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Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents

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0. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with $n \geqq 3$. We are concerned with the problem of existence of a function u satisfying the nonlinear elliptic equation

$$(0.1)\qquad \begin{aligned} -\Delta u &= u^p + f(x,u) && \text{on } \Omega,\\ u &> 0 && \text{on } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $p=(n+2)/(n-2)$, $f(x,0)=0$ and $f(x,u)$ is a lower-order perturbation of u^p in the sense that $\lim_{u\to+\infty} f(x,u)/u^p = 0$. A typical example is $f(x,u)=\lambda u$, where λ is a real constant. The exponent $p=(n+2)/(n-2)$ is critical from the viewpoint of Sobolev embedding. Indeed solutions of (0.1) correspond to critical points of the functional

$$\Phi(u) = \tfrac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1} - \int F(x,u),$$

where $F(x,u)=\int_0^u f(x,t)\,dt$. Note that $p+1=2n/(n-2)$ is the limiting Sobolev exponent for the embedding $H_0^1(\Omega)\subset L^{p+1}(\Omega)$. Since this embedding is *not compact*, the functional Φ *does not satisfy the* (PS) *condition.* Hence there are serious difficulties when trying to find critical points by standard variational methods. In fact, there is a sharp contrast between the case $p<(n+2)/(n-2)$ for which the Sobolev embedding is compact, and the case $p=(n+2)/(n-2)$. Many *existence* results for problem (0.1) are known when $p<(n+2)/(n-2)$ (see the review article by P. L. Lions [20] and the abundant list of references in [20]). On the other hand, a well-known *nonexistence* result of Pohozaev [24] asserts that if Ω is starshaped there is *no* solution of the problem

$$\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} && \text{on } \Omega,\\ u &> 0 && \text{on } \Omega,\\ u &= 0 && \text{on } \partial\Omega; \end{aligned}$$

see (1.4). But, as we shall see, *lower-order terms can reverse this situation.*

Our motivation for investigating (0.1) comes from the fact that it resembles some variational problems in geometry and physics where lack of compactness also occurs. The most notorious example is *Yamabe's problem*: find a function u satisfying

$$\begin{aligned} -4\frac{(n-1)}{(n-2)}\Delta u &= R' u^{(n+2)/(n-2)} - R(x)u && \text{on } M,\\ u &> 0 && \text{on } M, \end{aligned}$$

for some constant R'. Here M is an n-dimensional Riemannian manifold, Δ its Laplacian, and $R(x)$ is the scalar curvature.

But there are many other examples:

(a) Existence of extremal functions for isoperimetric inequalities, Hardy–Littlewood–Sobolev inequalities, trace inequalities, etc.; see Jacobs [14][1], Lieb [19], P. L. Lions [21].

(b) Existence of non-minimal solutions for Yang–Mills functionals; see C. Taubes [29].[2]

(c) Existence of non-minimal solutions for H-systems[3] (Rellich's conjecture concerning the existence of "large" surfaces of constant prescribed mean curvature spanned by a given curve in $\mathbf{R}^3$); see [5].

(d) See K. K. Uhlenbeck [31] for still more.

Our paper is organized as follows. In Section 1, we investigate the model problem

$$(0.2)\qquad \begin{aligned} -\Delta u &= u^p + \lambda u && \text{on } \Omega,\\ u &> 0 && \text{on } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $p=(n+2)/(n-2)$ and λ is a real constant. Surprisingly, the cases where $n=3$ and $n\geqq 4$ turn out to be quite different:

(a) *when* $n \geqq 4$, problem (0.2) has a solution for every $\lambda\in(0,\lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$; moreover it has no solution if $\lambda \notin (0,\lambda_1)$ and Ω is starshaped (see Theorem 1.1),

[1] This reference was brought to our attention by L. Carleson.
[2] This reference was brought to our attention by M. Atiyah.
[3] This problem was mentioned to us by S. Hildebrandt.

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(b) *when* $n=3$, problem (0.2) is much more *delicate* and we have a complete solution only when Ω is a *ball*. In that case, problem (0.2) has a solution *if and only if* $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$ (see Theorem 1.2).

This unexpected phenomenon can perhaps shed some light on Yamabe's problem which was solved by Th. Aubin [3] in high dimensions, namely $n \geqq 6$, in case the Weyl curvature tensor of the Riemannian metric is not identically zero. (In case it is identically zero, and the manifold has finite Poincaré group, the problem is also solved in [3].)

Our approach for proving the above results is the following. The solutions of (0.2) correspond to nontrivial critical points of the functional

$$\Phi(u)=\frac{1}{2}\int|\nabla u|^2-\frac{1}{p+1}\int|u|^{p+1}-\frac{1}{2}\lambda\int u^2.$$

Another viewpoint—which we shall use—is to look for critical points of the functional $\int|\nabla u|^2-\lambda\int u^2$ on the sphere $\|u\|_{p+1}=1$. Such a critical point u satisfies the equation

$$-\Delta u-\lambda u=\mu u^p,$$

where μ is a Lagrange multiplier. After "stretching" the Lagrange multiplier we obtain a solution of (0.2). We prove indeed that for suitable λ's we have:

$$\inf_{\substack{u\in H_0^1\\ \|u\|_{p+1}=1}}\left\{\int|\nabla u|^2-\lambda\int u^2\right\} \text{ is achieved.} \tag{0.3}$$

The major difficulty in proving (0.3) stems from the fact that the function $u\mapsto\|u\|_{p+1}$ is *not* continuous under weak convergence in $H_0^1(\Omega)$. The decisive device in order to overcome this lack of compactness is to establish that for suitable λ's we have

$$\inf_{\substack{u\in H_0^1\\ \|u\|_{p+1}=1}}\left\{\int|\nabla u|^2-\lambda\int u^2\right\}<\inf_{\substack{u\in H_0^1\\ \|u\|_{p+1}=1}}\int|\nabla u|^2\equiv S, \tag{0.4}$$

where S corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega)\subset L^{p+1}(\Omega)$.

