 0}$$

(Coupled equations for $n_i(\omega\pm\ell\omega_0)$ $(\ell=0,\pm1,\cdots)$ correct to all orders in $|\mathbf{E}_0|$.
P.K. Kaw (1976) in Advances in Plasma Physics Vol. 6.)

If ion response to $\mathbf{E}_0$ is ignorable,

$$\boxed{\left\{1+\chi_i\Delta_e^{-1}(1+\chi_e)^{-1}\Delta_e\right\}n_i = 0}$$

Unmagnetized plasma ($\omega_c = 0$)

$Y_0 = 0$, $\theta = \pi/2$, $\mathbb{r}_0(t) = -\frac{2q}{m\omega_0^2}\mathbb{E}_0 \cos\omega_0 t$

$a = -\frac{2q}{m\omega_0^2}\mathbb{k}\cdot\mathbb{E}_0$ = (excursion length)/ k^{-1} along $\mathbb{E}_0$

$a_e \gg a_i \longrightarrow$ ion response neglected

$$\Delta_e(\mathbb{k},\omega) = \sum_{\ell=-\infty}^{\infty}(-1)^\ell J_\ell(a_e)\,\delta[\omega+\ell\omega_0]$$

$$0 = \{1 + \chi_i\Delta_e^{-1}(1+\chi_e)^{-1}\Delta_e\}\,n_i$$

$$= n_i(\omega) + \chi_i(\omega)\int d\omega'\int d\omega''\Delta_e^{-1}(\omega')[1+\chi_e]^{-1}(\omega-\omega')\Delta_e(\omega'')n_i(\omega-$$

$$= n_i(\omega)\left\{1 + \chi_i(\omega)\frac{\varepsilon_0}{\varepsilon_e(\omega)}J_0^2(a_e)\right.$$

$$+ \chi_i(\omega)\sum_{\ell=1}^{\infty}\varepsilon_0 J_\ell^2(a_e)\left[\frac{1}{\varepsilon_e(\omega+\ell\omega_0)} + \frac{1}{\varepsilon_e(\omega-\ell}\right.$$

where $\varepsilon_e(\omega) = \varepsilon_0[1+\chi_e(\omega)]$ (electron dielectric fun

<u>Weak pump case $a_e^2 \ll 1$:</u>

$J_0(a_e) \doteqdot 1$, $J_1(a_e) \doteqdot a_e/2$, $J_\ell(a_e) \doteqdot 0$ ($\ell > 1$)

$$1 + \frac{\chi_i(\omega)\varepsilon_0}{\varepsilon_e(\omega)} = -\varepsilon_0\chi_i(\omega)\frac{a_e^2}{4}\left[\frac{1}{\varepsilon_e(\omega+\omega_0)} + \frac{1}{\varepsilon_e(\omega-\omega_0)}\right]$$

$$1 + \frac{\varepsilon_0\chi_i(\omega)}{\varepsilon_e(\omega)} = \frac{\varepsilon(\omega)}{\varepsilon_e(\omega)}$$ where $\varepsilon(\omega) = \varepsilon_0[1+\chi_e(\omega)+\chi_i$

(dielectric function)

$$\boxed{1 = -\frac{\varepsilon_0\chi_i(\mathbb{k},\omega)\varepsilon_e(\mathbb{k},\omega)}{\varepsilon(\mathbb{k},\omega)}\frac{a_e^2}{4}\left[\frac{1}{\varepsilon_e(\mathbb{k},\omega+\omega_0)} + \frac{1}{\varepsilon_e(\mathbb{k},\omega-\omega_0}\right.}$$

Note: ion response ($\chi_i \neq 0$) is necessary for the dipole pu

Example 1. Parametric decay into electron plasma and ion waves: photon $\longrightarrow$ plasmon + phonon

