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Morse Theory for Harmonic Maps

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These are preliminary lecture notes, intended only for distribution to participants

Morse Theory for Harmonic Maps

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In the previous paper [Ch 1], we studied the Minimax Principle as well as the Ljusternik-Schnirelman category theory for harmonic maps with prescribed boundary data defined on Riemann surfaces, by the heat flow method. In this paper, we shall continue our study on the Morse theory. Our main results are the Morse inequalities (Theorem 1) for isolated harmonic maps, and the Morse handle body decomposition for nondegenerate harmonic maps (Theorem 2). These results are extensions of the work of K. Uhlenbeck [Ul], where the harmonic maps are defined on manifolds without boundary, and are all assumed to be nondegenerate. Our method is based on the heat flow, by which the deformation is constructed. In contract to the perturbation method developed by K. Uhlenbeck [Ul], our approach seems more direct than hers.

1. Preliminaries

Let (M,g) and (N,h) be two Riemannian manifolds. For a smooth map $u : M \to N$, let $e(u) = \frac{1}{2}|\nabla u|^2$ be the energy density, and let

$$E(u) = \int_M e(u)\,dV_g$$

be the energy. The critical points of the energy functional are harmonic maps, which satisfy the following Euler-Lagrange equation:

$$\Delta u^\alpha = \Delta_M u^\alpha + g^{ij}(x)\, {}^N\Gamma^\alpha_{\beta\gamma}(u)\, u^\gamma_{,i}\, u^\beta_{,j} = 0$$

where ${}^N\Gamma$ is the Christoffel symbol of N, (g^{ij}) is the inverse of the metric g, Δ_M is the Laplace-Betrami operator w.r.t. g, and Δu is the trace of the tension tensor field.

The associate heat flow equation reads as follows

(1.1) $$\partial_t f = \Delta f$$

where $f : [0,\infty) \times M \to N$.

We consider the initial-boundary value problem. Find $f \in C^{1+\frac{\gamma}{2}, 2+\gamma}((0,\infty) \times \bar{M}, N)$

(1.2) $$f(0,x) = \phi(x) ,$$

(1.3) $$f(t,\cdot)\big|_{\partial M} = \psi(\cdot) ,$$

where $\psi \in C^{2,\gamma}(\partial M, N)$, $\gamma > 0$, and $\phi \in C^{2,\gamma}_{\psi}(\bar{M},N)$. The later is the class of $C^{2,\gamma}(\bar{M},N)$ functions, which have the common boundary value ψ .

Let

$$m = \inf\{E(u) \,|\, u \in C^1_{\psi}(\bar{M},N)\} ,$$

$$b = \inf\{E(v) \,|\, v : S^2 \to N \text{ , nonconstant harmonic}\}$$

(if there is no nonconstant harmonic map from S^2 to N , then we define $b = +\infty$) , and let $\mathcal{F}$ be a component of $C^{2,\gamma}_{\psi}(\bar{M},N)$.

Assume that $\dim M = 2$, and that

$$E(\phi) < m + b \text{ or } m_{\mathcal{F}} + b \text{ if } \pi_2(N) = 0 \text{ , and } \phi \in \mathcal{F} ,$$

where

$$m_{\mathcal{F}} = \inf\{E(u) \,|\, u \in \mathcal{F}\} .$$

Then we have

(1) The heat flow, i.e. the solution of (1.1), (1.2) and (1.3), globally exists.

(2) $\exists$ a harmonic map $\tilde{u} \in C^{2,\gamma}_{\psi}(\bar{M},N)$, and a sequence $t_j \uparrow +\infty$ such that

$$f(t_j,\cdot) \to \tilde{u}(\cdot) \text{ in } C^1(\bar{M},N) .$$

(3) If the infinitely dimensional manifold $C^{2,\gamma}_{\psi}(\bar{M},N)$ is endowed with a weaker topology $W^2_p(M,N)$, $p > \frac{4}{1-\gamma}$, the flow

$$(t,\phi) \longmapsto f_{\phi}(t,\cdot)$$

is continuous from $[0,\infty) \times C^{2,\gamma}_{\psi}(\bar{M},N) \to C^{2,\gamma}_{\psi}(\bar{M},N)$, where $f_{\phi}(t,\cdot)$ denotes the flow with initial data ϕ .

(4) The set

$$K_c = \{u \in C_\psi^{2,\gamma}(\bar{M},N) \mid \Delta u = 0,\ E(u) = c\}$$

is compact under the above topology, if $c < m+b$. (or $c < m_{\mathcal{F}} + b$, if $\pi_2(N) = 0$, and $\phi \in \mathcal{F}$).

(5) Let $K = \bigcup_{c<m+b} K_c$ (or $\bigcup_{c<m_{\mathcal{F}}+b} K_c$, if $\pi_2(N) = 0$, and $\phi \in \mathcal{F}$).