Our arguments are inspired by the work [3] of Aubin. The main point of the proof consists in evaluating the ratio

$$Q_\lambda(u)=\frac{\|\nabla u\|_2^2-\lambda\|u\|_2^2}{\|u\|_{p+1}^2}$$

for

$$u(x)=\frac{\varphi(x)}{(\varepsilon+|x|^2)^{(n-2)/2}},\qquad \varepsilon>0, \tag{0.5}$$

where φ is a cut-off function. The functions $(\varepsilon+|x|^2)^{-(n-2)/2}$ play a natural role because they are extremal functions for the Sobolev inequality in $\mathbf{R}^n$. This approach has served as a source of inspiration in [5] where a similar method is used; in [5] it is not the Sobolev inequality but a certain isoperimetric inequality that plays the key role.

Finally, for the nonexistence part of Theorem 1.2 (i.e., $\lambda\leqq\frac{1}{4}\lambda_1$) we use an argument "à la Pohozaev" with more complicated multipliers.

In Section 2, we turn to the general form of problem (0.1). Once more there is a difference between the cases $n=3$ and $n\geqq4$. We summarize our result on the following simple example:

$$\begin{aligned} -\Delta u &= u^p+\mu u^q &&\text{on } \Omega,\\ u&>0 &&\text{on } \Omega,\\ u&=0 &&\text{on } \partial\Omega,\end{aligned} \tag{0.6}$$

where $p=(n+2)/(n-2)$, $1<q<p$, and $\mu>0$ is a constant. When $n\geqq4$, problem (0.6) has a solution for every $\mu>0$. When $n=3$ ($p=5$), problem (0.6) is again much more delicate:

(a) if $3<q<5$, problem (0.6) has a solution for *every* $\mu>0$;

(b) if $1<q\leqq3$, it is only for *large values* of μ that (0.6) possesses a solution.

The proofs involve a combination of various ingredients. We start with a geometrical result which is an expression of the Ambrosetti-Rabinowitz [1] mountain pass theorem without the (PS) condition:

THEOREM 2.2. *Let Φ be a C^1 function on a Banach space E. Suppose*

(0.7) *there exists a neighborhood U of 0 in E and a constant ρ such that $\Phi(u)\geqq\rho$ for every u in the boundary of U,*

(0.8) $\Phi(0)<\rho$ *and* $\Phi(v)<\rho$ *for some* $v\notin U$.

Set

$$c=\inf_{P\in\mathcal{P}}\max_{w\in P}\Phi(w)\geqq\rho,$$

where $\mathcal{P}$ denotes the class of paths joining 0 to v.

Conclusion:

there is a sequence (u_j) in E such that $\Phi(u_j)\to c$ and $\Phi'(u_j)\to0$ in E^*.

When applying Theorem 2.2 to (0.6) we choose $E=H_0^1(\Omega)$ and

$$\Phi(u)=\frac{1}{2}\int|\nabla u|^2-\frac{1}{p+1}\int|u|^{p+1}-\frac{\mu}{q+1}\int|u|^{q+1}.$$

Condition (0.7) is clearly satisfied (U being a small ball). The major difficulty lies in using the conclusion of Theorem 2.2. For this purpose we prove (see Theorem 2.1) that if

$$c<\frac{1}{n}S^{n/2}, \tag{0.9}$$

then one can pass to the limit in the sequence (u_j) and obtain a nontrivial critical point of Φ. Thus we are left with the question: can one find a v such that the corresponding c satisfies (0.9)?[4] This last step is rather technical; it is achieved by choosing some special v's, for example of the form (0.5). We believe that this method can be useful in solving other problems where one is in a borderline situation for the (PS) condition—so that the standard approach fails.

Our thanks to E. Lieb for his kind help (see Lemma 1.2), to F. Browder and P. Rabinowitz for stimulating discussions, and to O. Bristeau (at INRIA) for suggestive numerical computations at stages where we could not guess the answer.

1. Existence of Positive Solutions for $-\Delta u=u^p+\lambda u$ on Ω, $u=0$ on $\partial\Omega$ with $p=(n+2)/(n-2)$

Let $\Omega\subset\mathbb{R}^n$, $n\geqq 3$, be a bounded domain. We are concerned with the problem of existence of a function u satisfying:

$$\begin{aligned} -\Delta u &= u^p+\lambda u && \text{on } \Omega,\\ u&>0 && \text{on } \Omega,\\ u&=0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $p=(n+2)/(n-2)$ and λ is a real constant. As we have indicated, the cases $n=3$ and $n\geqq 4$ are different and will be treated separately.

In subsections 1.1 and 1.2 we consider the cases $n\geqq 4$ and $n=3$, respectively.

In subsection 1.3 we have collected a number of additional properties and open problems. We denote by λ_1 the first eigenvalue of $-\Delta$ with zero Dirichlet condition on Ω.

1.1. The case $n\geqq 4$.

Our main result is the following:

THEOREM 1.1. *Assume $n\geqq 4$. Then for every $\lambda\in(0,\lambda_1)$ there exists a solution of* (1.1).

Remark 1.1. There is *no* solution of (1.1) when $\lambda\geqq\lambda_1$. Indeed, let φ_1 be the eigenfunction of $-\Delta$ corresponding to λ_1 with $\varphi_1>0$ on Ω. Suppose u is a

[4] Note that c depends on v.

solution of (1.1). We have

$$-\int(\Delta u)\varphi_1=\lambda_1\int u\varphi_1=\int u^p\varphi_1+\lambda\int u\varphi_1>\lambda\int u\varphi_1$$

and thus $\lambda<\lambda_1$.