$\omega_0 \doteqdot \omega_{pe}$ $\quad k\lambda_D \ll 1$ $\quad T_e \gg T_i$

$|\omega| < \omega_{pi}$

$$\varepsilon_e(\mathbb{k},\omega\pm\omega_0) \doteqdot \varepsilon_0/\omega_0^2\,[(\omega\pm\omega_0)^2 - \omega_k^2]$$

$$\omega_k^2 = \omega_{pe}^2[1+3k^2\lambda_D^2]\ (\omega_k\text{: Bohm-Gross freq.})$$

$$\varepsilon_e(\mathbb{k},\omega) \doteqdot \varepsilon_e(\mathbb{k},0) \doteqdot \varepsilon_0\chi_e(\mathbb{k},0) = \varepsilon_0/k^2\lambda_D^2$$

$$\chi_i(\mathbb{k},\omega) \doteqdot -\omega_{pi}^2/\omega^2$$

$$\varepsilon(\mathbb{k},\omega) \doteqdot \varepsilon_e(\mathbb{k},0) + \chi_i(\mathbb{k},\omega) = \frac{\varepsilon_0}{k^2\lambda_D^2}[1-\Omega_k^2/\omega^2]$$

$$\Omega_k^2 = k^2C_s^2/(1+k^2\lambda_D^2)\ (\Omega_k\text{: ion acoustic freq.})$$

$$C_s^2 = ZT_e/m_i\ (C_s\text{: ion acoustic velocity})$$

Dispersion relation

$$1 = \frac{a_e^2}{4}\frac{\omega_0^2\omega_{pi}^2}{\omega^2-\Omega_k^2}\left\{\frac{1}{[(\omega+\omega_0)^2-\omega_k^2]} + \frac{1}{[(\omega-\omega_0)^2-\omega_k^2]}\right\}$$

$$\lambda\mu Z_0^2 = \frac{a_e^2}{4}\omega_0^2\omega_{pi}^2$$

$$\varepsilon^2 = \frac{a_e^2}{4}\frac{\omega_{pi}^2}{\Omega_k^2} \doteqdot \frac{a_e^2}{4k^2\lambda_D^2} = \left(\frac{V_0}{V_{Te}}\right)^2\cos^2\theta\ \ (k^2\lambda_D^2 \ll$$

where $V_0 = \frac{eE_0}{m_e\omega_0}$, (electron quivering velocity)

$$\cos\theta = \mathbb{k}\cdot\mathbb{E}_0/kE_0$$

$$\gamma_{max}^2 = \frac{\varepsilon^2\omega_0\omega_L}{4} = \frac{\omega_0\Omega_k}{4}\left(\frac{V_0}{V_{Te}}\right)^2 \text{ at } \theta = 0 \text{ or } \pi$$

MEASUREMENTS OF ION ACOUSTIC DECAY INSTABILITIES (IADI) IN LASER PELLET INTERACTIONS

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MAIN SUBJECTS

(1) IADI threshold is quite low.

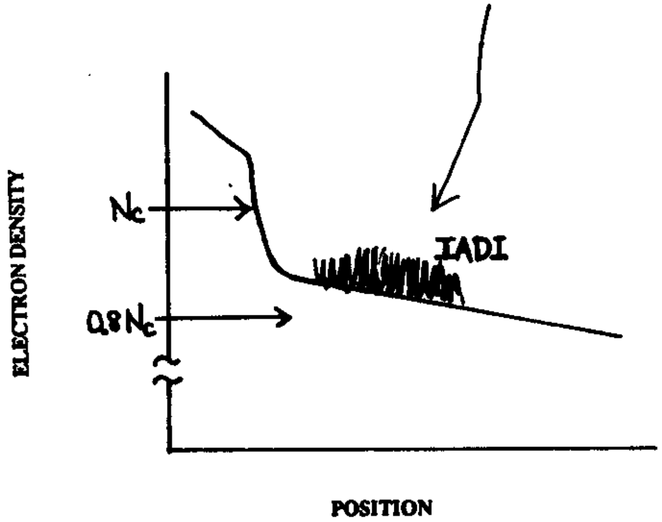
5×10^{12} W/cm^2 for IR laser.
3×10^{13} W/cm^2 for green laser.