Suppose that

$$\operatorname{dist}_{W_p^2}(f_\phi(t,\cdot),K) \geq \delta > 0 \qquad \forall t \in \mathbb{R}_+^1$$

then we have $\varepsilon = \varepsilon(\delta) > 0$ such that

$$\|\Delta f_\phi(t,\cdot)\|_{L^2(M,N)} \geq \varepsilon \ .$$

(6) For any closed neighbourhood $U \subset C_\psi^{2,\gamma}(\bar{M},N)$ of K_c , under the W_p^2-topology, where $c < m+b$ (or $m_{\mathcal{F}} + b$ if $\pi_2(N) = 0$ and $\phi \in \mathcal{F}$), $\exists\, \varepsilon > 0$, a closed neighbourhood $V \subset U$, and a W_p^2- $(p > \frac{4}{1-\gamma})$ strong deformation retract $\eta : [0,1] \times E_{c+\varepsilon} \to E_{c+\varepsilon}$, satisfying

$$\eta(1, E_c \cap V) \subset E_c \cap U \ , \text{ and}$$

$$\eta(1, E_{c+\varepsilon} \setminus V) \subset E_{c-\varepsilon} \ ,$$

where $E_a = \{u \in C_\psi^{2,\gamma}(\bar{M},N) \mid E(u) \leq a\}$ is the level set, $\forall a \in \mathbb{R}_+^1$.

It follows from the proof of Th. 7.1 in [Ch 1].

2. The Morse inequalities

In this section, we establish the Morse inequalities for harmonic maps under the assumption, that all harmonic maps are isolated. As shown in [Ch 2] , the crucial step in the proof is to prove the following deformation lemma.

Lemma 2.1. Let $\mathcal{F}$ be a component of $C_\psi^{2+\gamma}(\bar{M},N)$. Suppose that there is no harmonic maps with energy in the interval $(c,d]$, where $d < m_{\mathcal{F}} + b$, and that there are at most finitely many harmonic maps on the level $E^{-1}(c)$. Assume that $\pi_2(N) = 0$, then E_c is a strong deformation retract of E_d .

In order for give the proof, first we turn out to improve the conclusion (2) in section 1, under the condition that the set of smooth harmonic maps is isolated. Namely

<u>Lemma 2.2</u>. Let $E(\phi) < m_{\mathcal{F}} + b$, and let

$$c = \lim_{t\to+\infty} E(f_\phi(t,\cdot)) \ .$$

If K_c is isolated, then $f_\phi(t,\cdot) \to \tilde{u} \in K_c$ in the W^2_p-topology, $\forall p > \frac{4}{1-\gamma}$, as $t \to +\infty$.

<u>Proof</u>. According to the conclusion (2), in combining with a bootstrap iteration, one shows that $\exists \tilde{u} \in K_c$ and $t_j \uparrow +\infty$ such that

$$f_\phi(t_j,\cdot) \to \tilde{u} \ , \quad C^{2,\gamma'}_\psi(\bar{M},N) \ , \quad \forall \gamma' \in (0,\gamma) \ .$$

If our conclusion was not true, then there would be a $\delta > 0$, such that the neighbourhood $U_\delta = \{u \in C^{2,\gamma}_\psi(\bar{M},N) \,|\, \mathrm{dist}_{W^2_p}(u,\tilde{u}) \le \delta\}$ contains the single element $\tilde{u}$ in K_c , and a sequence $t'_j \uparrow +\infty$ such that $f_\phi(t'_j,\cdot) \notin U_\delta$. Therefore $\exists\ (t^*_i, t^{**}_i)$ satisfying

(1) $t^*_i, t^{**}_i \to +\infty$,

(2) $f_\phi(t^*_i,\cdot) \in \partial U_{2\delta}$, $f_\phi(t^{**}_i,\cdot) \in \partial U_\delta$,

and

(3) $f_\phi(t,\cdot) \in U_{2\delta} \setminus U_\delta \qquad \forall t \in (t^*_i, t^{**}_i)$.

On one hand, we had

$$\delta \le \|f_\phi(t^*_i,\cdot) - f_\phi(t^{**}_i,\cdot)\|_{W^2_p} \le C_\delta |t^*_i - t^{**}_i|^{\gamma/2} \ ,$$

provided by the embedding theorem. On the other hand, according to conclusion (5),

$$\begin{aligned} &E(f_\phi(t^{**}_i,\cdot)) - E(f_\phi(t^*_i,\cdot)) \\ &= \int_{t^*_i}^{t^{**}_i}\int_M |\partial_t f(t,\cdot)|^2 dV_g dt \\ &= \int_{t^*_i}^{t^{**}_i}\int_M |\Delta f(t,\cdot)|^2 dV_g dt \\ &> \varepsilon(\delta)\, |t^{**}_i - t^*_i| \ . \end{aligned}$$

Since the LHS of the inequality tends to zero as $i \to \infty$, this is a contradiction. ∎

Now we return to the proof of lemma 2.1., the basic idea is to reparametrize the heat flow $f_\phi(t,\cdot)$.

