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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE CENTRATOM TRIESTE

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Critical Exponents and Critical Dimensions for Polyharmonic Operators

James Serrin
University of Minnesota
School of Mathematics
Minneapolis, Minnesota
U.S.A.

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PATRIZIA PUCCI JAMES SERRIN

Introduction

In the well known work of Brezis and Nirenberg on positive solutions of semilinear elliptic equations involving critical exponents there occurs a remarkable phenomenon relating to the dimension $n = 3$ and the possibility of solving the Dirichlet problem in a bounded domain. The purpose of this paper is to show that the same phenomenon also appears in the study of higher order elliptic boundary value problems.

More specifically, for the representative problem

$$\text{(I)} \qquad \begin{aligned} -\Delta u &= \lambda u + u^s \quad \text{in } B \\ u &= 0 \quad \text{on } \partial B, \end{aligned}$$

where λ is a real constant, $s = (n+2)/(n-2)$ is the critical Sobolev exponent, and B is the unit ball in $\mathbb{R}^n$, $n > 2$, it was shown in [1] that the conditions for existence of a positive radial solution are surprisingly different when $n = 3$ and when $n \geq 4$. When $n \geq 4$, problem (I) has a positive solution if and only if $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$ in B with homogeneous Dirichlet boundary conditions. On the other hand, when $n = 3$ the range $(0, \lambda_1)$ is no longer correct, but must instead be replaced by the *smaller interval* $(\lambda_1/4, \lambda_1)$. The lower bound for the set of values λ for which a positive solution can exist is therefore *greater than zero* precisely in the case $n = 3$.

Here we treat various polyharmonic problems analogous to (I). For example, for the model biharmonic problem

$$\text{(II)} \qquad \begin{aligned} \Delta^2 u &= \lambda u + u^s \quad \text{in } B \subset \mathbb{R}^n \\ u &= Du = 0 \quad \text{on } \partial B, \end{aligned}$$

where $s = (n+4)/(n-4)$, $n > 4$, the conditions for existence of a positive radial solution in dimensions $n = 5, 6, 7$ correspond to the earlier condition for the case $n = 3$, while the situation when $n \geq 8$ corresponds to the Laplace case for $n \geq 4$. The appearance of more than one critical dimension for the biharmonic operator, rather than a single critical dimension as in the case of the Laplace operator, indicates a further remarkable and unexpected feature of the phenomenon discovered by Brezis and Nirenberg.

The demonstration of the lower bound $\lambda_1/4$ given by Brezis and Nirenberg relies crucially on an identity – (1.37) in their paper – having implicit roots in Noetherian theory. We shall adopt a similar outlook here, examining for polyharmonic operators the consequences of various analogous identities of variational type. These identities in turn are most easily obtained as special cases of a general result established by the authors in [5].

To introduce our results, we first state a refinement of a central conclusion of [5] for the general polyharmonic problem

$$\text{(III)}_\Omega \qquad \begin{aligned} (-\Delta)^K u &= f(u) \quad \text{in } \Omega \\ u = Du = \cdots &= D^{K-1} u = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Here Ω is a bounded star-shaped domain in $\mathbb{R}^n$, $n > 2K$, and

$$f(u) = \lambda u + g(u).$$

This refinement, given formally in Theorem 8 of Section 3, asserts that, if $ug(u) \geq 0$ and if furthermore

$$(\alpha + 1)\, G(u) \leq ug(u)$$

for same real $\alpha \geq s$, $s = (n+2K)/(n-2K)$, then a lower bound for λ for the existence of non-trivial solutions of (III)_Ω is

$$\text{(IV)} \qquad \lambda \geq \frac{2}{s+1}\,\frac{\alpha - s}{\alpha - 1}\,\lambda_K.$$

Here $G(u) = \int_0^u g(t)\,dt$ and λ_K is the first eigenvalue of the polyharmonic operator $(-\Delta)^K$ in Ω with Dirichlet boundary conditions. The existence condition (IV) reduces precisely to the typical relation $\lambda \geq 0$ when α takes the limiting value s; for greater values of α the right hand side is positive.

Should g be a pure power, say $g(u) = u|u|^{p-1}$, then naturally we take $\alpha = p$. This gives the condition $p \geq s$ for the validity of (IV) and in turn the existence condition

$$\lambda \geq \frac{2}{s+1}\,\frac{p - s}{p - 1}\,\lambda_K.$$

The number

$$s = \frac{n+2K}{n-2K}$$

is *critical*: it is exactly at this value of the exponent p that (IV) becomes valid. In alternate terms, it is exactly at this value of p that the variational integral for problem (III)_Ω loses compactness, as follows from the Sobolev-Kondrachov embedding theorem, that is $W_0^{K,2}(\Omega) \overset{\text{comp}}{\subset} L^{p+1}(\Omega)$ fails exactly when $p \geq s$.

Returning to the strictly radial case – when $\Omega = B$ – we have the following main result, formally stated in Theorem 5. Let u be a radially symmetric solution of problem (III)_B in the case $n = 2K+1$, and suppose that $ug(u) \geq 0$ and

$$(s+1)\, G(u) \leq ug(u).$$

Then the condition $\lambda \geq 0$ given by (IV) can be replaced by the stronger restriction

$$\lambda > (2K - \tfrac{1}{2})\,\lambda_{K-1}.$$

This result corresponds closely with the conclusion obtained by Brezis and Nirenberg for the Laplace case. Indeed, if we take $K = 1$, $n = 2K + 1 = 3$, and $s = 5$, and also $ug(u) = u^6 = 6\,G(u)$, then our necessary condition for existence becomes $\lambda > 3/2$, while they have $\lambda > \lambda_1/4 = \pi^2/4$. That our bound is not best possible is due to the complexity of treating the problem when $K \geq 2$.

