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# Continuation and Limit Properties for Solutions of Strongly Nonlinear Second Order Differential Equations

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## Continuation and limit properties for solutions of strongly nonlinear second order differential equations

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### §1. Introduction.

The purpose of this paper is to study continuation, boundedness, and asymptotic behavior properties for solutions of nonlinear ordinary differential equations of second order, in which the nonlinearity occurs both in the solution variable and its derivative. The typical equation we consider has the form

$$(A(u')u')' + \delta(r)\,A(u')u' + f(r,u) = 0, \tag{1.1}$$

where the solution $u = u(r)$ is defined on some interval $I$ of the real line, the function $A = A(p)$ determines the nonlinearity of the equation in the derivative $u'$, and $f = f(r,u)$ is a continuous function containing the nonlinearity in $u$. Equation (1.1) can be also written in the equivalent form

$$(g(r)\,A(u')u')' + g(r)\,f(r,u) = 0, \tag{1.1$'$}$$

where $g(r) = \exp\int_{r_0}^{r} \delta(s)ds$.

An immediate and elementary example of (1.1) is the Bessel equation

$$u'' + \frac{1}{r}u' + \left(1 - \frac{\mu^2}{r^2}\right)u = 0, \tag{1.2}$$

in which $A(p) \equiv 1$ and $f(r,u)$ is linear in $u$. More generally, (1.1) includes the Lane-Emden equation of astrophysics

$$u'' + \frac{2}{r}u' + u^{q-1} = 0, \qquad u > 0, \qquad q > 2,$$

and various generalizations of this equation for composite stellar densities; the Emden-Fowler equation, either in the form given by Fowler

$$u'' + r^{\nu}\,u|u|^{q-2} = 0, \qquad \nu \in \mathbb{R}, \qquad q \geq 2,$$

or its forerunner

$$u'' + \frac{n-1}{r}u' + u|u|^{q-2} = 0, \qquad n \neq 2, \quad q \geq 2;$$

and the Haraux-Weissler equation [6]

$$u'' + \left(\frac{n-1}{r} + \frac{r}{2}\right)u' + \frac{\ell}{2}u + u|u|^{q-2} = 0, \qquad n > 1, \qquad \ell > 0, \qquad q \geq 2.$$

Another interesting example occurs when $A(p) = |p|^{m-2}$, $m > 1$, for which (1.1) has the form

$$(1.3) \qquad \left(|u'|^{m-2}u'\right)' + \delta(r)\,|u'|^{m-2}u' + f(r,u) = 0.$$

In this case the defining function $A(p)$ is usually called the degenerate Laplace operator, or the "m-Laplacian". This equation of course has a strongly singular behavior when $u' = 0$.

Equation (1.1), when $A(p)$ depends only on $|p|$ and $\delta(r) = (n-1)/r$, can also be considered as the *radial version* of the partial differential equation in $\mathbf{R}^n$

$$\operatorname{div}\{A(|Du|)\,Du\} + f(r,u) = 0,$$

an equation whose study has been initiated in [5] and in [9]. An important case occurs here when $A$ is the mean curvature operator, that is $A(p) = 1/\sqrt{1+|p|^2}$.

From a more abstract point of view, perhaps the most fruitful way to visualize (1.1) is as the Euler-Lagrange equation for extremals of the variational problem

$$(1.4) \qquad \delta \int_J g(r)\left[G(u') - F(r,u)\right]dr = 0.$$

Here $J$ is an open interval of the real line, $g = g(r)$ is a *positive, differentiable function* on $J$ with

$$g(r) = \exp \int_{r_0}^{r} \delta(s)ds, \quad r_0 \in J, \qquad \delta(r) = \frac{g'(r)}{g(r)},$$

the function $G = G(p)$ is related to $A$ by

$$G(p) = \int_0^p tA(t)dt, \qquad \text{and} \qquad F(r,u) = \int_0^u f(r,t)dt.$$

That (1.1) or (1.4) be *regular* is expressed by the condition that $p\,A(p)$ be a strictly increasing continuous function on $\mathbf{R}$ which vanishes at $p = 0$, or equivalently that $G$ be a strictly convex, continuously differentiable function on $\mathbf{R}$ with $dG/dp = 0$ at $p = 0$. The strict convexity of $G$ is itself equivalent to the strong Weierstrass condition for the integrand in (1.4).

In the main part of the paper we shall frequently frame our hypotheses in terms of the variational problem (1.4); the reader may easily interpret them directly for the differential equation (1.1). We shall suppose always that (1.1) and (1.4) are regular.

The principal conclusions of the paper are that under quite weak conditions on the functions $F$ and $G$ any extremal of (1.4) can be extended over the full interval $J$, and that under slightly stronger hypotheses on $g$ and $F$, including the condition $\delta(r) \geq 0$, both $u$ and $u'$ are bounded as $r \to \infty$ when $J = \mathbf{R}^+$. Finally, again in the case $J = \mathbf{R}^+$, if $F(r,u)$ satisfies the further principal condition $u\,f(r,u) > 0$ for $u \neq 0$, then, for any solution $u$ which is bounded as $r \to \infty$, both $u$ and $u'$ tend to zero as $r \to \infty$. (For results in case the principal condition fails, cf. for example [5].)



The conclusions can be illustrated most simply for the special case $A(p) = |p|^{m-2}$, $m > 1$, with the function $F$ independent of $r$, that is, $F = F(u)$. Then equation (1.3) becomes

$$(1.5) \qquad \left(|u'|^{m-2}u'\right)' + \delta(r)\,|u'|^{m-2}u' + f(u) = 0$$

and $G(p) = |p|^m/m$. By Theorem 3, if $F(u) \geq -M$ then all solutions of (1.5) can be extended to the full interval $J$ in which $g(r) > 0$. By Theorem 4, if $J = \mathbf{R}^+$ and $\delta(r) \geq 0$ for $r$ suitably large, and if furthermore

$$F(u) = \int_0^u f(t)dt \to \infty$$

when $u \to \pm\infty$, then *all solutions of* (1.5) *are bounded as* $r \to \infty$. From Theorem 5, if there are positive constants $\beta$, $\beta'$ such that

$$(1.6) \qquad \beta \leq r\,\delta(r) \leq \beta' r^m \qquad \text{as } r \to \infty,$$

and if

$$u\,f(u) > 0 \qquad \text{for } u \neq 0,$$

then *all solutions u which are bounded as* $r \to \infty$ *satisfy*

$$(1.7) \qquad u(r),\ u'(r) \to 0 \qquad \text{as } r \to \infty.$$

Simple examples show that this conclusion may fail if the upper bound $\beta' r^m$ in (1.6) is replaced by $\beta' r^{m+\epsilon}$ with $\epsilon > 0$. As a case in point, the function

$$u(r) = \exp\left(\frac{m-1}{\epsilon\, r^{\epsilon/(m-1)}}\right)$$

is a bounded solution on $[1,\infty)$ of equation (1.5), with

$$\delta(r) = r^{m-1+\epsilon}\left(1 + \frac{m-1+\epsilon}{r^{m+\epsilon}} + \frac{m-1}{r^{m+\epsilon m/(m-1)}}\right), \qquad f(u) = u\,|u|^{m-2},$$

yet $u(r) \to 1$ as $r \to \infty$ (see also Artstein and Infante [1]). Similarly, the lower bound in (1.6) is also essentially best possible; it cannot for example be weakened even to $\delta \in L^1[1,\infty)$, as is shown by an example of Levin and Nohel ([7], Remark 2.1).