Let $\tau = \rho(t)$, where

$$\rho(t) = (E(\phi)-c)^{-1}\int_0^t \|\Delta f_\phi(s,\cdot)\|^2_{L^2} ds ,$$

if $E(\phi) > c$ and let

$$g(\tau,\cdot) = f(t,\cdot) .$$

Then we have the following relations:

(1) $$\partial_\tau g(\tau,\cdot) = \frac{dt}{d\tau}\partial_t f(t,\cdot)$$

$$= \frac{(E(\phi)-c)}{\|\Delta g(\tau,\cdot)\|^2_{L^2}} \Delta g(\tau,\cdot) ,$$

(2) $$\frac{d}{d\tau}E(g(\tau,\cdot)) = -\int_M <\partial_\tau g(\tau,\cdot),\Delta g(\tau,\cdot)>dV_g$$

$$= -(E(\phi)-c) .$$

Therefore

$$E(g(\tau,\cdot)) = (1-\tau)E(\phi) + \tau c , \forall\tau \in [0,1] .$$

(3) The function $\rho : [0,\infty) \to \mathbb{R}^1$, is continuous and monotone increasing, which satisfies the following properties:

$$\rho(0) = 0 ,$$

$$\rho(+\infty) = 1 \quad \text{if} \quad f_\phi(t,\cdot) \to \tilde{u} \in K_c \quad \text{as} \quad t \to +\infty ,$$

$$\rho(+\infty) > 1 \quad \text{if} \quad \lim_{t\to+\infty} E(f_\phi(t,\cdot)) < c .$$

Let us define a function $\eta : [0,1] \times E_d \to E_d$ as follows:

$$\eta(\tau,\phi) = \begin{cases} g_\phi(\tau,\cdot) & \text{if} \quad (\tau,\phi) \in [0,1] \times (E_d \setminus E_c) , \\ \phi & \text{if} \quad (\tau,\phi) \in [0,1] \times E_c . \end{cases}$$

In order to show that E_c is a deformation retract of E_d, only the continuity at the following sets is needed:

(1) $\{1\}\times A$, where $A = \{\phi \in E_d \setminus E_c \,|\, f_\phi(\infty,\cdot) \in K_c\}$

(2) $[0,1]\times E^{-1}(c)$

Verification for case (1). $\forall \phi_o \in A$, $\forall \varepsilon > 0$, want to find $\delta > 0$ such that

$$\left.\begin{array}{l} \mathrm{dist}_{W_p^2}(\phi,\phi_o) < \delta \\ \tau > 1-\delta \end{array}\right\} \text{ implies } \mathrm{dist}_{W_p^2}(g_\phi(\tau,\cdot),\tilde{u}) < \varepsilon$$

where $\tilde{u} = f_{\phi_o}(\infty,\cdot)$.

Choose $\varepsilon_o = \varepsilon_o(\delta_1)$ as in the conclusion (5), i.e.

$$\|\Delta f_\phi(t,\cdot)\|_{L^2} \geq \varepsilon_o \quad \text{if} \quad \mathrm{dist}_{W_p^2}(f_\phi(t,\cdot),K) \geq \delta_1 \ \forall t .$$

and choose

$$0 < \delta_1 < \left(\frac{\varepsilon}{2C_\varepsilon}\right)^{2/\gamma} \frac{\varepsilon_o^2}{E(\phi)-c}$$

such that

$$\mathrm{dist}_{W_p^2}(g_{\phi_o}(1-\delta_1,\cdot),\tilde{u}) < \frac{\varepsilon}{2} .$$

Again, we choose $\delta_2 > 0$ such that $\mathrm{dist}_{W_p^2}(\phi,\phi_o) < \delta_2$ implies

$$\mathrm{dist}_{W_p^2}(g_{\phi_o}(1-\delta_1,\cdot),g_\phi(1-\delta_1,\cdot)) < \frac{\varepsilon}{2} .$$

Therefore we have

$$\mathrm{dist}_{W_p^2}(g_\phi(1-\delta_1,\cdot),\tilde{u}) < \varepsilon \quad \forall \phi \in B_{\delta_2}(\phi_o) .$$

We want to prove

$$\mathrm{dist}_{W_p^2}(g_\phi(\tau,\cdot),\tilde{u}) < \varepsilon \quad \forall (\tau,\phi) \in (1-\delta_1,1]\times B_{\delta_2}(\phi_o) .$$

If not, $\tau'' > \tau' > 1-\delta_1$ and $\phi_1 \in B_{\delta_2}(\phi_o)$ such that

$$g_{\phi_1}(\tau',\cdot) \in \partial B_{\varepsilon/2}(\tilde{u}) , \quad g_{\phi_1}(\tau'',\cdot) \in \partial B_\varepsilon(\tilde{u}) ,$$

and

$$g_{\phi_1}(\tau,\cdot) \in B_\varepsilon(\tilde{u}) \setminus B_{\varepsilon/2}(\tilde{u}) .$$

Then we have

$$\frac{\varepsilon}{2} \leq \operatorname{dist}_{W_p^2}(g_{\phi_1}(\tau',\cdot),g_{\phi_1}(\tau'',\cdot))$$

$$= \operatorname{dist}_{W_p^2}(f_{\phi_1}(t',\cdot),f_{\phi_1}(\ '',\cdot))$$

$$\leq C_\varepsilon |t'-t''|^{\gamma/2} \ .$$

On the other hand

$$\varepsilon_o^2|t'-t''| \leq \int_{t'}^{t''} \|\Delta f_{\phi_1}(t,\cdot)\|_{L^2}^2 dt$$

$$= E(f_{\phi_1}(t'',\cdot)) - E(f_{\phi_1}(t',\cdot))$$

$$= E(g_{\phi_1}(\tau'',\cdot)) - E(g_{\phi_1}(\tau',\cdot))$$

$$= (E(\phi)-c)\,|\tau''-\tau'|$$

$$< \delta_1(E(\phi)-c) \ ,$$

which implies that

$$\delta_1 > \frac{\varepsilon_o^2}{E(\phi)-c}\,|t'-t''| \geq \frac{\varepsilon_o^2}{(E(\phi)-c)}\left(\frac{\varepsilon}{2C_\varepsilon}\right)^{2/\gamma} .$$

This is a contradiction.