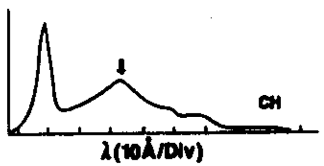
(2) IADI results scale very well with λ_L (IR to green lasers).

(3) IADI can be a useful tool of plasma diagnostics at critical density.

Ionic charge state Z can be measured fairly accurately.

IADI is excited on the shallow underdense shelf



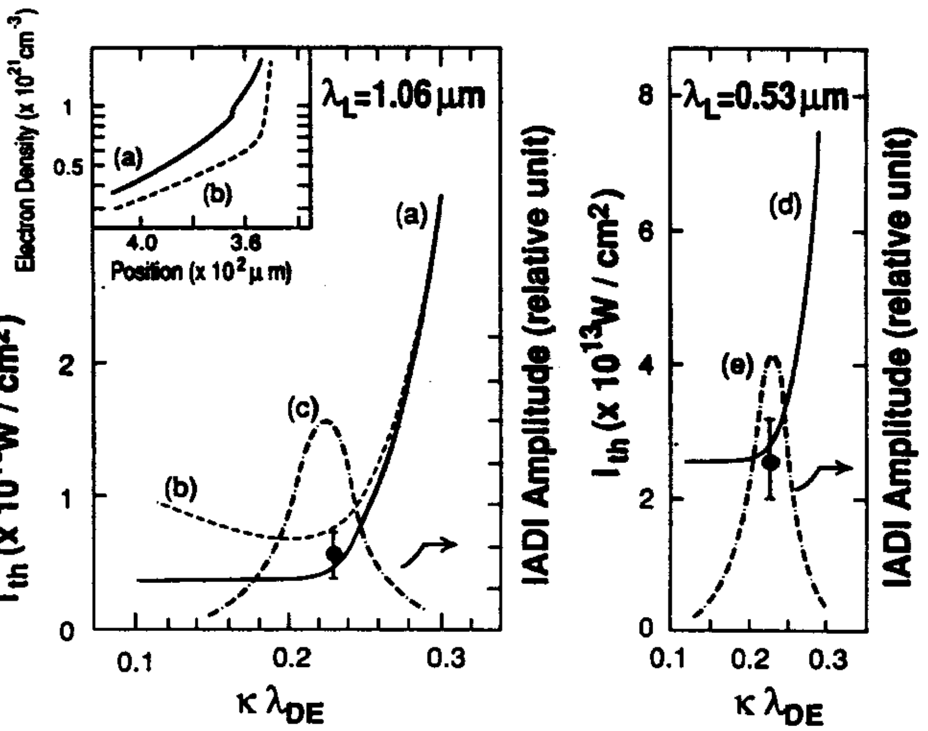


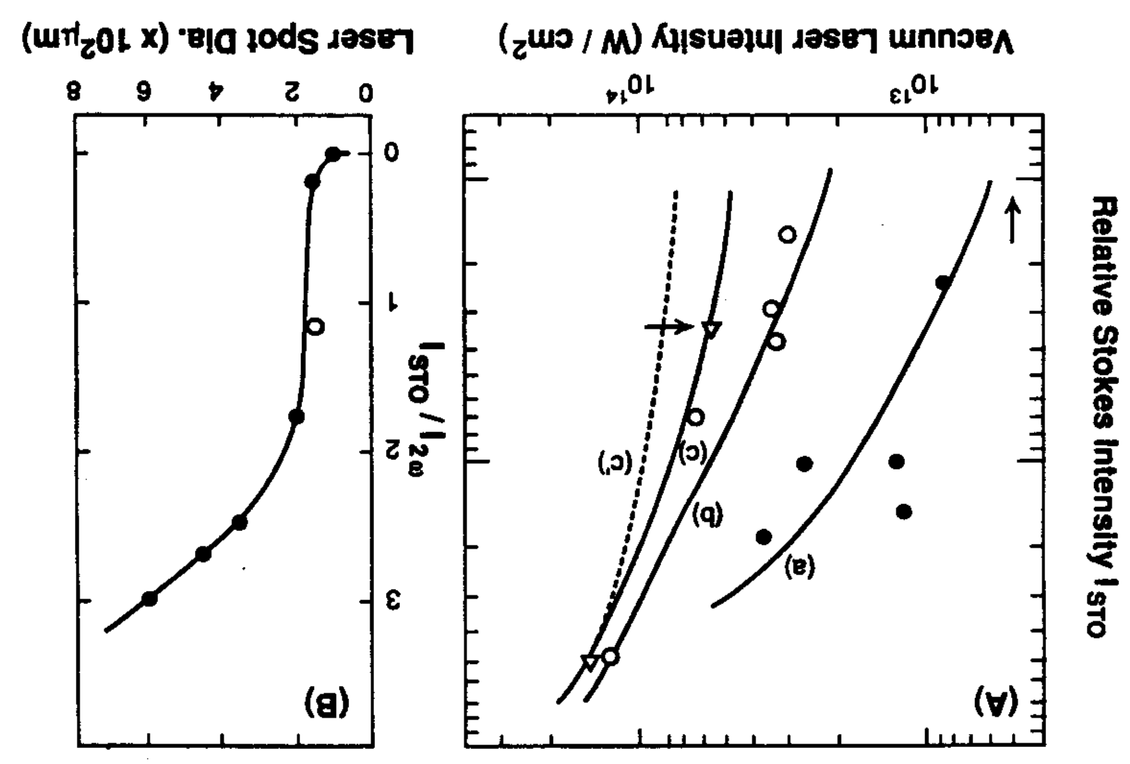
$$\Delta\lambda = \begin{cases} 23\,\text{Å (IR laser)} \\ 12\,\text{Å (Green)} \end{cases} \Longrightarrow \frac{N}{N_c} \approx 0.84$$

Bohm-Gross dispersion

vs. $k\lambda_{DE}$

Experimental Results Agree with the Collisional Theory.





STOKES INTENSITY vs. LASER INTENSITY AS A PARAMETER OF LASER SPOT SIZE

Example 2. Stimulated conversion of photon to plasmon or Compton scattering on ions.

$$\begin{cases} \text{plasmon: weakly damped } (k\lambda_D \ll 1) \\ \text{phonon: strongly damped } (T_e \sim T_i) \end{cases}$$

