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Existence of Steady Vortex Rings in an Ideal Fluid

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Existence of Steady Vortex Rings in an Ideal Fluid

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Abstract. We prove the existence of global steady vortex rings in an ideal fluid with given propagation speed $W>0$, flux constant $k \geq 0$ and any bounded, positive, nondecreasing vorticity function.

1. INTRODUCTION AND THE MAIN RESULT.

Let $(r = \sqrt{x_1^2 + \ldots + x_4^2}\ , z = x_5)$ denote cylindrical coordinates in $\mathbb{R}^5$. h is the Heaviside function $h(s) = 0$ $(s \leq 0)$, $h(s) = 1$ $(s > 0)$. Following [11], [14], steady vortex rings in an ideal (that is, inviscid and incompressible) fluid can be obtained from solutions $u = u(r,z)$ of the problem

$$-\Delta u = \lambda g(r^2(u - \frac{W}{2}) - k), \quad u(r, z) \to 0 \ ((r, z) \to \infty), \tag{P}$$

with $W > 0$, $k \geq 0$ denoting propagation speed and flux of the vortex, coupling strength $\lambda > 0$, and with $g = hf$, for a given vorticity function f.

In theory, any non-negative $f \not\equiv 0$ can appear; in practice, the vorticity function is determined by how the vortex is created. Positive, non-decreasing functions seem to be of particular physical interest. Note that in this case, g is singular at 0.

THEOREM 1: *Suppose $f \geq 0$, f is not identically zero, non-decreasing and bounded. Then for any $\lambda > 0$, $W > 0$, $k \geq 0$ problem (P) admits a positive solution $u = u(r,z) \in H^{2,p}_{loc}(\mathbb{R}^5)$, $\forall p < \infty$, with $\nabla u \in L^2(\mathbb{R}^5)$, which is symmetric about $z = 0$ and non-increasing in $|z|$, giving rise to a vortex ring with non-empty, bounded core $A = supp(\Delta u)$.*

REMARKS: i) For $k = 0$, $\lambda = 1$, $f \equiv 1$ an explicit solution was obtained by Hill [10] ("Hill's spherical vortex"). Moreover, there are bifurcation results for small $k \geq 0$ ([5], [12]) and global existence results for superlinear vorticity functions with $f(0) = 0$, ([1], [11]) or $f(0) << 1$ ([7]). By a constrained minimization technique, Fraenkel-Berger [9] solved (P) for a broad class of monotone functions f; however, in their work the coupling constant λ arises as a Lagrange parameter which is left undetermined.
ii) Our approach extends to unbounded functions f satisfying suitable growth conditions at infinity. The case of bounded f appears to be the most difficult and we adhere to this case for ease of exposition.
iii) By the uniqueness result of [4], for $k = 0$, $\lambda = 1$, $f \equiv 1$ we re-obtain Hill's solution.

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2. APPROXIMATE SOLUTIONS.

We may normalize $\lambda = 1$, $W = 2$. For $R > 0$ let $B_R = B_R(0)$ and define

$$H(R) = \{u \in H_o^{1,2}(B_R);\ u = (r, |z|)\} \subset D^{1,2}(R^5)$$

where $D^{1,2}(R^5)$ denotes the completion of $C_o^\infty(R^5)$ $(\int = \int_{R^5})$

$$\|u\|^2 = \int |\nabla u|^2 dx.$$

By Rellich's theorem $H(R) \to L^2(B_R)$ compactly. We seek to approximate a solution $\bar{u}$ of (P) by solutions $u_R \in H(R)$ of

$$-\Delta u = g(r^2(u-1) - k) \text{ in } B_R\ ,\ u = 0 \text{ on } \partial B_R. \qquad (P_R)$$

Consider the related functional E on $H(R)$ given by

$$E(u) = \frac{1}{2}\int |\nabla u|^2 dx - \int \int_0^u g(r^2(v-1) - k)dv\, dx \ =: \frac{1}{2}\|u\|^2 - J(u).$$

Since g is bounded and monotone, J is uniformly Lipschitz continuous on $L^2(B_R)$ and convex with (set-valued) sub-differential

$$\partial J(u) = \{ v \in L^2(B_R)\ ;\ v \in \bar{g}(r^2(u-1) - k) \text{ a.e.}\},$$