Theorem 5 is closely related to Theorem 1 of Levin and Nohel and to the results of Artstein and Infante, and indeed provides a considerable generalization of this work, see the remarks and Corollary following Theorem 5 in Section 5.

## §2. Preliminary results.

Thoughout the paper we consider extremals of the variational problem

$$\delta \int_J g(r)[G(u') - F(r,u)]\,dr = 0, \tag{2.1}$$

where $J$ is an open interval of R, say $J = (a_0, a_1)$ with $a_0 = -\infty$ and $a_1 = +\infty$ allowed. We suppose always, and without further comment, that

(a) $g \in C^1(J)$, $\quad g(r) > 0$ *for all* $r \in J$;

(b) $G \in C^1(\mathbf{R})$, $\quad G(p) > 0$ *for all* $p \neq 0$, $G(0) = 0$, *and* $G$ *strictly convex in* R;

(c) $F \in C^1(J \times \mathbf{R})$, $\quad F(r,0) = 0$ *for all* $r \in J$.

The first parts of assumptions (a), (b), (c) are just the condition that the *integrand* $\mathcal{F}(r,u,p)$ in (2.1) be of class $C^1$ in $J \times \mathbf{R} \times \mathbf{R}$.

The fact that $G$ is strictly convex shows that the associated variational problem is weakly elliptic. Clearly the derivative $G_p$ of $G$ is strictly increasing in R, so that $s = G_p(p)$ is invertible; we denote its inverse function by $p = G_p^{-1}(s)$. Of course $G_p(0) = 0$.

Now let $H = H(p)$ be the Legendre transform of $G = G(p)$, namely the continuous function defined by

$$H(p) = pG_p(p) - G(p) \qquad \text{for every } p \in \mathbf{R}. \tag{2.2}$$

**Lemma 1.** *We have*

$$H(p) = \int_0^{G_p(p)} G_p^{-1}(s)\,ds \qquad \text{for all } p \in \mathbf{R}. \tag{2.3}$$

*In turn,* $H(p) > 0$ *for* $p \neq 0$ *and* $(\operatorname{sign} p)H(p)$ *is a strictly increasing function in* R. *Moreover* $H(p)/p \to 0$ *as* $p \to 0$.

**Proof:** By Riemann-Stieltjes integration

$$H(p) = pG_p(p) - \int_0^p G_p(t)\,dt = \int_0^p t\,dG_p(t) = \int_0^{G_p(p)} G_p^{-1}(s)\,ds.$$

**Remark.** If $H$, the *energy* function of the problem, is considered to be the fundamental element of the theory, rather than $G$, then we must assume that $H$ is continuous, that $(\operatorname{sign} p)H(p)$ is strictly increasing in R, that $H(p)/p \to 0$ as $p \to 0$, and that the integral

$$\int_0^p \frac{H(t)}{t^2}\,dt$$



is convergent. In this case $G(p) = p\int_0^p \frac{H(t)}{t^2}\,dt$, as follows directly from (2.2).

**Lemma 2.** *For every* $(u,p) \in \mathbf{R}^2$ *we have*

$$G(p) \le pG_p(p), \qquad H(p) \le pG_p(p), \qquad |uG_p(p)| \le G(\sigma u) + H(p), \tag{2.4}$$

*where* $\sigma = \operatorname{sign}(pu)$.

**Proof:** The first two inequalities follow at once from assumption (b) and Lemma 1. To prove the third one we first assume that $u, p \ge 0$. Then by Young's inequality and Lemma 1 we find that

$$u\,G_p(p) \le \int_0^u G_p(t)\,dt + \int_0^{G_p(p)} G_p^{-1}(t)\,dt = G(u) + H(p),$$

taking (2.3) into account. The alternative case $u, p \le 0$ can be treated similarly; the remaining cases are easily derived from the two already obtained.

We now turn to the variational equation associated with problem (2.1). Define

$$f(r,u) = F_u(r,u) \qquad \text{for every } (r,u) \in J \times \mathbf{R},$$

and

$$A(p) = \frac{1}{p}G_p(p) \qquad \text{for all } p \neq 0.$$

Clearly $p\,A(p)$ can be extended by continuity to be zero at $p = 0$. Also of course $A(p) > 0$ for all $p \neq 0$.

An *extremal* for (2.1) is by definition a $C^1$ function $u$ defined on some subinterval $I$ of $J$, such that

$$G_p(u'(r)) = A(u'(r))u'(r) \in C^1(I) \tag{2.5}$$

and for which the corresponding (weak) Euler-Lagrange equation is satisfied on $I$, namely

$$(gA(u')u')' + g\,f(r,u) = 0, \tag{2.6}$$

or equivalently

$$g(r)A(u'(r))u'(r) + \int_{r_0}^r g(t)f(t,u(t))\,dt = \text{Constant},$$

where $r_0$ is any fixed point of $I$. In the special case when $g(r) = r^{n-1}$, $n > 1$, and $G(p) = \frac{1}{2}p^2$, we have $A(p) = 1$ and (2.6) reduces to the radial Laplace equation

$$u'' + \frac{n-1}{r}u' + f(r,u) = 0,$$

in which $n$ can be thought of as a real parameter.

As noted in the introduction, if the function $A = A(p)$ in (2.6) is considered to be the fundamental element of the theory, rather than $G$, then we have to *define*

$$G(p) = \int_0^p t\,A(t)dt \qquad \text{for every } p \in \mathbf{R}.$$

In this case (b) is equivalent to the assumption that $p\,A(p)$ be continuous and strictly increasing in $\mathbf{R}$.

Rather surprisingly, in spite of the fact that neither $u'(r)$ nor $H(p)$ need be separately differentiable, the composite function $H(u'(r))$ is differentiable in $I$ along an extremal of (2.1). Indeed by (2.3), (2.5) and (2.6) we have for every $r \in I$

$$\begin{aligned}(H(u'(r)))' &= u'(r)\frac{d}{dr}G_p(u'(r)) = u'(r)\left[-\frac{g'(r)}{g(r)}G_p(u'(r)) - f(r,u(r))\right]\\ &= -\frac{g'(r)}{g(r)}A(u'(r))u'^2(r) - u'(r)f(r,u(r)).\end{aligned}$$

Therefore, along an extremal $u$ of (2.1) in $I$ we have the identity

$$(H(u'(r)) + F(r,u(r)))' = -\frac{g'(r)}{g(r)}A(u'(r))u'^2(r) + F_r(r,u(r)).$$

For simplicity in what follows we shall frequently use the common notation of ordinary differential equations, where $u = u(r)$ and $u' = u'(r)$ denote the solution and its derivative. Thus the above formula can be written simply as

$$(H(u') + F(r,u))' = -\frac{g'}{g}A(u')u'^2 + F_r(r,u). \tag{2.7}$$

Since along an extremal $u$ of (2.1) the function $H(u')$ is differentiable with respect to $r$, the main identity of Proposition 1 of [10] holds even for $C^1$ extremals of (2.1), yielding the following useful result.