<u>Verification for case (2)</u>. $\forall \phi_o \in E^{-1}(c)$, $\forall \varepsilon > 0$ want to find $\delta > 0$ such that $\operatorname{dist}(\phi,\phi_o) < \delta$ implies $\operatorname{dist}(\eta(\tau,\phi),\phi_o) < \varepsilon$.

Similar to the above argument, let us choose

$$0 < \delta_1 < \varepsilon_o^2 \left(\frac{\varepsilon}{2C_\varepsilon}\right)^{2/\gamma} .$$

Find $0 < \delta < \varepsilon/2$ such that

$$E(\phi)-c < \delta_1 \qquad \forall \phi \in B_\delta(\phi_o) \ .$$

If our conclusion was not true, by the same procedure, we would have

(i) $\frac{\varepsilon}{2} \leq C_\varepsilon |t'-t''|^{\gamma/2}$,

and

(ii) $\varepsilon_o^2|t'-t''| \leq (E(\phi)-c)\,|\tau''-\tau| < \delta_1$.

This is again a contradiction.

The continuity of η is proved, so that E_c is a strong deformation retract of E_d, $d < m_{\mathcal{F}} + b$. ∎

Before going to set up the Morse inequalities, we define the critical groups for an isolated harmonic map.

Let G be an Abelian group. Let $u_o \in \mathcal{F}$ be an isolated harmonic map $c = E(u_o)$. Choose a neighbourhood U of u_o such that $K \cap U \cap \mathcal{F} = \{u_o\}$.

<u>Definition</u>

$$C_q(u_o;G) = H_q(E_c \cap U, (E_c \setminus \{u_o\}) \cap U; G)$$

$q = 0,1,2,\ldots$ are defined to be the critical groups with coefficient group G of E at u_o, where $H_*(X,Y;G)$ stands for the singular relative homology group with coefficient group G.

The excision property of the relative homology groups assures that these critical groups are well defined, i.e. they do not depend on the special choice of U.

Suppose that $\forall d < m_{\mathcal{F}} + b$, there are only isolated harmonic maps. Since $K \cap E_d$ is compact, they are finite. There are only isolated critical values (at most with accumulate point $m_{\mathcal{F}} + b$)

$$m_{\mathcal{F}} = c_o < c_1 < \ldots < c_i < \ldots < m_{\mathcal{F}} + b .$$

For each c_i, there are finitely many harmonic maps:

$$K_{c_i} = \{u_{ij} \mid j = 1,2,\ldots,m_i\} .$$

$\forall d < m_{\mathcal{F}} + b$, let

$$M_q^d = \sum_{c_i < d} \sum_{j=i}^{m_i} \text{rank } C_q(u_{ij};G)$$

be the q-th Morse type number, $q = 0,1,2,\ldots,$ for the manifold $E_d \cap \mathcal{F}_1$ and let

$$\beta_q^d = \text{rank } H_q(E_d \cap \mathcal{F}, G)$$

be the q^{th} Betti number, $q = 0,1,2,\ldots$, for $E_d \cap \mathcal{F}$.

It follows from a direct computation (cf. [Ch 2]) that there exists a formal power series with nonnegative coefficients $Q^d(t)$ such that

$$\sum_{q=o}^{\infty} M_q^d t^q = \sum_{q=o}^{\infty} \beta_q^d t^q + (1+t)Q^d(t) .$$

This includes a series of Morse inequalities. Namely, we have proved:

Theorem 1. Let $\mathcal{F}$ be a component of $C_{\psi}^{2,\gamma}(\bar{M}^2,N)$, and let $d < m_{\mathcal{F}} + b$. Assume that $\pi_2(N) = 0$, and that in the level set $E_d \cap \mathcal{F}$ there are only isolated harmonic maps. Then we have the following Morse inequalities:

$$M_o^d \geq \beta_o^d ,$$

$$M_1^d - M_o^d \geq \beta_1^d - \beta_o^d ,$$

$$\cdots \qquad \cdots$$

$$M_n^d - M_{n-1}^d + \ldots + (-1)^n M_o^d \geq \beta_n^d - \beta_{n-1}^d + \ldots + (-1)^n \beta_o^d ,$$

$$\cdots \qquad \cdots .$$

Theorem 1'. Let $d < m + b$. Assume that there are only isolated harmonic maps in the level set E_d , then the above inequalities hold, wherein, the Morse type numbers M_q^d count all harmonic maps in the level set E_d , and β_q^d is the Betti number of E_d , $q = 0,1,2,\ldots$.