$\bar{g}$ denoting the maximal monotone extension of g.
Hence E possesses a super-differential ∂E and there holds:

LEMMA 2: *If $u \in H(R)$ satisfies $0 \in \partial E(u)$, then $u \in H^{2,p}(B_R)$ for all $p < \infty$ and solves (P_R) almost everywhere.*

Since J is Lipschitz in L^2-norm and since $H(R) \to L^2(B_R)$ is compact, E is weakly lower semi-continuous and coercive on $H(R)$. Hence

LEMMA 3: $\forall R > 0\ \exists\, v_R \in H(R);\ E(v_R) = \min_{H(R)} E.$

However, choose $\varphi \in C_o^\infty(R^5)$ with $J(\varphi) > 0$ and for $R \geq 1$ let $\varphi_R(x) = \varphi(\frac{x}{R})$ Then $\|\varphi_R\|^2 = R^3\|\varphi\|^2$ while by monotonicity of g

$$J(\varphi_R) = \int \Big(\int_0^{\varphi_R(x)} g(r^2(v-1) - k)dv\Big)\, dx \ \geq \int \Big(\int_0^{\varphi(x/R)} g\,\big((\tfrac{r}{R})^2(v-1) - k\big)dv\Big)dx = R^5 J(\varphi),$$

and $\inf\limits_{H(R)} E \to -\infty\ (R \to \infty)$. Hence v_R cannot converge.

For suitable R_o fix $u_1 \in H(R_o)$; with $E(u_1) < 0$ and for $R \geq R_o$ let

$$\Gamma(R) = \{p \in C([0,1]; H(R));\ p(0) = 0,\ p(1) = u_1\}$$

$$\gamma(R) = \inf_{p \in \Gamma(R)} \sup_{u \in p} E(u).$$

Note that ∂J is uniformly bounded in L^2, hence compact in $H^{-1}(B_R)$ for any R. Thus E satisfies the Palais-Smale condition for Lipschitz maps (see [10]). Moreover since by Sobolev's embedding theorem

$$J(u) \leq \int_{\{u \geq 1\}} (\sup f)|u| \leq c \int |u|^{10/3} dx \leq c\|u\|^{10/3},$$

Chang's [10] version of the mountain pass lemma [2] may be applied to yield saddle-point-type solutions u_R of (P_R) for any $R \geq R_o$. Employing a device from [6] the solutions u_R can be obtained Steiner-symmetric, that is $u = u(r, |z|)$ and non-increasing in $|z|$. Finally, adapting an idea from [13], one can obtain a uniform a-priori estimate $\|u_{R_m}\| \leq c < \infty$ for a sequence $R_m \to \infty$ from the observation that $R \to \gamma(R)$ is monotone, whence γ is a.c. differentiable and $R_m \frac{d}{dR}\gamma(R_m) \to 0$ $(m \to \infty)$ for a suitable sequence $R_m \to \infty$. See [3] for details. Hence

LEMMA 4: *There exists a sequence $R_m \to \infty$ and constants $C, R^* > 0$ such that for any m there is a solution u_m of (P_{R_m}) with $E(u_m) = \gamma(R_m)$, $\|u_m\| \leq C$, $u_m = u_m(r, |z|)$ is non-increasing in $|z|$, and $\emptyset \neq supp(\Delta u_m) \subset B_{R^*}$ for all m.*

Observe that since $E(u_m) = \gamma(R_m) > 0$ we have $u_m \not\equiv 0$ whence $u_m > 0$ by the maximum principle.

PASSING TO THE LIMIT.

Since f and hence g is u[illegible]y bounded, from (P_R) we see that (u_m) is equicontinuous. Hence we may assume that $u_m \to u$ weakly in $D^{1,2}(\mathbb{R}^5)$ and locally uniformly. Passing to the limit in (P_R), u solves (P) with $supp(\Delta u) \subset \overline{B}_{R^*}$, $u = u(r, |z|)$ and is non-increasing in $|z|$. Moreover u cannot vanish identically; otherwise $u_m < 1$ on B_{R^*} for large m, whence $\Delta u_m = 0$ by (P_R) and then also $u_m \equiv 0$, which is impossible. Thus u is not identically zero, and hence $u > 0$ by the maximum principle.

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