*Let $u$ be an extremal of problem (2.1) and let $a$, $h$ be scalar functions of class $C^1(I)$. Then the following identity holds in $I$,*

$$\begin{aligned}&\left\{gh\left[H(u') + F(r,u) + \frac{a}{h}A(u')uu'\right]\right\}'\\ &= g\left[\left(h' + h\frac{g'}{g}\right)F(r,u) + h\,F_r(r,u) - a\,u\,f(r,u)\right.\\ &\quad \left. - \left(h' + h\frac{g'}{g}\right)G(u') + (a + h')A(u')u'^2 + a'A(u')uu'\right].\end{aligned} \tag{2.8}$$



*In the particular case in which $a(r) = \alpha\,r^{k-1}/g(r)$ and $h(r) = r^k/g(r)$, where $\alpha$ and $k$ are constants, the identity* (2.8) *reduces to*

$$\begin{aligned}&\left\{r^k\left[H(u') + F(r,u) + \frac{\alpha}{r}A(u')uu'\right]\right\}'\\ &= r^{k-1}\left[k\,F(r,u) + r\,F_r(r,u) - \alpha\,u\,f(r,u)\right.\\ &\quad \left. - k\,G(u') + \left(\alpha + k - r\frac{g'}{g}\right)A(u')u'^2 + \frac{\alpha}{r}\left(k - 1 - r\frac{g'}{g}\right)A(u')uu'\right].\end{aligned} \tag{2.9}$$

*The identity* (2.7) *is the special case of* (2.9) *where $k = \alpha = 0$.*

The question of *existence* of extremals of (2.1) has been avoided in the above discussion, since it is not in itself relevant to the conclusions of the paper. It is nevertheless worth noting that this question is settled by the results of [9], where it is (in essence) shown that the initial value problem

$$u(r_1) = u_1, \qquad u'(r_1) = u_1'$$

for (2.6) always has a solution in some neighborhood of $r_1$. An outline of the proof is indicated in Section 3 below, as part of the demonstration of Theorem 1.

## §3. Continuation properties for extremals.

In this section we shall treat nonlinearities $F$ which satisfy growth conditions of the following type:

(i) *there is a sequence $(r_j)$ with $r_j \nearrow a_1$ such that $F(r_j,\cdot)$ is bounded below for every $j$;*

(ii) *there is a nonnegative function $\psi \in L^1_{loc}(J)$ such that $F_r(r,u) \le (1 + |F(r,u)|)\psi(r)$ a.e. in $J$ and for all $u \in \mathbf{R}$;*

(i)′ *there is a sequence $(\rho_j)$ with $\rho_j \searrow a_0$ such that $F(\rho_j,\cdot)$ is bounded below for all $j$;*

(ii)′ *there is a nonnegative function $\psi \in L^1_{loc}(J)$ such that $F_r(r,u) \ge -(1 + |F(r,u)|)\psi(r)$ a.e. in $J$ and for all $u \in \mathbf{R}$.*

In the important special case when $f(r,u) = \rho(r)\phi(u)$ condition (c) of Section 2 is satisfied when $\rho \in C^1(J)$ and $\phi \in C(\mathbf{R})$, and the hypotheses (i)–(ii)′ when $\rho(r)$ is positive and $\Phi(u) = \int_0^u \phi(t)dt$ is bounded below in $\mathbf{R}$. Indeed since $F(r,u) = \rho(r)\Phi(u)$ it is enough to take $\psi(r) = \rho'(r)/\rho(r)$.

The following preliminary result will be useful throughout the section.

LEMMA. *Suppose* (i) *and* (ii) *hold. Then the function $F$ is bounded below in $I \times \mathbf{R}$, for any interval $I$ whose closure is in $J$.*

PROOF: Let $I = [b_0, b_1]$ be a compact subinterval of $J$. Consider any fixed value $u = u_0$ and any fixed $\bar{r}$ in $I$. We shall show that there exists a real number $M$, depending only on $I$, such that $F(\bar{r}, u_0) \ge M$, which will complete the proof.

If $F(\bar{r}, u_0) \geq 0$ then $M = 0$ satisfies the claim. Hence suppose $F(\bar{r}, u_0) < 0$. Let $b = \inf_{r_j > b_1} r_j$. We consider two cases:

$$(3.1) \qquad F(r, u_0) < 0 \quad \text{for all } r \in [\bar{r}, b),$$

$$(3.2) \qquad \text{there is a point } \bar{b} \in [\bar{r}, b) \text{ such that } F(r, u_0) < 0 \text{ for } r \in [\bar{r}, \bar{b}),\ F(\bar{b}, u_0) = 0.$$

Obviously one of these must occur.

Consider case (3.1). Then for $r \in [\bar{r}, b]$ we have by (ii)

$$F_r(r, u_0) \leq (1 - F(r, u_0))\psi(r).$$

Integrating from $\bar{r}$ to $b$, we get

$$F(\bar{r}, u_0) \geq 1 - (1 - F(b, u_0)) \exp \int_{\bar{r}}^{b} \psi(s)ds$$
$$\geq 1 - (1 - F(b, u_0)) \exp \int_{b_0}^{b} \psi(s)ds.$$

Clearly $b$ depends only on the interval $I$, the value $F(b, u_0)$ is bounded below by a number independent of $u_0$ by virtue of (i), and the exponential is bounded since $\psi \in L^1_{loc}(J)$. Hence $F(\bar{r}, u_0)$ is bounded below by a number depending only on $I$.

Next consider case (3.2). By the same procedure, integrating from $\bar{r}$ to $\bar{b}$ we obtain

$$F(\bar{r}, u_0) \geq 1 - \exp \int_{\bar{r}}^{\bar{b}} \psi(s)ds \geq 1 - \exp \int_{b_0}^{b} \psi(s)ds,$$

a lower bound depending only on $I$. This completes the proof.

The conclusion of the Lemma continues to hold when (i), (ii) are replaced by (i)′, (ii)′, the argument requiring only obvious changes.

Our first result gives a simple continuation property of solutions to the right endpoint $a_1$ of $J$. We recall that the functions $g$, $G$, $F$ always satisfy conditions (a), (b), (c) of Section 2.

**Theorem 1.** *Assume that $g$ is an increasing function in $J$, that is $\delta(r) \geq 0$. Suppose also that conditions* (i) *and* (ii) *hold and that the Legendre transform of $G$ satisfies*

$$(3.3) \qquad H(p) \to \infty \quad \text{as } |p| \to \infty.$$

*Then any extremal of problem* (2.1) *can be continued to the right endpoint of $J$.*

**Proof:** Let $u$ be an extremal of problem (2.1) in some domain $I$ and denote by $r_0$ any fixed point of $I$. By (2.7), for any $r \in I$ we have

$$(3.4) \qquad (H(u') + F(r, u))' = -\frac{g'}{g} A(u')u'^2 + F_r(r, u).$$



Thanks to assumptions (a), (ii) and the fact that $g$ is increasing, from (3.4) we see that

$$(3.5) \qquad (H(u') + F(r, u))' \leq F_r(r, u) \leq (1 + |F(r, u)|)\psi(r) \quad \text{a.e. in } I.$$

Now, assume for contradiction that the extremal $u$ cannot be continued beyond some point $r_1$ in $J$, $r_0 < r_1 < a_1$.