3. Morse decomposition

In this section, we study the handle body decomposition of the level sets of the energy function, under the assumption that all harmonic maps in these level sets are nondegenerate. As a consequence, we explain the Morse type numbers M_q^d which was studied before.

Let u_o be a harmonic map from M to N.. Let $E = u_o^*TN$ be the pull back bundle over M . Let $\mathcal{O}$ be a neighbourhood of $C^{\infty}(M,N)$ which contains the section $u_o(M)$. It is obvious that $\mathcal{O}$ is diffeomorphic to a neighbourhood $\mathcal{O}_E$ of the zero section of the tangent space $T_{u_o}(E)$. The diffeomorphism is realized by the exponential map:

$$\begin{array}{ccc} \sigma \in \mathcal{O}_E & \xrightarrow{\exp_{u_o(x)}} & \mathcal{O} \\ \cap & & \cap \\ T_{u_o}(E) & & C^\infty_\psi(M,N) \end{array} .$$

Since then, we do not distinguish the tangent vector σ with its exponential map $\exp_{u_o(x)}\sigma(x)$. We shall restrict our studies in the neighbourhood $\mathcal{O}_E$ of the vector space $T_{u_o}(E)$. The Taylor expansion of the energy functional at u_o is as follows:

$$E(u) = E(u_o) + \frac{1}{2} d^2E(u_o)(\sigma,\sigma) + R(\sigma)$$

where $u(x) = \exp_{u_o(x)}\sigma(x)$, and the remainder $R(\sigma)$ satisfies

$$|R(\sigma)| = o(\int_M |\nabla\sigma|^2) ,$$

and

$$|dR(\sigma)| = o((\int_M |\nabla\sigma|^2)^{1/2}) .$$

As to the Hessian $d^2E(u_o)$, it is well known (see Eells-Lemaire [EL1]) that, $\forall\sigma,\eta \in C^\infty(T_{u_o}(E))$,

$$d^2E(u_o)(\sigma,\eta) = \int_M \langle J_{u_o}\sigma,\eta\rangle dV_g ,$$

ahere

$$J_{u_o}\sigma = -\Delta^{u_o}\sigma - \text{Trace } R^N(du_o,\sigma)du_o ,$$

is the Jacobi operator.

Noticing that J_{u_o} is a linear self-adjoint elliptic differential operator, with domain $W^2_2 \cap \overset{\circ}{W}{}^1_2(T_{u_o}(E))$, J_{u_o} can be extended to be a continuous bilinear form on the Hilbert space $\overset{\circ}{W}{}^1_2(T_{u_o}(E))$. And since

$$d^2E(u_o)(\sigma,\sigma) \geq \|\sigma\|^2_{W^1_2} - C_1(u_o)\|\sigma\|^2$$

where $C_1(u_o)$ is a constant depending on u_o, the negative eigenspace of J_{u_o} must be finitely dimensional. The dimension of the negative eigenspace of J_{u_o} is called the Morse index of the harmonic map u_o, and is denoted by $\text{ind}(u_o)$.

u_o is called nondegenerate, if J_{u_o} is invertible.

For the self adjoint operator J_{u_o}, it is well known that we have a spectral decomposition E_λ and two projections P_+ and P_-, which correspond the positive and negative eigenspaces respectively. For any $\sigma \in C^{2,\gamma}_\psi(u_o^*TN)$, we have

$$\sigma_{\pm} := P_{\pm}\sigma \in C^{2,\gamma}_\psi(u_o^*TN) .$$

The two square roots

$$A_{\pm} := (P_{\pm}(\pm J_{u_o})P_{\pm})^{1/2}$$

are well defined, and we have that

$$\|A_{\pm}\sigma\|_{L^2} \text{ is equivalent to } \|\sigma_{\pm}\|_{\overset{\circ}{W}{}^1_2} .$$

In the following, we shall denote $\|A_{\pm}\sigma\|_{L^2}$ by $|\sigma_{\pm}|$, and let $|\sigma|^2 = |\sigma_+|^2+|\sigma_-|^2$.
Thus, the energy function is written as follow:

$$E(u) = c + \frac{1}{2}(|\sigma_+|^2-|\sigma_-|^2) + R(\sigma) .$$

For any given $0<\gamma<1$, we choose $\tau > 0$, satisfying

$$\frac{\tau}{1-\tau} < \sqrt{\frac{1-\gamma}{1+\gamma}} ,$$

and $\delta > 0$ such that for a W^2_p-ball B_δ with radius δ, centered at the zero section of $C^{2,\gamma}(u_o^*TN)$, we have,

(3.1) $$|R(\sigma)| < \frac{1}{2}\gamma\,|\sigma|^2$$

and

(3.2) $$|dR(\sigma)| < \tau\,|\sigma| .$$

$\forall\sigma \in U = B_\delta$. (In the following we always denote B_δ by U). These imply that

(3.3) $$\frac{1}{2}(1-\gamma)|\sigma_+|^2 - \frac{1}{2}(1+\gamma)|\sigma_-|^2 \le E(u) - c \le \frac{1}{2}(1+\gamma)|\sigma_+|^2 - \frac{1}{2}(1-\gamma)|\sigma_-|^2 .$$

Now we are going to construct a series of deformations, which deform the level set $E_{c+\varepsilon}$ (for suitable $\varepsilon > 0$) to $E_{c-\varepsilon}$ attached with cells:

(1) According to lemma 2.1 we have a strong deformation retract η_1, which deforms $E_{c+\varepsilon}$ into E_c, for $\varepsilon > 0$ small, if $E^{-1}(c,c+\varepsilon] \cap K = \emptyset$.