Retain only one of $\varepsilon_e(\mathbf{k}, \omega \pm \omega_0)$

$$\varepsilon_e(\mathbf{k}, \omega \pm \omega_0) = -\frac{a_e^2}{4} \frac{\varepsilon_e(\mathbf{k},\omega)\,\varepsilon_0 \chi_i(\mathbf{k},\omega)}{\varepsilon(\mathbf{k},\omega)}$$

$$= -\frac{a_e^2}{4} \frac{\varepsilon_e(\mathbf{k},\omega)\,\varepsilon_0 \chi_i(\mathbf{k},\omega)\left[\varepsilon_e^*(\mathbf{k},\omega) + \varepsilon_0 \chi_i^*(\mathbf{k},\omega)\right]}{|\varepsilon(\mathbf{k},\omega)|^2}$$

$$k\lambda_D \ll 1 \;\to\; \varepsilon_e(\mathbf{k},\omega) \doteqdot \varepsilon_e(\mathbf{k},0) \doteqdot \varepsilon_0/k^2\lambda_D^2$$

$$\mathrm{Im}\,\varepsilon_e(\mathbf{k},\omega \pm \omega_0) = -\frac{a_e^2}{4}\left|\frac{\varepsilon_e(\mathbf{k},0)}{\varepsilon(\mathbf{k},\omega)}\right|^2 \varepsilon_0\, \mathrm{Im}\,\chi_i(\mathbf{k},\omega)$$