Integrating (3.5) on $[r_0, r]$, with $r_0 < r < r_1$, we have

$$(3.6) \qquad H(u') + F(r, u) \leq \int_{r_0}^{r} (1 + |F(s, u(s))|)\psi(s)ds + c_0,$$

where $c_0 = H(u'(r_0)) + F(r_0, u(r_0))$.

We now assert that there is a positive number $k$ such that

$$(3.7) \qquad |F(r, u(r))| \leq k \quad \text{for each } r \in [r_0, r_1).$$

Indeed, from (3.6) and the condition $\psi \in L^1_{loc}(J)$, it follows that for every $r \in [r_0, r_1)$

$$|F(r, u)| \leq \int_{r_0}^{r} |F(s, u(s))|\psi(s)ds + c_1,$$

where $c_1 = c_0 + \int_{r_0}^{r_1} \psi(s)ds - 2\min\{0, M\}$ and $M$ is a lower bound for $F$ in $[r_0, r_1] \times \mathbb{R}$. It is now immediate from Gronwall's inequality that (3.7) holds with $k = c_1 \exp \int_{r_0}^{r_1} \psi(s)ds$. From (3.6) and (3.7) also $H(u')$ is bounded in $[r_0, r_1)$. Hence by Lemma 1 and (3.3) there is a constant $P > 0$ such that

$$|u'(r)| \leq P \quad \text{for all } r \in [r_0, r_1).$$

Consequently $u$ is Lipschitz continuous in $[r_0, r_1)$ and so $u(r) \to u_1$ as $r \nearrow r_1$.

Now, from (3.5) and (3.7) we also have

$$(H(u') + F(r, u))' \leq (1 + k)\psi(r) \quad \text{a.e. in } (r_0, r_1).$$

Therefore, by assumption (ii),

$$H(u') + F(r, u) - (1 + k)\int_{r_0}^{r} \psi(s)ds$$

is decreasing in $[r_0, r_1)$ with

$$F(r, u(r)) \to F(r_1, u_1) \quad \text{and} \quad \int_{r_0}^{r} \psi(s)ds \to \int_{r_0}^{r_1} \psi(s)ds \quad \text{as } r \nearrow r_1.$$

Thus the function $H(u'(r))$ also tends to a finite limit as $r \nearrow r_1$, and by Lemma 1 and (3.3) this means that

$$u'(r) \to u'_1 \quad \text{as } r \nearrow r_1,$$

for some finite number $u_1'$.

We now use the existence theorem given in [9], Proposition 1, to show that the Cauchy problem

$$\begin{cases} (g\,G_p(u'))' + g\,f(r,u) = 0 \\ u(r_1) = u_1, \qquad u'(r_1) = u_1' \end{cases}$$

admits a $C^1$ solution $\bar{u}$ in some interval $[r_1, R]$, $r_1 < R < a_1$. To do this requires only that we consider the operator

$$T[u](r) = u_1 + \int_{r_1}^{r} G_p^{-1}\left(-\int_{r_1}^{t} \frac{g(s)}{g(t)} f(s,u(s))ds + \frac{g(r_1)}{g(t)} G_p(u_1')\right) dt \tag{3.8}$$

defined on the set

$$C = \{u \in C[r_1, R] \,:\, \|u(\cdot) - u_1\|_\infty \le 1\},$$

where $C[r_1, R]$ denotes the space of continuous functions on $[r_1, R]$ endowed with the uniform norm $\|\cdot\|_\infty$. As in [9], the operator $T$ is compact and continuous on $C$, with $T(C) \subset C$, provided that $R - r_1$ is sufficiently small, and the existence of $\bar{u}$ is proved.

It now follows that $u$ on $[r_0, r_1)$ can be continued as an extremal to $[r_0, R]$ by putting

$$u(r) \equiv \begin{cases} u(r), & r_0 \le r < r_1 \\ \bar{u}(r), & r_1 \le r \le R. \end{cases}$$

This contradiction completes the proof.

In the next theorem we establish continuation of extremals without the assumption that $g$ is increasing.

THEOREM 2. *Suppose that the conditions* (i) *and* (ii) *hold. Assume also that there is a positive number $\vartheta$ such that*

$$G(p) \le \vartheta\, H(p) \qquad \textit{for all } p \in \mathbb{R}. \tag{3.9}$$

*Then any extremal of problem* (2.1) *can be continued to the right endpoint of $J$.*

PROOF: Let $u$ be an extremal of problem (2.1) and let $r_0$ be a point of its domain $I$. Assume for contradiction that $u$ cannot be continued beyond some point $r_1$, $r_0 < r_1 < a_1$. Integrating (3.4) on $[r_0, r]$, with $r_0 < r < r_1$, and setting $c_0 = H(u'(r_0)) + F(r_0, u(r_0))$ as before, we obtain

$$H(u') + F(r,u) = -\int_{r_0}^{r} \frac{g'}{g} A(u') u'^2 ds + \int_{r_0}^{r} F_r(s,u) ds + c_0.$$

By (2.2) and (3.9) we have $p^2 A(p) \le (1+\vartheta) H(p)$ for every $p \in \mathbb{R}$. Let $c$ be such that $|g'(r)/g(r)| \le c$ for $r \in [r_0, r_1]$. Then, using assumptions (i) and (ii), we see that for every $r \in [r_0, r_1)$

$$\begin{aligned} H(u') + |F(r,u)| &\le c(1+\vartheta)\int_{r_0}^{r} H(u')ds + \int_{r_0}^{r} |F(s,u)|\psi(s)ds + c_1 \\ &\le \int_{r_0}^{r} (H(u') + |F(s,u)|)\varphi(s)ds + c_1, \end{aligned}$$



where $c_1 = c_0 + \int_{r_0}^{r_1} \psi(s)ds + 2\min\{0, M\}$, $M$ is a lower bound for $F$ in $[r_0, r_1] \times \mathbb{R}$, and $\varphi(r) = c(1+\vartheta) + \psi(r)$, so that $\varphi \in L^1_{loc}(J)$. It is now a consequence of Gronwall's inequality that $H(u') + |F(r,u)|$ is bounded in $[r_0, r_1)$, say by $c_1 \exp \int_{r_0}^{r_1} \varphi(s)ds$. Thus, since (3.9) implies (3.3), there is a positive number $P$ such that $|u'(r)| \le P$ for every $r \in [r_0, r_1)$.

The rest of the proof is now essentially the same as in Theorem 1, that is the extremal $u$ can here be continued to some interval $[r_0, R]$, with $r_1 < R < a_1$. This contradiction completes the proof.

Continuation of extremals to the left endpoint $a_0$ of $J$ can be obtained as in Theorems 1 and 2, using (i)′, (ii)′ instead of (i), (ii). Regarding continuation of extremals over the entire interval $J$ where $g > 0$, we state the following

THEOREM 3. *Suppose that* (i), (i)′, (ii) *and* (ii)′ *hold and also that* (3.9) *is satisfied. Then any extremal $u$ of* (2.1) *can be continued to all of $J$.*

PROOF: Continuation to $a_1$ follows from Theorem 2. To prove continuation to $a_0$ one can apply the proof of Theorem 2 with obvious modifications, and in particular the use of (i)′, (ii)′ rather than (i), (ii).