We shall say that the dimension $n = 2K + 1$ is *critical*, in view of the fact that the natural existence condition $\lambda \geq 0$ can *in this dimension* be replaced by $\lambda >$ Pos. Const.

It is worth observing finally that if $g(u) > 0$ for $u > 0$, then *a necessary condition for the existence of non-negative, non-trivial solutions of problem* $(\mathrm{III})_B$ *is that*

$$\lambda < \lambda_K.$$

This follows, exactly as in the Laplace case, from the positivity of the first eigenfunction of $(-\Delta)^K$ in B, the latter result having been established by P.L. Lions in a recent paper [4].

Since the eigenvalues λ_K are not generally tabulated, it is worth expressing the main estimate above in more accessible terms. In this respect the following eigenvalue inequality, obtained in [6], is useful:

$$\lambda_K \geq \begin{cases} \lambda_1(\lambda_1\mu_2)^m & K = 2m+1 \quad \text{odd} \\ (\lambda_1\mu_2)^m & K = 2m \quad \text{even,} \end{cases}$$

where λ_1 and μ_2 are respectively the first eigenvalue of the Laplace operator in B with Dirichlet boundary conditions and the first non-zero eigenvalue for the radial Laplace operator in B with Neumann boundary conditions. Since $\lambda_1 = (j_{(n-2)/2})^2$ and $\mu_2 = (j_{n/2})^2$, it is clear that the eigenvalues λ_K increase rapidly with the order of the equation and with the dimension n. For example, when $K = 3$ and $n = 2K + 1 = 7$ we already have $\lambda_K \geq 1622$.

For its intrinsic curiosity, one may also consider the case $K = 5$. Here the critical dimension is $n = 2K + 1 = 11$ and the critical exponent is $s = 21$. This yields the equation

$$\Delta^5 u + \lambda u + u^{21} = 0$$

and the necessary condition for the existence of a non-trivial radial solution of the Dirichlet problem in the unit ball in $\mathbf{R}^{11}$,

$$\lambda \geq 9.5\,\lambda_4 \geq 9.5\,(j_{9/2})^4 (j_{11/2})^4 \approx 326 \cdot 10^6,$$

a very large number indeed.

When $K = 2$ the results are more extensive, as we have noted above. Our most general conclusion in this direction is Theorem 3′, where we show for the biharmonic operator that the dimensions $n = 5, 6, 7$ are critical, or, considering n as a real parameter, the values $4 < n < 8$. In particular, if $ug(u) \geq 0$ and

$$(s+1)\,G(u) \leq ug(u),$$

then a lower bound for λ for the existence of non-trivial radial solutions of problem $(\mathrm{III})_B$ is

$$\lambda > \frac{n}{4}(n^2 - 16)(8 - n), \quad 4 < n < 8.$$

It seems almost certain that as the order K increases there should be increasingly large intervals of critical dimensions. The computational difficulties in verifying this appear extreme. Nevertheless, with the results for $K = 2$ in mind, and taking into account the variational techniques introduced by Brezis and Nirenberg in [1], we conjecture that for any $K \geq 1$ the critical dimensions n are *precisely* those in the range

$$2K < n < 4K,$$

while for $n \geq 4K$ problem $(\mathrm{III})_B$ – at least in the case $g(u) = u|u|^{s-1}$ – has a positive radial solution if and only if $\lambda \in (0, \lambda_K)$. Recently we have learned from Edmunds, Fortunato and Jannelli [2] that our conjecture is valid for $K = 2$; that is, for $n \geq 8$ positive radial solutions corresponding to the case $g(u) = u|u|^{s-1}$ exist for all $\lambda \in (0, \lambda_K)$.

The outline of the paper is as follows. In Section 1 we consider the simplest model case for the polyharmonic problem $(\mathrm{III})_B$, namely

$$f(u) = \lambda u + u|u|^{s-1}, \quad s = \frac{n+2K}{n-2K}.$$

We begin by reformulating, for the model case of polyharmonic operators, the general identity given for higher order variational integrals on pages 700-701 of [5]. The particular results of Section 1, as well as those of later sections, then follow by suitably specializing this polyharmonic identity to radially symmetric functions of various types.

In Section 2 we treat the case of the biharmonic operator, both in the model case $f(u) = \lambda u + u|u|^{s-1}$ and in the more general algebraic case

$$f(u) = \lambda u|u|^{q-1} + u|u|^{s-1}, \qquad 1 \leq q < s.$$

It can be expected that even for $q > 1$ there is some range of values for which each of the dimensions $n = 5, 6, 7$ remains critical, particularly since in [1], where $n = 3$ and $s = 5$, it was shown that the dimension 3 stays critical for all $q \in [1, 3]$. This expectation is indeed correct, as shown in Theorem 4, with the following ranges:

$$\begin{aligned} n &= 5, & s &= 9, & q &\in [1, 5.77] \\ n &= 6, & s &= 5, & q &\in [1, 3) \\ n &= 7, & s &= 11/3, & q &\in [1, 5/3); \end{aligned}$$

we observe that the upper limiting value for q is exactly $s - 2$ when $n \geq 6$.

Finally in Section 3 we turn to the general situation when

$$f(u) = \lambda u + g(u)$$

or when

$$f(u) = \lambda u|u|^{q-1} + g(u), \qquad q \geq 1.$$

The computations in the latter cases are direct extensions of those in the earlier parts of the paper; they have been deferred to the end in order not to obscure the main ideas presented in Sections 1 and 2. We remark that the case of general functions $g(u)$ does not seem to have been treated previously, even in the case $K = 1$ of the Laplace operator.