(2) By the conclusion (b), we have $\varepsilon > 0$ and a strong deformation retract η_2, which deforms E_c into $E_{c-\varepsilon} \cup (E_c \cap U)$, and satisfies $\eta_2(1,E_c \cap V) \subset E_c \cap U$, $\eta_2(1,E_c \setminus V) \subset E_{c-\varepsilon}$.

(3) Let us define two conical neighbourhoods:

$$C_\gamma = \{\sigma \in U \mid |\sigma_+| \le \sqrt{\frac{1-\gamma}{1+\gamma}}\, |\sigma_-|\},$$
$$\hat{C}_\gamma = \{\sigma \in U \mid |\sigma_+| \le \sqrt{\frac{1+\gamma}{1-\gamma}}\, |\sigma_-|\}.$$

The inequality (3.3) implies that

$$C_\gamma \subset E_c \cap U \subset \hat{C}_\gamma .$$

<u>Lemma 3.1</u>. There exists a strong deformation retract η_3, which deforms $E_{c-\varepsilon} \cup (E_c \cap U)$ into $E_{c-\varepsilon} \cup C_\gamma$.

<u>Proof</u>. Noticing that $\forall \sigma \notin E_{c-\varepsilon} \cup C_\gamma$, but $\sigma \in U$, we have

$$|\sigma_-| \le \sqrt{\frac{1+\gamma}{2}}\, \delta .$$

Let $K = \sqrt{\frac{2}{1+\gamma}} - 1$ (> 0), and define a flow on U as follows:

$$\eta(t,\sigma) = (1-t)\sigma_+ + (1+tK)\sigma_- .$$

We have

(a) $\eta(0,\sigma) = \sigma$.

(b) $\eta(1,\sigma) = \sqrt{\frac{2}{1+\gamma}}\, \sigma_- \in U$ if $\sigma \notin C_\gamma$

(c) Let $\phi(t) = E(\eta(t,\cdot))$, we have

$$\phi'(t) = -\frac{|\eta_+|^2}{1-t} - K|\eta_-|^2 + \langle dR(\eta(t,\cdot)), -\sigma_+ + K\sigma_-\rangle$$
$$\le (1-\tau)\left[-\frac{|\eta_+|^2}{1-t} - K|\eta_-|^2 + \frac{\tau}{1-\tau}\left(\frac{1}{1-t} + K\right)|\eta_+||\eta_-|\right]$$
$$= (1-\tau)\left[-\frac{|\eta_+|}{1-t}\left(|\eta_+| - \frac{\tau}{1-\tau}|\eta_-|\right) - K|\eta_-|\left(|\eta_-| - \frac{\tau}{1-\tau}|\eta_+|\right)\right]$$

where $\eta = (\eta_+, \eta_-)$. If $\eta \in (E_c \cap U) \setminus C_\gamma \subset \hat{C}_\gamma \setminus C_\gamma$, then we have

$$|\eta_+| \geq \sqrt{\frac{1-\gamma}{1+\gamma}}|\eta_-| \geq \frac{\tau}{1-\tau}|\eta_-| ,$$

and

$$|\eta_-| \geq \sqrt{\frac{1-\gamma}{1+\gamma}}|\eta_+| \geq \frac{\tau}{1-\tau}|\eta_+| .$$

It follows

$$\phi'(t) < 0 \qquad \forall \eta \in \hat{C}_\gamma \setminus C_\gamma . \tag{3.4}$$

Combining (a), (b) with (c), we obtain

$$\eta(t,\cdot) \in (E_c \cap U) ,$$

provided by the fact that $C_\gamma \subset E_c \cap U$.

From (a) and (b), we see that if $\sigma \notin E_{c-\varepsilon} \cup C_\gamma$, but $\sigma \in E_c \cap U$, then there is a unique $t^* \in (0,t)$ such that $\eta(t^*,\sigma) \in E^{-1}(c-\varepsilon) \cup \partial C_\gamma$. The uniqueness and the continuous dependence of t^* to σ are verified by the transversality: $\eta \overline{\pitchfork} E^{-1}(c-\varepsilon) \cup \partial C_\gamma$, which follows from the inequality (3.4).