**Examples.** *1.* The degenerate Laplace operator $G(p) = |p|^m/m$, $m > 1$, has

$$G_p(p) = p|p|^{m-2}, \qquad A(p) = |p|^{m-2} \qquad \text{and} \qquad H(p) = \frac{m-1}{m}|p|^m.$$

Hence (3.9) holds with $\vartheta = m - 1$.

*2.* The mean curvature operator $G(p) = \sqrt{1+p^2} - 1$ has

$$G_p(p) = \frac{p}{\sqrt{1+p^2}}, \qquad A(p) = \frac{1}{\sqrt{1+p^2}} \qquad \text{and} \qquad H(p) = 1 - \frac{1}{\sqrt{1+p^2}}.$$

Thus both (3.3) and (3.9) fail.

*3.* The function $G(p) = p\tan^{-1} p - \frac{1}{2}\log(1+p^2)$ has $G_p(p) = \tan^{-1} p$, $G_{pp}(p) = 1/(1+p^2) > 0$,

$$A(p) = \frac{1}{p}\tan^{-1} p \qquad \text{and} \qquad H(p) = \frac{1}{2}\log(1+p^2).$$

Here (3.3) holds while obviously (3.9) fails for any $\vartheta > 0$. Therefore Theorem 1 can be applied, but both Theorems 2 and 3 are inapplicable.

## §4. Boundedness of extremals.

Here we discuss the boundedness of extremals on $J$. The results vary slightly depending on whether we consider boundedness as $r \to a_1$, the right endpoint of $J$, or as $r \to a_0$, the left endpoint. For simplicity, we shall present two results for the case $r \to a_1$, leaving other examples to the reader.

THEOREM 4. *Let $u$ be an extremal of problem* (2.1) *whose domain $I$ contains $I_1 = [r_0, a_1)$. Assume that there exists a sequence $(r_j)$ with $r_j \nearrow a_1$, such that $F(r_j, \cdot)$ is bounded below uniformly in $j$ and*

$$F(r_j, u) \to \infty \qquad \text{as } |u| \to \infty \text{ uniformly in } j. \tag{4.1}$$

*Suppose also that* (ii) *holds with $\psi \in L^1(I_1)$ and that $g$ is increasing. Then $u$ is bounded in $I_1$.*

PROOF: The proof is the same as for Theorem 1, with $r_1$ taken to be $a_1$. We find as there, by using the Gronwall inequality, that

$$|F(r,u)| \le c_1 \exp \int_{I_1} \psi(s)ds \qquad \text{for every } r \in I_1,$$

where $c_1 = c_0 + \int_{I_1} \psi(s)ds - 2\min\{0, M\}$, and $M$ is a lower bound for $F$ in $I_1 \times \mathbf{R}$. Existence of this bound follows exactly as in the proof of the lemma in Section 3. In the same way (4.1) and (ii) imply that

$$F(r,u) \to \infty \qquad \text{as } |u| \to \infty \text{ uniformly in } r \in I_1.$$

It now follows at once that $u$ is bounded as $r$ tends to $a_1$, completing the proof.

**Remarks.** Theorem 4 can be applied even when $G(p) = \sqrt{1+p^2} - 1$, in which case equation (2.6) becomes the mean curvature equation

$$\left(g(r)\frac{u'}{\sqrt{1+u'^2}}\right)' + g(r) f(r,u) = 0.$$

Here the associated function $H$ does not satisfy either (3.3) or (3.9).

If hypothesis (4.1) fails then the conclusion need not hold, as shown by the simple example $u'' + \frac{u'}{r} = 0$ with $a_1 = +\infty$ and $u(r) = \log r$.

We next give an alternative version of Theorem 4 for $F$ depending only on $u$ and for the special case when the extremal $u$ is oscillatory as $r \to a_1$, that is, possesses an infinite number of zeros which accumulate at $a_1$ (the usual case is $a_1 = +\infty$).

THEOREM 4'. *Let $u$ be an extremal of problem* (2.1) *whose domain contains $I_1 = [r_0, a_1)$ and which is oscillatory as $r \to a_1$. Assume also that $g$ is increasing and that the function $F$ given in* (c) *depends only on $u$ and satisfies the assumption*

$$\begin{aligned} F(u) &< \lim_{t\to\infty} F(t) \qquad \text{for } u \ge 0 \\ F(u) &< \lim_{t\to-\infty} F(t) \qquad \text{for } u \le 0. \end{aligned} \tag{4.2}$$



*Then $u$ is bounded in $I_1$.*

PROOF: Since $F_r(r,u) \equiv 0$ and $g'(r) \ge 0$, from (3.4) we have

$$(H(u') + F(u))' \le 0 \qquad \text{for all } r \in I_1.$$

Therefore the function $H(u') + F(u)$ is decreasing in $I_1$ and so is bounded above. We denote by $u_{crit}$ the values of $u$ at local maxima and local minima in $I_1$. Thus $F(u_{crit})$ is decreasing and by the growth condition (4.2) this shows that $|u_{crit}|$ is bounded. Hence $u$ is bounded in $I_1$.

**Remarks.** If $H$ satisfies (3.3), then, under the assumptions of either Theorem 4 or Theorem 4', the extremal $u$ and its derivative $u'$ are bounded in $I_1$.

If moreover the interval $J$ is bounded on the right, that is $a_1 < \infty$, then clearly $u$ is Lipschitz continuous in $I_1$, and so has a finite limit as $r \to a_1$. Thus the extremal $u$ can be extended to a function of class $C(\overline{I_1})$. The derivative $u'$ need not approach a limit unless further assumptions are made concerning the continuity of $F$ at $r = a_1$. If, however, we assume also that $F \in C(\overline{J} \times \mathbf{R})$ then by the arguments in the proof of Theorem 1 we see that $u'$ also approaches a finite limit as $r \to a_1$. The condition $F \in C(\overline{J} \times \mathbf{R})$ holds in particular when $f(r,u) = \rho(r)\phi(u)$ and $\rho(r) \to$ limit as $r \to a_1$.

In both Theorems 4 and 4' it is evident that $g$ is bounded from zero near the endpoint $a_1$. Some condition of this sort is in fact essential for the boundedness of extremals, as is clear even from the case of Bessel's equation, where $g(r) = r^2 \to 0$ and the corresponding solutions may become unbounded as $r \to 0$.

## §5. Global asymptotic stability of extremals.

In the preceding section we noted, for intervals $J$ bounded on the right, that extremals can be continuously extended, under certain conditions, to $\overline{I_1} \ni a_1$.

When $J$ is unbounded on the right the situation is more delicate. The purpose of this section is to present several results in this direction, under mild assumptions on the functions $g = g(r)$ and $F = F(r,u)$. The following properties of $F$ will be crucial in the discussion. Without loss of generality we assume from here on that $J = \mathbf{R}^+$.

(P) *There exists a number $R > 0$ such that for each pair of values $u_0$, $U$, with $0 < u_0 \le U$, there is a constant $\kappa > 0$ for which $u f(r,u) \ge \kappa$ when $r \ge R$ and $|u| \in [u_0, U]$.*

(P)' *For every $U > 0$ there is a nonnegative function $\psi \in L^1[R,\infty)$ such that $F_r(r,u) \le \psi(r)$ for almost all $r \in [R,\infty)$ and all $|u| \le U$.*

In the important case when $f(r,u) = \rho(r)\phi(u)$, it is easy to see that property (P) is equivalent to the simple condition

$$\rho(r) \ge \text{Const.} > 0 \quad \text{for } r \ge R, \qquad u\,\phi(u) > 0 \quad \text{for } u \ne 0.$$

Moreover, condition (P)' holds when $\rho' \in L^1[R,\infty)$. Of course when $f$ does not depend on $r$ property (P)' is irrelevant.