Remark 1.2. There is *no* solution of (1.1) when $\lambda\leqq 0$ and Ω is a (smooth) *starshaped* domain. This follows from *Pohozaev's identity* (see Pohozaev [24]) which we now recall. Suppose u is a (smooth) function satisfying

$$\begin{aligned} -\Delta u&=g(u) && \text{on } \Omega,\\ u&=0 && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where g is a continuous function on $\mathbb{R}$. Then we have

$$(1-\tfrac{1}{2}n)\int_\Omega g(u)\cdot u+n\int_\Omega G(u)=\tfrac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2, \tag{1.3}$$

where

$$G(u)=\int_0^u g(t)\,dt$$

and ν denotes the outward normal to $\partial\Omega$. In particular, when $g(u)=u^p+\lambda u$ we deduce from (1.3) that

$$\lambda\int_\Omega u^2=\tfrac{1}{2}\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2. \tag{1.4}$$

If Ω is starshaped about the origin we have $(x\cdot\nu)>0$ a.e. on $\partial\Omega$. When $\lambda<0$ it follows immediately from (1.4) that $u\equiv 0$. When $\lambda=0$ we deduce from (1.4) that $\partial u/\partial\nu=0$ on $\partial\Omega$ and then by (1.1) we have

$$0=-\int_\Omega\Delta u=\int_\Omega u^p;$$

thus $u\equiv 0$.

The situation can be quite different when Ω is *not starshaped*. For example if Ω is an annulus, there exists a radial solution of (1.1) for every $\lambda\in(-\infty,\lambda_1)$; this fact was first pointed out by Kazdan and Warner [16]; see also subsection 1.3 (point 3) below.

Set

$$S_\lambda=\inf_{\substack{u\in H_0^1\\ \|u\|_{p+1}=1}}\{\|\nabla u\|_2^2-\lambda\|u\|_2^2\}\quad\text{with}\quad\lambda\in\mathbb{R}, \tag{1.5}$$

so that

(1.6)
$$S_0 = S = \inf_{\substack{u \in H_0^1 \\ \|u\|_{p+1}=1}} \|\nabla u\|_2^2$$

corresponds to the best constant for the Sobolev embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, $p+1 = 2n/(n-2)$. We start with some remarks concerning the best Sobolev constant S:

(a) S is independent of Ω and depends only on n. This follows from the fact that the ratio $\|\nabla u\|_2/\|u\|_{p+1}$ with $p+1=2n/(n-2)$ is invariant under scaling; in other words, the ratio $\|\nabla u_k\|_2/\|u_k\|_{p+1}$ is independent of k, where $u_k(x) = u(kx)$.

(b) The infimum in (1.6) is *never* achieved when Ω is a *bounded* domain. Indeed, suppose that S were attained by some function $u \in H_0^1(\Omega)$. We may assume that $u \geqq 0$ on Ω (otherwise replace u by $|u|$). Fix a ball $\tilde{\Omega} \supset \Omega$ and set

$$\tilde{u} = \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \tilde{\Omega} \setminus \Omega. \end{cases}$$

Thus S is also achieved on $\tilde{\Omega}$ by $\tilde{u}$ and $\tilde{u}$ satisfies $-\Delta \tilde{u} = \mu \tilde{u}^p$ for some constant $\mu > 0$; this contradicts Pohozaev's result.

(c) When $\Omega = \mathbf{R}^n$, the infimum in (1.6) is achieved by the function

(1.7)
$$U(x) = C(1+|x|^2)^{-(n-2)/2}$$

or (after scaling) by any of the functions

(1.8)
$$U_\varepsilon(x) = C_\varepsilon(\varepsilon + |x|^2)^{-(n-2)/2}, \qquad \varepsilon > 0,$$

where C and C_ε are normalization constants; see Th. Aubin [2], G. Talenti [28] (both are based on some earlier work of G. A. Bliss [4]) and also E. Lieb [19].

Our first lemma plays a crucial role in the proof of Theorem 1.1; it is an adaptation of an original argument due to Th. Aubin [3] in the context of Yamabe's conjecture.

LEMMA 1.1. *We have*

(1.9)
$$S_\lambda < S \quad \textit{for all} \quad \lambda > 0.$$

Proof: Without loss of generality we may assume that $0 \in \Omega$. We shall estimate the ratio

$$Q_\lambda(u) = \frac{\|\nabla u\|_2^2 - \lambda \|u\|_2^2}{\|u\|_{p+1}^2}$$

with

(1.10)
$$u(x) = u_\varepsilon(x) = \frac{\varphi(x)}{(\varepsilon + |x|^2)^{(n-2)/2}}, \qquad \varepsilon > 0,$$

where $\varphi \in \mathscr{D}_+(\Omega)$ is a *fixed* function such that $\varphi(x) \equiv 1$ for x in some neighborhood of 0. We claim that, as $\varepsilon \to 0$, we have

(1.11)
$$\|\nabla u_\varepsilon\|_2^2 = \frac{K_1}{\varepsilon^{(n-2)/2}} + O(1),$$

(1.12)
$$\|u_\varepsilon\|_{p+1}^2 = \frac{K_2}{\varepsilon^{(n-2)/2}} + O(\varepsilon),$$

(1.13)
$$\|u_\varepsilon\|_2^2 = \begin{cases} \dfrac{K_3}{\varepsilon^{(n-4)/2}} + O(1) & \text{if } n \geqq 5, \\ K_3|\log \varepsilon| + O(1) & \text{if } n = 4, \end{cases}$$

where K_1, K_2 and K_3 denote positive constants which depend only on n and such that $K_1/K_2 = S$.