Let us define

$$\eta_3(t,\sigma) = \begin{cases} \eta(t^*t,\sigma) & \text{if} \quad \sigma \in E_c \cap U \setminus C_\gamma \\ \sigma & \quad\quad \sigma \in E_{c-\varepsilon} \cup C_\gamma \end{cases}$$

This is the deformation we need. ∎

(4) Noticing that $\forall \sigma \in E_{c-\varepsilon} \cap C_\gamma$,

$$-\varepsilon \geq E(u) - c \geq \frac{1-\gamma}{2}|\sigma_+|^2 - \frac{1+\gamma}{2}|\sigma_-|^2 .$$

We have

$$|\sigma_-| > \sqrt{\frac{2\varepsilon}{1+\gamma}} \tag{3.5}$$

so $E_{c-\varepsilon} \cap C_\gamma \subset S := \{\sigma \in C_\gamma \,|\, |\sigma_-| > \sqrt{\frac{2\varepsilon}{1+\gamma}}\}$.

On the other hand $\forall \sigma \in S$,

$$|\sigma_-| \geq k_o|\sigma_+| + \delta_o ,$$

where

$$k_o = \frac{1}{2}\sqrt{\frac{1+\gamma}{1-\gamma}} , \text{ and } \quad \delta_o = \frac{1}{2}\sqrt{\frac{2\varepsilon}{1+\gamma}} .$$

Let us define

$$T_{k_o,\delta_o} = \{\sigma \in C_\gamma \mid |\sigma_-| \geq k_o|\sigma_+| + \delta_o\} .$$

In the following, we turn out to prove

<u>Lemma 3.2</u>. There is a strong deformation retract η_4 which deforms $E_{c-\varepsilon} \cup C_\gamma$ into $E_{c-\varepsilon} \cup T_{k_o,\delta_o} \cup \{\theta_+\} \times B^k_{\delta_o}$, where $k = \mathrm{ind}(u_o)$.

<u>Proof</u>. We define

$$\eta_4(t,\sigma) = \begin{cases} \sigma & \sigma \in E_{c-\varepsilon} \cup T_{k_o,\delta_o}, \\ \sigma_- + \left[1-t\left(1-\dfrac{|\sigma_-|-\delta_o}{k_o|\sigma_+|}\right)\right]\sigma_+ & \sigma \in C_\gamma, \delta_o \leq |\sigma_-| \leq k_o|\sigma_+| + \delta_o \\ \sigma_- + (1-t)\sigma_+ & \sigma \in C_\gamma \cap \{|\sigma_-| \leq \delta_o\} \end{cases} \blacksquare$$

(5) Choose $\varepsilon > 0$ so small, that

$$\varepsilon < \frac{\delta^2(1-\gamma)}{2} . \tag{3.6}$$

Define

$$0 < \mu < \sqrt{\frac{1-(\gamma+\frac{2\varepsilon}{\delta^2})}{1+(\gamma+\frac{2\varepsilon}{\delta^2})}} \tag{3.7}$$

we consider the energy function on the conical section of the sphere ∂B_δ : $S_\mu = \{\sigma \in \partial B_\delta \mid |\sigma_+| < \mu|\sigma_-|\}$. Let $\sigma \in S_\mu$, we have

$$\begin{aligned} E(u)-c &\leq \frac{1+\gamma}{2}|\sigma_+|^2 - \frac{1-\gamma}{2}|\sigma_-|^2 \\ &\leq \left(\frac{1+\gamma}{2}\mu^2 - \frac{1-\gamma}{2}\right)|\sigma_-|^2 \\ &= -\frac{1}{2}\left(\frac{1-\mu^2}{1+\mu^2} - \gamma\right)|\sigma_-|^2(1+\mu^2) . \end{aligned}$$

Since

$$\delta^2 = |\sigma_+|^2 + |\sigma_-|^2 \leq (1+\mu^2)|\sigma_-|^2 ,$$

and then

$$E(u)-c \leq -\frac{1}{2}\left(\frac{1-\mu^2}{1+\mu^2} - \gamma\right)\delta^2 < -\varepsilon ,$$

i.e. $S_\mu \subset \overset{c}{E}_{c-\varepsilon}$.

Lemma 3.3. The exit set of the flow

$$\eta(t,\sigma) = e^{-k_1 t}\sigma_+ + e^{k_2 t}\sigma_-$$

on the ball B_δ , is the set S_μ , where $\frac{k_2}{k_1} = \mu^2$, $k_1, k_2 > 0$.

Proof. The flow η remains on the plane generated by the two vectors σ_+ and σ_- . Suppose that η meets ∂B_δ at time t_o , and let $\eta_+ = e^{-k_1 t_o}\sigma_+$, $\eta_- = e^{k_2 t_o}\sigma_-$.

Choosing suitable coordinates $(\eta_+,\eta_-) = \delta(\cos\theta,\sin\theta)$ we assume that the flow η leaves the ball B_δ . By comparing the tangents of the ball with the tangents of the flow, we see

$$-\frac{k_2}{k_1}\,\mathrm{tg}\,\theta > -\mathrm{ctg}\,\theta ,$$

i.e.

$$|\eta_+| < \mu|\eta_-| .$$

In other words $(\eta_+,\eta_-) \in S_\mu$. ∎

Lemma 3.4. There is a strong deformation retract η_5 which deforms the set $E_{c-\varepsilon} \cup T_{k_o,\delta_o} \cup (\{\theta_+\} \times B^k_{\delta_o})$ into $E_{c-\varepsilon} \cup (\{\theta_+\} \times B^k_\delta)$.