Condition (P) implies $u\, f(r,u) > 0$ for $r \ge R$ and $u \ne 0$, and so in turn $F(r,u) > 0$ for those values of $r$ and $u$. The following lemma makes this statement more precise.

LEMMA. *Suppose that $F = F(r,u)$ has property (P). Then for every $U > 0$ there is a positive increasing function $\omega = \omega(t)$, $0 < t \le U$, such that*

$$F(r,u) \ge \omega(|u|) \qquad \text{for } r \ge R \text{ and } 0 < |u| \le U.$$

PROOF: Let $\kappa(u_0, U)$ be the value of $\kappa$ in (P) corresponding to the interval $[u_0, U]$. If this function is not already increasing in the variable $u_0$, we replace it by the new function

$$\bar\kappa(u_0, U) = \sup_{0<t\le u_0} \kappa(t, U),$$

which is positive and increasing and for which (P) is also satisfied.

Now for any $r \ge R$ and $u \in (0, U]$ we have

$$F(r,u) = \int_0^u f(r,t)dt \ge \int_{u/2}^u \frac{\bar\kappa(\frac12 u, U)}{u} dt = \tfrac12 \bar\kappa(\tfrac12 u, U).$$

Similarly $F(r,u) \ge \frac12 \bar\kappa(\frac12|u|, U)$ when $u \in [-U, 0)$. The proof is thus completed by taking $\omega(t) = \frac12 \bar\kappa(\frac12 t, U)$, $0 < t \le U$.

We recall the relation $\delta(r) = g'(r)/g(r)$. Our main theorem is now

THEOREM 5. *Let $u$ be an extremal of problem (2.1), bounded in $[r_0, \infty)$ for some $r_0 > 0$. Suppose that there are positive numbers $\beta$ and $\beta'$ such that*

$$\beta \le r\,\delta(r) \le \beta' r \qquad \text{for every } r \in [r_0, \infty). \tag{5.1}$$

*Assume also that conditions (P) and (P)′ hold.*

*Then both $u$ and $u'$ tend to zero as $r \to \infty$.*

Condition (5.1) can be significantly weakened if the operator $A$ satisfies a mild algebraic growth condition as $p \to 0$, see the remarks at the end of the proof.

PROOF OF THEOREM 5: Let $u$ be an extremal of problem (2.1), with $|u(r)| \le L$ for every $r \in [r_0, \infty)$. Without loss of generality we shall assume that the number $R$ given in (P) is less than $r_0$, and that $L > 0$. Let $\psi$ denote the function given in (P)′ corresponding to $U = L$. From (2.7), (5.1) and (P)′ we have $\left(H(u') + F(r,u)\right)' \le \psi(r)$ a.e. in $[R, \infty)$. Therefore

$$H(u') + F(r,u) + \int_r^\infty \psi(s)ds$$



is decreasing in $[R, \infty)$. Hence, since $H$ and $F$ are nonnegative, there is a number $\ell \ge 0$ such that

$$H(u') + F(r,u) \to \ell \qquad \text{as } r \to \infty. \tag{5.2}$$

If $\ell = 0$, then by Lemma 1 of Section 2 and the lemma above we immediately derive that $u'(r) \to 0$ and $u(r) \to 0$ as $r \to \infty$, completing the proof.

Let us therefore assume for contradiction that $\ell > 0$ in (5.2). We choose $r_1 \ge r_0$ so that

$$\tfrac34 \ell \le H(u') + F(r,u) \le 2\ell \quad \text{for } r \ge r_1 \qquad \text{and} \qquad \int_{r_1}^\infty \psi(s)ds \le \tfrac18 \ell \tag{5.3}$$

Consider the two disjoint subsets $I_1$ and $I_2$ of $[r_1, \infty)$ such that

$$F(r, u(r)) \le \tfrac14 \ell \qquad \text{whenever } r \in I_1, \tag{5.4}$$

$$F(r, u(r)) > \tfrac14 \ell \qquad \text{whenever } r \in I_2. \tag{5.5}$$

By (5.3) and Lemma 1 there is a positive number $p_0$ such that

$$H(u'(r)) \ge \tfrac12 \ell \qquad \text{and} \qquad |u'(r)| \ge p_0 > 0 \qquad \text{for all } r \in I_1. \tag{5.6}$$

We claim moreover that there is a positive number $u_0$ such that

$$|u(r)| \ge u_0 > 0 \qquad \text{for every } r \in I_2. \tag{5.7}$$

Indeed, by (5.5), (P)′ and (5.3) we have for $r \in I_2$

$$\begin{aligned} \tfrac14 \ell < F(r, u(r)) &= F(r_1, u(r)) + \int_{r_1}^r F_r(s, u(r))ds \le F(r_1, u(r)) + \int_{r_1}^\infty \psi(s)ds \\ &\le F(r_1, u(r)) + \tfrac18 \ell; \end{aligned} \tag{5.8}$$

in other words $F(r_1, u(r)) > \ell/8$ for every $r \in I_2$. The claim now follows since $F$ is continuous in the variable $u$ and $F(r_1, 0) = 0$ by (c).

We shall show that for a suitable choice of the constants $\alpha$ and $k$ the right-hand side of the main identity (2.9) is less than $r^k F_r(r,u)$ for all $r$ sufficiently large. We take in particular

$$0 < \alpha \le \tfrac14 \beta, \qquad 0 < k \le \tfrac14 \beta. \tag{5.9}$$

It is convenient in what follows to express the right-hand side $\mathcal{R} = \mathcal{R}(r)$ of (2.9) in the form

$$\mathcal{R} = r^{k-1}\Gamma(r) + r^k F_r(r,u),$$

so that what we must show is that $\Gamma(r) < 0$ for $r$ sufficiently large. Moreover, we shall write $\xi = \xi(r) = r\,g'(r)/g(r) = r\,\delta(r)$. For later use, note by (5.9) and (5.1) that

$$\frac{\xi - k + 1}{\xi - k - \alpha} \le 2\frac{1+\beta}{\beta} \qquad \text{for } r \in [r_0, \infty). \tag{5.10}$$

We now divide the discussion according to whether $r \in I_1$ or $r \in I_2$.