VERIFICATION OF (1.11): We have

$$\nabla u_\varepsilon(x) = \frac{\nabla \varphi(x)}{(\varepsilon + |x|^2)^{(n-2)/2}} - \frac{(n-2)\varphi(x)x}{(\varepsilon + |x|^2)^{n/2}}.$$

Since $\varphi \equiv 1$ near 0, it follows that

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^2 &= (n-2)^2 \int_\Omega \frac{|x|^2\,dx}{(\varepsilon + |x|^2)^n} + O(1) \\ &= (n-2)^2 \int_{\mathbf{R}^n} \frac{|x|^2\,dx}{(\varepsilon + |x|^2)^n} + O(1) \\ &= \frac{K_1}{\varepsilon^{(n-2)/2}} + O(1), \end{aligned}$$

where

$$K_1 = (n-2)^2 \int_{\mathbf{R}^n} \frac{|x|^2\,dx}{(1+|x|^2)^n} = \|\nabla U\|_2^2.$$

VERIFICATION OF (1.12): We have

$$\begin{aligned} \int_\Omega |u_\varepsilon|^{p+1} &= \int_\Omega \frac{\varphi^{p+1}(x)\,dx}{(\varepsilon + |x|^2)^n} = \int_\Omega \frac{[\varphi^{p+1}(x) - 1]\,dx}{(\varepsilon + |x|^2)^n} + \int_\Omega \frac{dx}{(\varepsilon + |x|^2)^n} \\ &= O(1) + \int_{\mathbf{R}^n} \frac{dx}{(\varepsilon + |x|^2)^n} = \frac{K_2'}{\varepsilon^{n/2}} + O(1), \end{aligned}$$

where

$$K_2' = \int_{\mathbf{R}^n} \frac{dx}{(1+|x|^2)^n} = \|U\|_{p+1}^{p+1}.$$

Thus (1.12) follows with $K_2=\|U\|^2_{p+1}$, and $K_1/K_2=S$.

VERIFICATION OF (1.13): We have

$$\int_\Omega |u_\varepsilon|^2=\int_\Omega \frac{[\varphi^2(x)-1]\,dx}{(\varepsilon+|x|^2)^{n-2}}+\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}}=O(1)+\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}}.$$

When $n\geqq 5$, we have

$$\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^{n-2}}=\int_{\mathbf{R}^n} \frac{dx}{(\varepsilon+|x|^2)^{n-2}}+O(1)$$

and (1.13) follows with

$$K_3=\int_{\mathbf{R}^n} \frac{dx}{(1+|x|^2)^{n-2}}.$$

When $n=4$, we have, for some constants R_1 and R_2,

$$\int_{|x|\leq R_1} \frac{dx}{(\varepsilon-|x|^2)^2}\leqq\int_\Omega \frac{dx}{(\varepsilon+|x|^2)^2}\leqq\int_{|x|\leq R_2} \frac{dx}{(\varepsilon+|x|^2)^2}$$

and

$$\int_{|x|\leq R} \frac{dx}{(\varepsilon+|x|^2)^2}=\omega\int_0^R \frac{r^3dr}{(\varepsilon+r^2)^2}=\tfrac{1}{2}\omega\,|\log\varepsilon|+O(1),$$

where ω is the area of S^3; thus (1.13) follows with $K_3=\frac{1}{2}\omega$. Combining (1.11), (1.12) and (1.13), we obtain

$$Q_\lambda(u_\varepsilon)=\begin{cases} S+O(\varepsilon^{(n-2)/2})-\lambda\dfrac{K_3}{K_2}\varepsilon & \text{if } \quad n\geqq 5,\\[2mm] S+O(\varepsilon)-\lambda\dfrac{K_3}{K_2}\varepsilon\,|\log\varepsilon| & \text{if } \quad n=4.\end{cases}$$

In all cases we deduce that $Q_\lambda(u_\varepsilon)<S$ provided $\varepsilon>0$ is small enough.

LEMMA 1.2. (E. Lieb) *If $S_\lambda<S$, the infimum in* (1.5) *is achieved.*

Proof: Let $(u_j)\subset H^1_0$ be a minimizing sequence for (1.5), that is,

(1.14) $$\|u_j\|_{p+1}=1,$$

(1.15) $$\|\nabla u_j\|_2^2-\lambda\|u_j\|_2^2=S_\lambda+o(1)\quad\text{as}\quad j\to\infty.$$

Since u_j is bounded in H^1_0 we may extract a subsequence—still denoted by u_j—such that

$$u_j\rightharpoonup u\quad\text{weakly in}\quad H^1_0,$$
$$u_j\to u\quad\text{strongly in}\quad L^2,$$
$$u_j\to u\quad\text{a.e. on}\quad \Omega,$$

with $\|u\|_{p+1}\leqq 1$. Set $v_j=u_j-u$, so that

$$v_j\rightharpoonup 0\quad\text{weakly in}\quad H^1_0$$
$$v_j\to 0\quad\text{a.e. on}\quad \Omega.$$

By (1.6) and (1.14) we have $\|\nabla u_j\|_2\geqq S$. From (1.15) it follows that $\lambda\|u\|_2^2\geqq S-S_\lambda>\infty$ and therefore $u\neq 0$. Using (1.15) we obtain

(1.16) $$\|\nabla u\|_2^2+\|\nabla v_j\|_2^2-\lambda\|u\|_2^2=S_\lambda+o(1)$$

since $v_j\rightharpoonup 0$ weakly in H^1_0. On the other hand, we deduce from a result of Brezis and Lieb [8] that

$$\|u+v_j\|_{p+1}^{p+1}=\|u\|_{p+1}^{p+1}+\|v_j\|_{p+1}^{p+1}+o(1)$$

(which holds since v_j is bounded in L^{p+1} and $v_j\to 0$ a.e.). Thus (by (1.14)) we have

$$1=\|u\|_{p+1}^{p+1}+\|v_j\|_{p+1}^{p+1}+o(1)$$

and therefore

$$1\leqq\|u\|_{p+1}^2+\|v_j\|_{p+1}^2+o(1)$$

which leads to

(1.17) $$1\leqq\|u\|_{p-1}^2+\frac{1}{S}\|\nabla v_j\|_2^2+o(1).$$

We claim that

(1.18) $$\|\nabla u\|_2^2-\lambda\|u\|_2^2\leqq S_\lambda\|u\|_{p+1}^2;$$

this will conclude the proof of Lemma 1.2 since $u\neq 0$.