Proof. We use the flow η defined in the lemma 3.3. Because $S_\mu \subset \mathring{E}_{c-\varepsilon}$, if $\sigma \notin E_{c-\varepsilon}$, then there must be a $t^* \in (0,\infty)$ such that $\eta(t^*,\sigma) \in E^{-1}(c-\varepsilon)$. On the other hand $\eta(t,\cdot)$ is transversal to the level set $E^{-1}(c-\varepsilon)$, provided by the fact:

$$\begin{aligned}\frac{d}{dt}E(\eta(t,\sigma)) &= -\langle\eta_+,k_1\eta_+\rangle - \langle\eta_-,k_2\eta_-\rangle + \langle dR(\eta),-k_1\eta_+ + k_2\eta_-\rangle \\ &\le -k_1|\eta_+|^2 - k_2|\eta_-|^2 + \tau(|\eta_+|+|\eta_-|)(k_1|\eta_+|+k_2|\eta_-|) \\ &= -(1-\tau)[k_1|\eta_+|^2 + k_2|\eta_-|^2 - \frac{\tau}{1-\tau}(k_1+k_2)|\eta_+||\eta_-|] \\ &= -(1-\tau)k_1[|\eta_+|^2 + \mu^2|\eta_-|^2 - \frac{\tau}{1-\tau}(1+\mu^2)|\eta_+||\eta_-|] \\ &< 0 ,\end{aligned}$$

if we choose

$$(3.8) \qquad \frac{\tau}{1-\tau} < \frac{2\mu}{1+\mu^2} .$$

Therefore $\forall \sigma \in T_{k_o,\delta_o} \setminus E_{c-\varepsilon}$, $t^* = t^*(\sigma)$ is uniquely determined, and is continuous. We define our deformation retract as follows:

$$\eta_5(t,\sigma) = \begin{cases} \sigma & \text{if } \sigma \in E_{c-\varepsilon} \\ \eta(t^*(\sigma)t,\sigma) & \text{if } \sigma \in T_{k_o\delta_o} \setminus E_{c-\varepsilon} \\ e^{k_2 t^*(\delta_o \sigma_-/|\sigma_-|)}\sigma_- & \text{if } \sigma \in \{\theta_+\} \times B^k_{\delta_o} . \end{cases} \quad \blacksquare$$

For any two strong deformation retract

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} Y$$

we define their composition as follows

$$\phi(t,x) = \begin{cases} \phi_1(2t,x) & t \in [0,\frac{1}{2}] \\ \phi_2(2t-1,\phi_1(1,x)) & t \in [\frac{1}{2},1] . \end{cases}$$

This is again a strong deformation retract $\phi : X_1 \to Y$, which is denoted by $\phi = \phi_2 \circ \phi_1$.

Now we come to our main conclusion in this section.

<u>Theorem 2</u>. Assume that $\pi_2(N) = 0$, and let $\mathcal{F}$ be a component of $C^{2,\gamma}_{\psi}(M^2,N)$. Suppose that on the level $E^{-1}(c) \cap \mathcal{F}$, $c < m_{\mathcal{F}} + b$, there are only nondegenerate harmonic maps $u_1,\ldots,u_\ell$, with Morse indices $m_1,\ldots,m_\ell$ respectively. Then the level set $E_{c-\varepsilon} \cap \mathcal{F}$ attached with ℓ handles, which dimensions correspond to these indices, is a strong deformation retract of $E_{c+\varepsilon} \cap \mathcal{F}$, for suitable $\varepsilon > 0$.

<u>Proof</u>. We choose $\gamma = \frac{1}{5}$, $\tau = \frac{1}{3}$, and $\mu = \frac{1}{2}$. And then we have $\delta > 0$ small enough such that (3.1) and (3.2) hold. Choose $\varepsilon > 0$ small enough such that $\varepsilon < \frac{\delta^2}{10}$ and that the conclusion (6) holds. The inequalities (3.6), (3.7) and (3.8) are satisfied automatically. The strong deformation retract now is defined to be

$$\rho = \rho_5 \circ \rho_4 \circ \rho_3 \circ \rho_2 \circ \rho_1 .$$

Combining lemmas 3.1, 3.2 with 3.4, we obtain our conclusion.

Theorem 2'. Let $c < m+b$. Suppose that on the level $E^{-1}(c)$ there are only nondegenerate harmonic maps with Morse indices. Then the same conclusions as in Theorem 2 holds.

Corollary. Suppose that u_o is a nondegenerate harmonic map, with $E(u_o) = c$ and $u_o \in C_\psi^{2,\gamma}(\bar{M},N)$, $\gamma > 0$. Assume that $c < m+b$ (or $c < m_{\mathcal{F}} + b$, if $u_o \in \mathcal{F}$ and $\pi_2(N) = 0$). Then we have

$$C_q(u_o;G) = \delta_{qk} G$$

where $k = \text{ind}\,(u_o)$.

Thus the Morse type number M_g^d is the number of harmonic maps with index q in E_d, $q = 0,1,2,\ldots$, if there are only nondegenerate harmonic maps in the level set E_d.

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