*Case* $r \in I_1$. By (b) and the fact that $uf(r,u) > 0$ we have

$$\Gamma \le k\,F(r,u) - (\xi - \alpha - k)A(u')u'^2 - \frac{\alpha}{r}(\xi - k + 1)A(u')uu'$$
$$\le k\,F(r,u) - (\xi - \alpha - k)A(u')\,|u'|\left(|u'| - \frac{2\alpha}{r}\cdot\frac{1+\beta}{\beta}\,|u|\right)$$

by (5.10). We now take $r_2 \ge r_1$ so that $r_2 \ge (1+\beta)L/p_0$. Hence for $r \ge r_2$ in $I_1$ we have by $(5.6)_2$ and (5.9)

$$|u'| - \frac{2\alpha}{r}\cdot\frac{1+\beta}{\beta}\,|u| \ge \frac{1}{2}|u'|$$

and

$$\Gamma \le k\,F(r,u) - \frac{1}{4}\beta\,A(u')\,u'^2 \le \frac{1}{4}\beta\Big(F(r,u) - H(u')\Big)$$

by (5.9) and $(2.4)_2$. Hence $\Gamma(r) < 0$ for any $r \in I_1 \cap (r_2, \infty)$ by (5.4) and $(5.6)_1$. In other words, taking (2.9) and (P)$'$ into account, we have shown that

(5.11) $$\left\{r^k\left(H(u') + F(r,u) + \frac{\alpha}{r}A(u')\,uu'\right)\right\}' < r^k\psi(r) \quad \text{for almost all } r \in I_1 \cap (r_2, \infty).$$

*Case* $r \in I_2$. By (5.3), (P) and (5.9), when $r \in I_2$ there holds

(5.12) $$\Gamma \le 2k\ell - \alpha\kappa - (\xi - \alpha - k)A(u')\,u'^2 - \frac{\alpha}{r}(\xi - k + 1)A(u')\,uu',$$

where $\kappa = \kappa(u_0, L)$ is the number of property (P) corresponding to $u_0$ given in (5.7) and to $U = L$. As in the previous case, if

$$|u'| \ge \frac{2\alpha}{r}\cdot\frac{1+\beta}{\beta}\,L$$

we get

$$\Gamma \le 2k\ell - \alpha\kappa.$$

On the other hand, if

(5.13) $$|u'| < \frac{2\alpha}{r}\cdot\frac{1+\beta}{\beta}\,L$$

then from (5.12)

(5.14) $$\Gamma \le 2k\ell - \alpha\kappa + \frac{\alpha}{r}(1 + \beta' r)L\,A(u')\,u'.$$

Thus if $r$ is suitably large, say $r \ge r_3$, we get from (5.13), (5.14) and (2.5)

$$\Gamma < 2k\ell - \tfrac{1}{2}\alpha\kappa.$$



Hence for $r \ge r_3$ in $I_2$ and for $k \le \alpha\kappa/4\ell$ we have $\Gamma(r) < 0$.

In conclusion, recalling (5.11), we have shown that

$$\left\{r^k\left(H(u') + F(r,u) + \frac{\alpha}{r}A(u')uu'\right)\right\}' < r^k\psi(r) \qquad \text{for almost all } r > r_4,$$

where $r_4 = \max\{r_1, r_2, r_3\}$. Consequently the function

$$\Psi(r) = r^k\left(H(u') + F(r,u) + \frac{\alpha}{r}A(u')uu' - \frac{1}{r^k}\int_{r_4}^{r} s^k\psi(s)ds\right)$$

is decreasing in $[r_4, \infty)$. Moreover $\displaystyle\int_{r_4}^{r}\left(\frac{s}{r}\right)^k\psi(s)ds \le \int_{r_1}^{\infty}\psi(s)ds \le \frac{1}{8}\ell$ by (5.3), while also by (5.9) and $(2.4)_3$ we have

$$\frac{\alpha}{r}|A(u')uu'| = \frac{\alpha}{r}|u\,G_p(u')| \le \frac{\alpha}{r}\Big(G(\sigma\,u) + H(u')\Big) \le \frac{\alpha}{r}\Big(G(L) + G(-L) + 2\,\ell\Big),$$

thanks to (5.3). Hence for all sufficiently large $r$

$$\Psi(r) \ge r^k\left(\frac{3}{4}\ell - \frac{\ell}{8} - \frac{\ell}{8}\right) = \frac{\ell}{2}r^k,$$

which contradicts the fact that $\Psi$ is decreasing in $[r_4, \infty)$ and completes the proof of the theorem.

**Remarks.** If for some $m > 1$ we have

(5.15) $$A(p) \le \text{Const.}\,|p|^{m-2}, \qquad p \ne 0,$$

then condition (5.1) can be improved to

(5.1)$'$ $$\beta \le r\,\delta(r) \le \beta' r^m.$$

For example, in equation (1.5) we can allow $\delta(r) = r^{m-1}$ and still retain the conclusion $u(r), u'(r) \to 0$ as $r \to \infty$. The proof is essentially the same as before, except that (5.14) now becomes

$$\Gamma \le 2k\ell - \alpha\kappa + \text{Const.}\ \frac{\alpha}{r}(1 + \beta' r^m)\,L\,|u'|^{m-1}$$
$$\le 2k\ell - \alpha\kappa + \text{Const.}\ \alpha^m L^m \left(\frac{1+\beta}{\beta}\right)^{m-1}\frac{1+\beta' r^m}{r^m},$$

by (5.13). Thus if $\alpha$ is suitably small we again obtain

$$\Gamma < 2k\ell - \tfrac{1}{2}\alpha\kappa,$$

and the rest of the proof is unchanged.

A further refinement of the proof (details omitted) shows, even more, that (5.1) can be weakened to the form

$$r\,\delta(r) \geq \beta, \qquad r^{-k}\int_{r_0}^{r} s^{k-1}\delta(s)ds \leq \text{Constant}$$

for some number $k$ with $0 < k < \beta$ and all $r \geq r_0$. Similarly, if (5.15) holds then it is enough if

$$r\,\delta(r) \geq \beta, \qquad r^{-k}\int_{r_0}^{r} s^{k-m}\delta(s)ds \leq \text{Constant}$$

for some number $k$ with $0 < k < \beta$ and all $r \geq r_0$. If equality holds in (5.15), then $0 < k < m\beta/(m-1)$. If $\beta > m$, or if $A(p) = |p|^{m-2}$, $p \neq 0$, and $\beta > m-1$, then one can specifically choose $k = m$ to obtain the condition

$$r^{-m}\int_{r_0}^{r} \delta(s)ds \leq \text{Constant}$$

discovered by Artstein and Infante [1] in the case when $m = 2$, $\delta(r) \geq$Pos. Constant and $F$ is independent of $r$.

Corollary to Theorem 5. *Let $u$ be a bounded solution of the equation*

$$(5.16) \qquad (A(u')u')' + h(r,u,u')A(u')u' + f(r,u) = e(r) \qquad \text{on } [r_0,\infty).$$

*Suppose that conditions* (P) *and* (P)′ *are satisfied by the function $f = f(r,u)$ and $F(r,u) = \int_0^u f(r,t)dt$, and that for some positive constants $\beta$, $\beta'$*

$$(5.17) \qquad \beta \leq r\,h(r,u,p) \leq \beta' r \quad \text{for every } r \geq r_0 \text{ and all } u \in \mathbf{R},\ p \in \mathbf{R}.$$

*Finally, assume that either*

$$(5.18) \qquad e(r) \to 0 \quad \text{as } r \to \infty, \qquad e' \in L^1[r_0,\infty),$$

or

$$(5.19) \qquad e \in L^1[r_0,\infty), \qquad H(p) \geq \textit{Pos. Const. } |p| \quad \text{when } |p| \geq p_1$$