We distinguish two cases:

(a) $S_\lambda>0$ (i.e., $0<\lambda<\lambda_1$),

(b) $S_\lambda\leqq 0$ (i.e., $\lambda\geqq\lambda_1$).

In case (a) we deduce from (1.17) that

(1.19) $$S_\lambda\leqq S_\lambda\|u\|_{p+1}^2+(S_\lambda/S)\|\nabla v_j\|_2^2+o(1).$$

Combining (1.16) and (1.19) we obtain (1.18).

In case (b) we have $S_\lambda\leqq S_\lambda\|u\|_{p+1}^2$ since $\|u\|_{p+1}\leqq 1$. We deduce, again, (1.18) from (1.16).

F. Browder has pointed out that this argument proves more: in fact, $v_j\to 0$ *strongly* in H^1_0; in other words, every minimizing sequence for (1.5) is relatively compact in H^1_0 for the strong H^1_0 topology.

Proof of Theorem 1.1 concluded: Let $u\in H^1_0$ be given by Lemma 1.2, that is,

$$\|u\|_{p+1}=1\quad\text{and}\quad\|\nabla u\|_2^2-\lambda\|u\|_2^2=S_\lambda.$$

We may as well assume that $u \geqq 0$ on Ω (otherwise we replace u by $|u|$). Since u is a minimizer for (1.5) we obtain a Lagrange multiplier $\mu \in \mathbf{R}$ such that

$$-\Delta u - \lambda u = \mu u^p \quad \text{on} \quad \Omega.$$

In fact, $\mu = S_\lambda$, and $S_\lambda > 0$ since $\lambda < \lambda_1$. It follows that ku satisfies (1.1) for some appropriate constant $k > 0$ ($k = S_\lambda^{1/(p-1)}$); note that $u > 0$ on Ω by the strong maximum principle.

Remark 1.3. Our first proof of Theorem 1.1 did not involve Lemma 1.2. Instead, we considered, as in the works of N. Trudinger [30] and Th. Aubin [3]:

$$\mu_q = \inf_{\substack{u \in H_0^1 \\ \|u\|_{q+1}=1}} \{\|\nabla u\|_2^2 - \lambda \|u\|_2^2\} \quad \text{for} \quad q < p. \tag{1.20}$$

It is easy to check that $\lim_{q \to p} \mu_q = S_\lambda$. Moreover since the embedding $H_0^1 \subset L^{q+1}$ is compact, the infimum in (1.20) is achieved by some $u_q \in H_0^1$ such that $u_q \geqq 0$ on Ω, $\|u_q\|_{q+1} = 1$ and

$$-\Delta u_q - \lambda u_q = \mu_q u_q^q. \tag{1.21}$$

It follows that

$$S\|u_q\|_{p+1}^2 - \lambda \|u_q\|_2^2 \leqq \|\nabla u_q\|_2^2 - \lambda \|u_q\|_2^2 = \mu_q. \tag{1.22}$$

As $q \to p$ (through a subsequence), $u_q \rightharpoonup u$ weakly in H_0^1. Passing to the limit in (1.22) we obtain

$$S - \lambda \|u\|_2^2 \leqq S_\lambda$$

and thus (by Lemma 1.1), $u \not\equiv 0$. Finally, we deduce from (1.21) that u satisfies

$$-\Delta u - \lambda u = S_\lambda u^p.$$

Stretching S_λ, as above, we obtain a solution of (1.1).

1.2. The case $n = 3$. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain. We are concerned with the problem of existence of a function u satisfying

$$\begin{aligned} -\Delta u &= u^5 + \lambda u \quad &&\text{on} \quad \Omega, \\ u &> 0 &&\text{on} \quad \Omega, \\ u &= 0 &&\text{on} \quad \partial\Omega, \end{aligned} \tag{1.23}$$

where λ is a real constant. This problem turns out to be rather delicate and we have a complete solution only when Ω is a *ball* (see subsection 1.3 for more general domains). Our main result is the following:

THEOREM 1.2. *Assume Ω is a ball. There exists a solution of* (1.23) *if and only if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$.*

For simplicity we take

$$\Omega = \{x \in \mathbf{R}^3; |x| < 1\}$$

so that $\lambda_1 = \pi^2$ (the corresponding eigenfunction is $|x|^{-1} \sin(\pi|x|)$).

We already know that (1.23) has no solution for $\lambda \geqq \lambda_1$ and for $\lambda \leqq 0$ (see subsection 1.1). As in subsection 1.1 we set

$$S_\lambda = \inf_{\substack{u \in H_0^1 \\ \|u\|_6=1}} \{\|\nabla u\|_2^2 - \lambda \|u\|_2^2\} \quad \text{with} \quad \lambda \in \mathbf{R}, \tag{1.24}$$

and $S = S_0$.