Let $\omega = \omega_r + i\gamma$ (γ: growth rate)

$$\mathrm{Im}\,\varepsilon_e(\mathbf{k},\omega \pm \omega_0) \doteqdot \frac{\partial}{\partial \omega_H} \mathrm{Re}\,\varepsilon_e(\mathbf{k},\omega_H)\,[\gamma + \Gamma_{\mathbf{k}}]$$

$$\doteqdot \pm \frac{2\varepsilon_0}{\omega_0}(\gamma + \Gamma_{\mathbf{k}})$$

where $\Gamma_{\mathbf{k}}$: electron Landau and collisional damping rate.

$$\frac{\gamma + \Gamma_{\mathbf{k}}}{\omega_0} \doteqdot \mp \frac{a_e^2}{8}\left|\frac{\varepsilon_e(\mathbf{k},0)}{\varepsilon(\mathbf{k},\omega)}\right|^2 \mathrm{Im}\,\chi_i(\mathbf{k},\omega_r)$$

For Maxwellian distribution of ions:

$$\omega_r\, \mathrm{Im}\,\chi_i(\mathbf{k},\omega_r) > 0$$

Growth ($\gamma > 0$) is possible when $\omega_0 - \omega_{\mathbf{k}} = \omega_r \; (>0)$

(consistent with Manley-Rowe rel.)

—— typically $\omega_r \sim k v$ ... ——

Example 3. Silin's kinetic instability or Compton scattering on electrons.

phonons: weakly damped ($T_e \gg T_i$)

plasmons: strongly damped ($k\lambda_{De} \gtrsim 1 \gg k\lambda_{Di}$

$$\varepsilon(\mathbf{k},\omega) = -\frac{a_e^2}{4}\varepsilon_e(\mathbf{k},\omega)\,\varepsilon_0 \chi_i(\mathbf{k},\omega)\left[\frac{1}{\varepsilon_e(\mathbf{k},\omega+\omega_0)} + \frac{1}{\varepsilon_e(\mathbf{k},\omega-\omega_0)}\right.$$

$$\varepsilon_e(\mathbf{k},\omega) \doteqdot \varepsilon_e(\mathbf{k},0)$$

$$\chi_i(\mathbf{k},\omega) \doteqdot -\omega_{pi}^2/\omega^2 + i\,\mathrm{Im}\,\chi_i(\mathbf{k},\omega_r) \doteqdot -\omega_{pi}^2/\Omega$$

$$\omega \to \omega_r + i\gamma, \quad \mathrm{Im}\,\chi_i(\mathbf{k},\omega_r) = 2\Gamma_{ik}/\Omega_k$$

Γ_{ik}: ion Landau and collisional dampin

$$\varepsilon(\mathbf{k},\omega) \doteqdot \varepsilon_e(\mathbf{k},0)\left[1 - \frac{\Omega_k^2}{(\omega_r + i\gamma)^2} + \frac{2i\Gamma_{ik}}{\Omega_k}\right]$$

$$\mathrm{Im}\,\varepsilon(\mathbf{k},\omega) = 2\varepsilon_e(\mathbf{k},0)(\gamma + \Gamma_{ik})/\Omega_k$$

$$\frac{\gamma + \Gamma_{ik}}{\Omega_k} = \frac{a_e^2}{8}\frac{\omega_{pi}^2}{\Omega_k^2}\varepsilon_0\, \mathrm{Im}\left\{\frac{1}{\varepsilon_e(\mathbf{k},\omega_r+\omega_0)} + \frac{1}{\varepsilon_e(\mathbf{k},\omega_r-\ldots}\right.$$

$$k\lambda_{De} \gtrsim 1 \;\to\; \Omega_k \sim \omega_{pi}$$

For Maxwellian distribution of electrons:

$$\mathrm{Im}\,\frac{1}{\varepsilon_e(\mathbf{k},\omega_r-\omega_0)} > 0, \quad \mathrm{Im}\,\frac{1}{\varepsilon_e(\mathbf{k},\omega_r+\omega_0)} <$$