*for some number $p_1 \geq 0$. Then*

$$u(r) \to 0, \qquad u'(r) \to 0 \qquad \text{as } r \to \infty.$$

*In case (5.19) the condition that $u$ is bounded can be replaced by*

$$(5.20) \qquad F(r,u) \to \infty \qquad \text{as } |u| \to \infty, \text{ uniformly in } r.$$



Proof: Corresponding to the given solution $u = u(r)$, we define

$$\delta(r) \equiv h(r,u(r),u'(r)), \qquad r \geq r_0.$$

Then by (5.17) the condition (5.1) is satisfied. Now set

$$\hat{f}(r,u) = f(r,u) - e(r) \qquad \text{for } r \geq r_0 \text{ and } u \in \mathbf{R},$$

so that in turn

$$u\,\hat{f}(r,u) = u\,f(r,u) - e(r)u \quad \text{and} \quad \hat{F}(r,u) = F(r,u) - e(r)u.$$

Suppose now that (5.18) is satisfied. Then, as in the first part of the proof of Theorem 5, we see that

$$H(u') + \hat{F}(r,u) + \int_r^\infty (\psi(s) + |e'(s)|\,L)ds$$

is decreasing in $[R,\infty)$. Hence (5.2) holds in view of (5.18). Thus we are done if $\ell = 0$.

Therefore, as before, assume for contradiction that $\ell > 0$. We follow the proof of Theorem 5 with $F$ replaced by $\hat{F}$ and with the right hand side $\mathcal{R} = \mathcal{R}(r)$ of (2.9) in the form

$$\mathcal{R}(r) = r^{k-1}\,\hat{\Gamma}(r) + r^k\,\hat{F}_r(r,u),$$

where

$$\hat{\Gamma}(r) = \Gamma(r) + (\alpha - k)e(r)\,u.$$

Then as before there is $r_4 > 0$ such that $\hat{\Gamma}(r) < 0$ for all $r \geq r_4$, since $|e(r)u(r)| \leq |e(r)|\,L \to 0$ as $r \to \infty$. Consequently the function

$$\hat{\Psi}(r) = r^k\Big\{H(u') + F(r,u) + \frac{\alpha}{r}A(u')\,u\,u' - e(r)\,u - \frac{1}{r^k}\int_{r_4}^{r} s^k(\psi(s) + |e'(s)|\,L)ds\Big\}$$

is decreasing in $[r_4,\infty)$, which yields the same contradiction as before.

Next suppose that (5.19) is satisfied. We show first that $u'$ is bounded in $[r_0,\infty)$. By (2.7) with $F$ replaced by $\hat{F}$ we have

$$(H(u') + \hat{F}(r,u))' \leq \hat{F}_r(r,u),$$

which can be rewritten in the form

$$\mathcal{L}' = (H(u') + F(r,u))' \leq F_r(r,u) + e(r)\,u' \leq \psi(r) + |e(r)|\,|u'|$$

a.e. in $[R,\infty)$ by (P)′. In turn by (5.19)

$$\mathcal{L}' \leq \psi(r) + p_1\,|e(r)| + \text{Const. } |e(r)|\,\mathcal{L}.$$

The Gronwall inequality implies that there exists $M > 0$ such that

(5.21) $$\mathcal{L}(r) \le M \quad \text{on } [R, \infty).$$

Hence $H(u')$ is bounded and $u'$ must be bounded by $(5.19)_2$, say $|u'(r)| \le L'$.

As in the first part of the proof of Theorem 5 we then see that

$$H(u') + F(r,u) + \int_r^\infty (\psi(s) + |e(s)|\, L')ds$$

in decreasing in $[R, \infty)$. Hence (5.2) holds in view of $(5.19)_1$ and the proof is complete when $\ell = 0$. If $\ell > 0$ we continue as in the proof of Theorem 5. From (2.9) with $F$ replaced by $\hat{F}$ we derive after some calculations

$$\left\{ r^k \left( H(u') + F(r,u) + \frac{\alpha}{r} A(u')\, u\, u' \right) \right\}'$$
$$= r^{k-1}\Gamma(r) + r^k \left\{ F_r(r,u) + e(r)\left( u' + \frac{\alpha\, u}{r} \right) \right\}.$$

The rest of the proof in case (5.19) is the same as for Theorem 5 since

$$e\left( u' + \frac{\alpha\, u}{r} \right) \in \mathrm{L}^1[r_0, \infty)$$

by $(5.19)_1$ and the boundedness of $u$ and $u'$.

To obtain the final part of the corollary note that (5.21) implies also that $F(r,u)$ is bounded along the extremal. Therefore condition (5.20) gives the required boundedness of $u$, and the proof proceeds unchanged.

The case (5.19–20) of the preceding corollary generalizes Theorem 1 of [7]. In particular the latter result applies when $A \equiv 1$ (so $H(p) = \frac{1}{2}p^2$) and when $h = h(r,u,p)$ is bounded above and below by positive constants. In [7] the upper bound is stated only for bounded ranges of $u$ and $p$, but the *a priori* boundedness of $u = u(r)$ and $u' = u'(r)$ has been established here in the last part of the proof of the corollary using (5.21) together with $(5.19)_2$ and (5.20). (Note also that (5.20) corresponds to (1.7) in [7].) Finally, in [7] the function $f$ is assumed *independent* of r and Lipschitz continuous in $u$, and $e$ is supposed to be bounded and of class $\mathrm{L}^1[r_0, \infty)$, while we have assumed only that $e$ is of class $\mathrm{L}^1[r_0, \infty)$.

It almost goes without saying that condition (5.17) in the Corollary can be weakened in the directions indicated in the remarks following the proof of Theorem 5. In particular, in case $A \equiv 1$ the upper bound for $r\, h$ in (5.17) can be weakened to $\beta' r^2$.



**Remark.** The continuation property of Theorem 1 can be obtained under weaker conditions on $F$ if (3.3) is replaced by

(3.3)′ $$H(p) \ge \text{Pos. Const. } |p| \quad \text{for all } p \text{ sufficiently large.}$$

In particular. in this case it is enough if $F$ has the form

(3.10) $$F(r,u) = F_1(r,u) + F_2(r,u),$$

where $F_1$ satisfies conditions (i)–(ii) and $F_2$ is such that there exists a nonnegative function $\psi_2 \in \mathrm{L}^1_{\mathrm{loc}}(J)$ for which

$$\left| \frac{\partial F_2}{\partial u}(r,u) \right| \le \psi_2(r) \quad \text{a.e. in } J \text{ and for all } u \in \mathbb{R}.$$

The proof is essentially the same as before, except that (3.5) is replaced by

$$(H(u') + F_1(r,u))' \le F_{1r}(r,u) + F_{2u}(r,u)\, u',$$

which leads in turn to

$$|H(u') + F_1(r,u)| \le \text{Constant}$$

instead of (3.7). Thus $|u'(r)|$ is bounded in $[r_0, r_1)$ and the conclusion is then obtained exactly as before. The same remark holds for Theorems 2 and 3, since (3.9) implies (3.3)′.

Condition (3.10) is satisfied in particular when $F(r,u) = F_1(r,u) - e(r)\, u$ and $e \in \mathrm{L}^1_{\mathrm{loc}}(J)$. In this case equation (1.1) takes the form

$$(A(u')\, u')' + \delta(r)\, A(u')\, u' + f_1(r,u) = e(r);$$

cf. also the corollary to Theorem 5 in Section 5.

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