The counterpart of Lemma 1.1 is

LEMMA 1.3. *We have*

$$S_\lambda < S \quad \textit{for all} \quad \lambda > \tfrac{1}{4}\lambda_1. \tag{1.25}$$

Proof: We shall estimate the ratio

$$Q_\lambda(u) = \frac{\|\nabla u\|_2^2 - \lambda \|u\|_2^2}{\|u\|_6^2}$$

with

$$u(x) = u_\varepsilon(r) = \frac{\varphi(r)}{(\varepsilon + r^2)^{1/2}}, \qquad r = |x|, \ \varepsilon > 0, \tag{1.26}$$

where φ is a *fixed* smooth function such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(1) = 0$. We claim that, as $\varepsilon \to 0$, we have

$$\|\nabla u_\varepsilon\|_2^2 = \frac{K_1}{\varepsilon^{1/2}} + \omega \int_0^1 |\varphi'(r)|^2 \, dr + O(\varepsilon^{1/2}), \tag{1.27}$$

$$\|u_\varepsilon\|_6^2 = \frac{K_2}{\varepsilon^{1/2}} + O(\varepsilon^{1/2}), \tag{1.28}$$

$$\|u_\varepsilon\|_2^2 = \omega \int_0^1 \varphi^2(r) \, dr + O(\varepsilon^{1/2}), \tag{1.29}$$

where K_1 and K_2 are positive constants such that $K_1/K_2 = S$ and ω is the area of S^2.

VERIFICATION OF (1.27): We have

$$u_\varepsilon'(r) = \frac{\varphi'(r)}{(\varepsilon + r^2)^{1/2}} - \frac{r\varphi(r)}{(\varepsilon + r^2)^{3/2}}$$

and thus

$$\|\nabla u_\varepsilon\|_2^2 = \omega \int_0^1 \left[\frac{|\varphi'(r)|^2}{(\varepsilon + r^2)} - \frac{2r\varphi(r)\varphi'(r)}{(\varepsilon + r^2)^2} + \frac{r^2\varphi^2(r)}{(\varepsilon + r^2)^3}\right] r^2\, dr.$$

Integrating by parts we find

$$-2\int_0^1 \frac{\varphi(r)\varphi'(r)r^3}{(\varepsilon + r^2)^2}\, dr = \int_0^1 \varphi^2(r)\left[\frac{3r^2}{(\varepsilon + r^2)^2} - \frac{4r^4}{(\varepsilon + r^2)^3}\right] dr,$$

and therefore

(1.30) $$\|\nabla u_\varepsilon\|_2^2 = \omega \int_0^1 \frac{|\varphi'(r)|^2}{(\varepsilon + r^2)} r^2\, dr + 3\omega\varepsilon \int_0^1 \frac{\varphi^2(r) r^2}{(\varepsilon + r^2)^3}\, dr.$$

Using the fact that $\varphi(0) = 1$ and $\varphi'(0) = 0$ we obtain

(1.31) $$\int_0^1 \frac{|\varphi'(r)|^2 r^2}{(\varepsilon + r^2)}\, dr = \int_0^1 |\varphi'(r)|^2\, dr + O(\varepsilon),$$

(1.32) $$\int_0^1 \frac{\varphi^2(r) r^2}{(\varepsilon + r^2)^3}\, dr = \int_0^1 \frac{r^2}{(\varepsilon + r^2)^3}\, dr + O(\varepsilon^{-1/2}).$$

Also, we have

(1.33) $$\int_0^1 \frac{r^2}{(\varepsilon + r^2)^3}\, dr = \frac{1}{\varepsilon^{3/2}} \int_0^{\varepsilon^{-1/2}} \frac{s^2}{(1+s^2)^3}\, ds = \frac{1}{\varepsilon^{3/2}} \int_0^\infty \frac{s^2}{(1+s^2)^3}\, ds + O(1).$$

Combining (1.30)–(1.32) and (1.33) we obtain (1.27) with

$$K_1 = 3\omega \int_0^\infty \frac{s^2}{(1+s^2)^3}\, ds.$$

Finally we note that $K_1 = \int_{\mathbb{R}^3} |\nabla U|^2\, dx$, where $U(x) = 1/(1+|x|^2)^{1/2}$; here we use the fact that

$$\int_0^\infty \frac{s^2}{(1+s^2)^3}\, ds = \tfrac{1}{16}\pi \quad \text{and} \quad \int_0^\infty \frac{s^4}{(1+s^2)^3}\, ds = \tfrac{3}{16}\pi.$$

VERIFICATION OF (1.28): We have

$$\|u_\varepsilon\|_6^6 = \omega \int_0^1 \frac{\varphi^6(r) r^2}{(\varepsilon + r^2)^3}\, dr = \omega \int_0^1 \frac{(\varphi^6(r) - 1) r^2}{(\varepsilon + r^2)^3}\, dr + \omega \int_0^1 \frac{r^2}{(\varepsilon + r^2)^3}\, dr$$
$$= I_1 + I_2.$$

Since $\varphi(0) = 1$ and $\varphi'(0) = 0$ we obtain

$$|I_1| \le C \int_0^1 \frac{r^4}{(\varepsilon + r^2)^3}\, dr = O(\varepsilon^{-1/2}).$$

Next we have

$$I_2 = \frac{\omega}{\varepsilon^{3/2}} \int_0^{\varepsilon^{-1/2}} \frac{s^2}{(1+s^2)^3}\, ds = \frac{\omega}{\varepsilon^{3/2}} \int_0^\infty \frac{s^2}{(1+s^2)^3}\, ds + O(1).$$

Therefore we find

$$\|u_\varepsilon\|_6^6 = \frac{1}{\varepsilon^{3/2}}\left[\omega \int_0^\infty \frac{s^2}{(1+s^2)^3}\, ds + O(\varepsilon)\right]$$

and (1.28) follows with

$$K_2 = \left[\omega \int_0^\infty \frac{s^2}{(1+s^2)^3}\, ds\right]^{1/3} = \|U\|_6^2.$$