(consistent with Manley Rowe rel.)

$$\mathrm{Im}\,\frac{1}{\varepsilon_e(\mathbf{k},\omega_r-\omega_0)} > -\mathrm{Im}\,\frac{1}{\varepsilon_e(\mathbf{k},\omega_r+\omega_0)}$$

Magnetized Plasma

If $\omega_0 \gg \omega_{pi}, \omega_{ci}$ no ion response

$$\frac{1}{\chi_i(k,\omega)} + \sum_{\ell=-\infty}^{\infty} \frac{\varepsilon_0 J_\ell^2(a_e)}{\varepsilon_e(k,\omega+\ell\omega_0)} = 0$$

(the same as unmagnetized plasma)

$$a_e^2 = \left(\frac{2e}{m_e}\right)^2 \left\{ \Big[\underbrace{\frac{k_\perp \cdot E_{0\perp}}{\omega_0^2 - \omega_{ce}^2} + \frac{k_\parallel E_{0\parallel}}{\omega_0^2}}_{\text{(polarization drift)}} \Big]^2 + \Big| \underbrace{\frac{k_\perp \times E_{0\perp}}{\omega_0^2 - \omega_{ce}^2} \frac{\omega_{ce}}{\omega_0}}_{\text{(ExB drift)}} \Big|^2 \right\}$$

If $\omega_0 \sim \omega_{pi}$ or ω_{ci} ion response to be included

$$\{ 1 + \chi_i \Delta_i \Delta_e^{-1} (1+\chi_e)^{-1} \Delta_e \Delta_i^{-1} \} \underbrace{\Delta_i n_i}_{\tilde{n}_i} = 0$$

$$\Delta_i \Delta_e^{-1} \varepsilon_e^{-1} \Delta_e \Delta_i^{-1} \tilde{n}_i$$
$$= \sum_{\ell_i} \sum_{\ell_e} e^{i\ell_i\theta_i - i\ell_e\theta_e} J_{\ell_i}(a_i) J_{\ell_e}(a_e) \frac{1}{\varepsilon_e(\omega + [\ell_i - \ell_e]\omega_0)}$$
$$\times \sum_{\ell_i'} \sum_{\ell_e'} e^{-i\ell_i'\theta_i + i\ell_e'\theta_e} J_{\ell_i'}(a_i) J_{\ell_e'}(a_e) \tilde{n}_i(\omega + [\ell_i - \ell_e - \ell_i' + \ell_e']\omega_0)$$

Use $\displaystyle\sum_{\ell=-\infty}^{\infty} e^{i\ell\theta} J_{\ell+p}(a) J_\ell(b) = e^{ip\psi} J_p(\lambda)$

where $\lambda = [a^2 + b^2 - 2ab\cos\theta]^{1/2}$, $\psi = \tan^{-1}\left[\frac{b\sin\theta}{a - b\cos\theta}\right]$

Set $\ell_i - \ell_e = p$, $\ell_i - \ell_i' - \ell_e + \ell_e' = q$

$$\Delta_i \Delta_e^{-1} \varepsilon_e^{-1} \Delta_e \Delta_i^{-1} \tilde{n}_i$$
$$= \sum_p \sum_q e^{iq(\pi - \phi + \theta_e)} J_p(\mu) J_{p-q}(\mu) \varepsilon_e^{-1}(\omega + p\omega_0) \tilde{n}_i(\omega + q\omega_0)$$

where $\mu = [a_e^2 + a_i^2 - 2a_e a_i \cos(\theta_i - \theta_e)]^{1/2}$

$$\phi = \tan^{-1}\left[\frac{a_i \sin(\theta_i - \theta_e)}{a_e - a_i\cos(\theta_i - \theta_e)}\right]$$