VERIFICATION OF (1.29): We have

$$\|u_\varepsilon\|_2^2 = \omega \int_0^1 \frac{\varphi^2(r) r^2}{(\varepsilon + r^2)}\, dr = \omega \int_0^1 \varphi^2(r)\, dr + O(\varepsilon^{1/2}).$$

Combining (1.27), (1.28) and (1.29) we obtain

(1.34) $$Q_\lambda(u_\varepsilon) = S + \varepsilon^{1/2} \frac{\omega}{K_2}\left[\int_0^1 |\varphi'(r)|^2\, dr - \lambda \int_0^1 \varphi^2(r)\, dr\right] + O(\varepsilon).$$

Choosing $\varphi(r) = \cos(\frac{1}{2}\pi r)$ we have

$$\int_0^1 |\varphi'(r)|^2\, dr = \tfrac{1}{4}\pi^2 \int_0^1 \varphi^2(r)\, dr$$

and thus

$$Q_\lambda(u_\varepsilon) = S + (\tfrac{1}{4}\pi^2 - \lambda) C\varepsilon^{1/2} + O(\varepsilon)$$

for some positive constant C. The conclusion of Lemma 1.3 follows by choosin $\varepsilon > 0$ small enough.

The next Lemma is a crucial step in the proof of Theorem 1.2:

LEMMA 1.4. *There is no solution of* (1.23) *for* $\lambda \le \frac{1}{4}\lambda_1$.

Proof: Suppose u is a solution of (1.23); by a result of Gidas–Ni–Nirenber [13] we know that u must be spherically symmetric. We write $u(x) = u(r)$ where $r = |x|$, and thus u satisfies

(1.35) $$-u'' - \frac{2}{r} u' = u^5 + \lambda u \quad \text{on} \quad (0, 1),$$

(1.36) $$u'(0) = u(1) = 0.$$

We claim that

$$\int_0^1 u^2(\lambda\psi'+\tfrac14\psi''')r^2\,dr=\tfrac23\int_0^1 u^6(r\psi-r^2\psi')\,dr+\tfrac12|u'(1)|^2\psi(1) \tag{1.37}$$

for every smooth function ψ such that $\psi(0)=0$.[5] Indeed, we first multiply (1.35) by $r^2\psi u'$ and obtain

$$\begin{aligned}&\int_0^1 |u'|^2(\tfrac12 r^2\psi'-r\psi)\,dr-\tfrac12|u'(1)|^2\psi(1)\\ &=-\tfrac16\int_0^1 u^6(2r\psi+r^2\psi')\,dr-\tfrac12\lambda\int_0^1 u^2(2r\psi+r^2\psi')\,dr.\end{aligned} \tag{1.38}$$

Next we multiply (1.35) by $(\frac12 r^2\psi'-r\psi)u$ and obtain

$$\begin{aligned}&\int_0^1 |u'|^2(\tfrac12 r^2\psi'-r\psi)\,dr-\tfrac14\int_0^1 u^2r^2\psi'''\,dr\\ &=\int_0^1 u^6(\tfrac12 r^2\psi'-r\psi)\,dr+\lambda\int_0^1 u^2(\tfrac12 r^2\psi'-r\psi)\,dr.\end{aligned} \tag{1.39}$$

Combining (1.38) and (1.39) we obtain (1.37). We already know that there is no solution of (1.23) for $\lambda\leqq 0$; thus we may assume that $0<\lambda\leqq\frac14\pi^2$. In (1.37) we choose $\psi(r)=\sin((4\lambda)^{1/2}r)$ so that $\psi(1)\geqq 0$,

$$\lambda\psi'+\tfrac14\psi'''=0,$$

and

$$r\psi-r^2\psi'=r\sin((4\lambda)^{1/2}r)-r^2(4\lambda)^{1/2}\cos((4\lambda)^{1/2}r)>0\quad\text{on}\quad(0,1]$$

(since $\sin\theta-\theta\cos\theta>0$ for all $\theta\in(0,\pi]$) and we obtain a contradiction.

Proof of Theorem 1.2 concluded: If $\lambda>\frac14\lambda_1$ we know that $S_\lambda<S$ (see Lemma 1.3). We may proceed exactly as in the proof of Theorem 1.1 (Lemma 1.2) and conclude that the infimum in (1.24) is achieved. Thus we obtain some $u\in H_0^1$ with $u\geqq 0$ on Ω, $\|u\|_6=1$ and

$$-\Delta u-\lambda u=S_\lambda u^5.$$

If, in addition, $\lambda<\lambda_1$, then $S_\lambda>0$ and after stretching, we obtain a solution of (1.23).

1.3. Additional properties, miscellaneous remarks and open problems.

(1). REGULARITY OF SOLUTIONS. The solution u of (1.1) given by Theorem 1.1 (respectively Theorem 1.2) lies in $H_0^1(\Omega)$. In fact, u belongs to

[5] Note that Pohozaev's identity corresponds to the case where $\psi(r)=r$.

$L^\infty(\Omega)$. This is proved by Trudinger [30] for Yamabe's problem (on a manifold without boundary) but the same argument applies here. Alternatively, one could also invoke the following Lemma which is essentially contained in Brezis-Kato [7]:

LEMMA 1.5. *Assume $u\in H_0^1(\Omega)$ satisfies*

$$-\Delta u=au\quad\textit{in}\quad\mathcal{D}'(\Omega),$$

where $a(x)\in L^{n/2}(\Omega)$ and $n\geqq 3$. Then $u\in L^t(\Omega)$ for all $t<\infty$.

For our purpose we use Lemma 1.5 with $a=\lambda+u^{p-1}\in L^{n/2}$ (since $u\in L^{p+1}$).

Finally we note that $u\in C^\infty(\Omega)$ (since $u>0$ in Ω) and, up to the boundary, u is as smooth as $\partial\Omega$ and p permit.