Weak coupling case: $\mu^2 \ll 1$

$J_0(\mu) \doteqdot 1$, $J_{\pm 1}(\mu) \doteqdot \pm \mu/2$, $J_{\pm\ell}(\mu) \doteqdot 0$ $(|\ell| > 1)$

where

$$\mu^2 = 4e^2 \left\{ \Big[\underbrace{\frac{1}{m_e}\Big(\frac{k_\perp \cdot E_{0\perp}}{\omega_0^2 - \omega_{ce}^2} + \frac{k_\parallel E_{0\parallel}}{\omega_0^2}\Big) + \frac{Z}{m_i}\Big(\frac{k_\perp \cdot E_{0\perp}}{\omega_0^2 - \omega_{ci}^2} + \frac{k_\parallel E_{0\parallel}}{\omega_0^2}\Big)}_{\text{[polarization drift]}} \Big]^2 \right.$$
$$\left. + \Big| \underbrace{\frac{1}{m_e} \frac{\omega_{ce}}{\omega_0} \frac{k_0 \times E_{0\perp}}{\omega_0^2 - \omega_{ce}^2} - \frac{Z}{m_i} \frac{\omega_{ci}}{\omega_0} \frac{k_\perp \times E_{0\perp}}{\omega_0^2 - \omega_{ci}^2}}_{\text{[E × B drift]}} \Big|^2 \right\}$$

⇒ coupling of $\tilde{n}_i(\omega)$ and $\tilde{n}_i(\omega \pm \omega_0)$

$$\left\{ \frac{\varepsilon(k,\omega)}{\varepsilon_e(k,\omega)} + \frac{\mu^2}{4} \varepsilon_0 \chi_i(k,\omega) \Big[\frac{1}{\varepsilon_e(k,\omega+\omega_0)} + \frac{1}{\varepsilon_e(k,\omega-\omega_0)} \Big] \right\} \tilde{n}$$
$$= \frac{\mu}{2} e^{i(\theta_e - \phi)} \varepsilon_0 \chi_i(k,\omega) \Big[\frac{1}{\varepsilon_e(k,\omega+\omega_0)} - \frac{1}{\varepsilon_e(k,\omega)} \Big] \Big\} \tilde{n}_i($$
$$- \frac{\mu}{2} e^{-i(\theta_e - \phi)} \varepsilon_0 \chi_i(k,\omega) \Big[\frac{1}{\varepsilon_e(k,\omega-\omega_0)} - \frac{1}{\varepsilon_e(k,\omega)} \Big] \Big\} \tilde{n}_i(k,$$

$$\frac{\varepsilon(k,\omega\pm\omega_0)}{\varepsilon_e(k,\omega\pm\omega_0)} \tilde{n}_i(k,\omega\pm\omega_0)$$
$$= \pm \frac{\mu}{2} e^{\mp i(\theta_e - \phi)} \varepsilon_0 \chi_i(k,\omega\pm\omega_0) \Big[\frac{1}{\varepsilon_e(k,\omega\pm\omega_0)} - \frac{1}{\varepsilon_e(k,\omega)} \Big]$$

For $|\omega| \ll \omega_{pi}, \omega_{ci}$ and $k^2\lambda_D^2 \gg 1 \Rightarrow \varepsilon_e(k,\omega) \doteqdot 1/k^2\lambda_D^2 \gg 1$

then we have

$$\boxed{1 = \frac{\mu^2}{4k^2\lambda_D^2} \frac{\varepsilon_0 \chi_i(k,\omega)}{\varepsilon(k,\omega)} \left\{ \frac{1}{\varepsilon_e(k,\omega+\omega_0)} \Big[\frac{\varepsilon_0 \chi_i(k,\omega+\omega_0)}{\varepsilon(k,\omega+\omega_0)} - 1 \Big] + \frac{1}{\varepsilon_e(k,\omega-\omega_0)} \Big[\frac{\varepsilon_0 \chi_i(k,\omega-\omega_0)}{\varepsilon(k,\omega-\omega_0)} - 1 \Big] \right\}}$$

[dispersion relation]