(2). THE CASE $p>(n+2)/(n-2)$ WITH $n\geqq 3$. It follows from general bifurcation theory—see e.g. Rabinowitz [25]—that for any $p>1$ (even $p>(n+2)/(n-2)$) problem (1.1) possesses a *component* $\mathscr{C}$ of solutions (λ,u) which meets $(\lambda_1,0)$ and which is unbounded in $\mathbf{R}\times L^\infty(\Omega)$. Theorem 1.1 *suggests* that, when $p=(n+2)/(n-2)$ and $n\geqq 4$, the projection of $\mathscr{C}$ on the λ-axis contains the interval $(0,\lambda_1)$ (with the obvious modification when $n=3$ and $p=5$).

On the other hand *when $p>(n+2)/(n-2)$ and Ω is starshaped*, problem (1.1) has *no* solution if $\lambda\leqq\lambda^*$, where λ^* is some *positive* constant which depends on Ω and p. This was pointed out by Rabinowitz [26] in the case $n=3$ and $p=7$, but the same argument works in the general case: suppose u satisfies (1.1); Pohozaev's identity leads to (assuming star-shapedness about the origin)

$$(1-\tfrac12 n)\int_\Omega(u^{p+1}+\lambda u^2)+n\int_\Omega\left(\frac{u^{p+1}}{p+1}+\tfrac12\lambda u^2\right)=\tfrac12\int_{\partial\Omega}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\right)^2>0$$

and thus we find

$$\left(-1+\tfrac12 n-\frac{n}{p+1}\right)\int_\Omega u^{p+1}<\lambda\int_\Omega u^2. \tag{1.40}$$

We deduce from (1.1) and (1.40) that

$$\begin{aligned}\lambda_1\int_\Omega u^2\leqq\int_\Omega|\nabla u|^2&=\int_\Omega u^{p+1}+\lambda\int_\Omega u^2\\ &<\lambda\left(-1+\tfrac12 n-\frac{n}{p+1}\right)^{-1}\int_\Omega u^2+\lambda\int_\Omega u^2,\end{aligned}$$

that is,

$$\lambda>\lambda_1\cdot\frac{n-2}{n}\cdot\frac{p-(n+2)/(n-2)}{p-1}.$$

(3). UNIQUENESS–NONUNIQUENESS When Ω is a *ball*, every solution of (1.1) is spherically symmetric (see [13]). Even in this case we do not know whether (1.1) has a unique solution. Uniqueness results for some semilinear elliptic equations in *all of* $\mathbf{R}^n$ have been obtained by Coffman [9], L. A. Peletier and J. Serrin [23], and K. McLeod and J. Serrin [22].[6] On the other hand, if Ω is *an annulus*, say $\Omega=\{x\in\mathbf{R}^n;\ 1<|x|<2\}$ with $n\geqq 4$, then (1.1) admits both radial and nonradial solutions for all $\lambda>0$ sufficiently small.[7] Indeed, set

$$\Sigma_\lambda=\inf_{\substack{u\in H_r\\ \|u\|_{p+1}=1}}\{\|\nabla u\|_2^2-\lambda\|u\|_2^2\}, \tag{1.41}$$

where $H_r=\{u\in H_0^1;\ u \text{ is radial}\}$. Since the injection $H_r\subset L^{p+1}$ is *compact*, the infimum in (1.41) is achieved (for any $\lambda\in\mathbf{R}$) by some $u_\lambda\in H_r$ such that

$$u_\lambda\geqq 0 \quad\text{on}\quad \Omega, \qquad \|u_\lambda\|_{p+1}=1$$

and $-\Delta u_\lambda-\lambda u_\lambda=\Sigma_\lambda u_\lambda^p$ on Ω. If $\lambda<\lambda_1$, then $\Sigma_\lambda>0$ and, after stretching Σ_λ, we obtain a solution of (1.1). Next we consider S_λ defined by (1.5). It is easy to check that the functions $\lambda\mapsto\Sigma_\lambda$ and $\lambda\mapsto S_\lambda$ are continuous (even Lipschitz continuous). We have $S=S_0<\Sigma_0$ (otherwise the best Sobolev constant would be achieved—which is impossible; see subsection 1.1). Thus for $\lambda>0$ sufficiently small, $S_\lambda<\Sigma_\lambda$, and the infimum in (1.5) is achieved (see Lemma 1.2) by some *nonradial* function; in this way we obtain a nonradial solution of (1.1). We do not know whether the nonradial solutions occur by secondary bifurcation from the branch of radial solutions

A similar argument shows that the problem

$$\begin{aligned} -\Delta u &= u^q \quad\text{on the annulus}\quad \Omega,\\ u&>0 \quad\text{on}\quad \Omega,\\ u&=0 \quad\text{on}\quad \partial\Omega, \end{aligned} \tag{1.42}$$

admits both radial and nonradial solutions for all $q<(n+2)/(n-2)$ sufficiently close to $(n+2)/(n-2)$.

(4). EQUATIONS WITH VARIABLE COEFFICIENTS. Let $\Omega\subset\mathbf{R}^n$, $n\geqq 4$, be a bounded domain. Assume $a(x)\in L^\infty(\Omega)$ is given such that

$$a(x)\geqq\delta \quad\text{on some open subset of}\quad \Omega, \tag{1.43}$$

[6] Other uniqueness results have been obtained by W. M. Ni: *Uniqueness of solutions of nonlinear Dirichlet problems*, J. Diff. Eqns., to appear, and in a paper by Ni and R. Nussbaum (in preparation).

[7] Of course this fact does not contradict the result on spherical symmetry of [13] which holds only on balls. Nonradial solutions for some semilinear equations on the annulus have also been investigated by D. Schaeffer [27] and C. Coffman [10].