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Strongly Non-linear Elliptic Problems Near Resonance: A Variational Approach

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STRONGLY NONLINEAR ELLIPTIC PROBLEMS NEAR RESONANCE : A VARIATIONAL APPROACH

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1. INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^N$, $f(x,s)$ a function on $\Omega\times\mathbb{R}$ and $h(x)$ a function or distribution on Ω. We consider the quasilinear Dirichlet problem

$$(1.1)\qquad \begin{cases} -\Delta_p u=f(x,u)+h(x) \text{ in } \Omega \ , \\ u=0 \text{ on } \partial\Omega. \end{cases}$$

Here $-\Delta_p$, $1<p<\infty$, is the p-Laplacian

$$-\Delta_p u=-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \ ,$$

where $|\nabla u|$ denotes the Euclidian norm of the gradient of u. The nonlinearity $f(x,s)$ in (1.1) generates a potential

$$(1.2)\qquad F(x,s):=\int_0^s f(x,t)dt$$

which, in this paper, will always be assumed to lie asymptotically as $s\to\pm\infty$ to the left side of the first eigenvalue λ_1 of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, i.e.

$$(1.3)\qquad F^{\pm}(x) := \limsup_{s\to\pm\infty} \frac{p\ F(x,s)}{|s|^p} \le \lambda_1 \ .$$

We are interested in the additional conditions to be imposed on $f(x,s)$, $F(x,s)$, and possibly $h(x)$, in order that (1.1) admits at least one solution.

These conditions turn out to be of two different types and we will refer to them as *nonresonance or resonance conditions on one side and growth conditions on the other side*. The nonresonance or resonance conditions bear on the potential $F(x,s)$ and possibly the forcing term $h(x)$ and involve the spectrum of $-\Delta_p$. They are used in the present problem to guarantee the coercivity of the associated functional

$$\Phi(u) := \frac{1}{p} \int_\Omega |\nabla u|^p - \int_\Omega F(x,u) - \int_\Omega h(x)u$$

on $W_0^{1,p}(\Omega)$. The terminology "resonance-nonresonance" is attached to these conditions by analogy with the linear O.D.E. situation and according to whether they involve $h(x)$ or not. They are briefly described below and discussed in details in section 2. The growth conditions bear on the nonlinearity $f(x,s)$ itself. They are used to permit some sort of differentiation of Φ. One feature of this paper is their relative generality which allows the consideration of strong nonlinearities. They are briefly described below and discussed in details in section 3.

The simplest nonresonance condition consists in requiring that strict inequality holds in (1.3) on subsets of positive measure (cf.proposition 2.2). This extends to the quasilinear case conditions which in the semilinear case p=2 were considered by [Li],[Ha],[M-W-W] (see [Ma_1] for an historical survey). More subtile *nonresonance* conditions can however be considered, which allow $F^+(x)\equiv\lambda_1$ and $F^-(x)\equiv\lambda_1$, as shown in proposition 2.4 where an assumption is made on the speed with which $pF(x,s)/|s|^p$ approaches λ_1. When this speed reaches some critical level, resonance can occur and some restriction must be imposed on $h(x)$ (cf.proposition 2.6). The restriction imposed here looks like the classical Landesman-Lazer condition. However it is expressed in terms of the limits of $(F(x,s)-\lambda_1|s|^p/p)|s|$ instead of, as

usual when p=2, the limits of $f(x,s)-\lambda_1 s$. This turns out to be more general (cf.remark 2.9 and example 2.11) and in addition it is more closely related to the necessary condition of Ahmad-Lazer-Paul [A-L-P] (cf.remark 2.12).

The growth condition usually imposed on $f(x,s)$ is of the form

$$(1.4) \qquad |f(x,s)| \le a|s|^{p^*-1}+b(x) ,$$

where p^* denotes the Sobolev conjugate exponent: $1/p^*=1/p-1/N$ (for simplicity we suppose for the moment $p < N$). Condition (1.4) implies the C^1 character of Φ. In this paper we only assume either an one-sided growth condition from above of the form

$$(1.5) \qquad \operatorname{sgn} s\, f(x,s) \le \zeta(s)+b(x)$$

or an one-sided growth condition from below of the form

$$(1.6) \qquad \operatorname{sgn} s\, f(x,s) \ge -\zeta(s)-b(x) ,$$

where the function $\zeta(s)$ satisfies $\zeta(s)=o(|s|^{p^*})$ as $s\to\pm\infty$. Of course these conditions do not suffice to guarantee the differentiability of Φ, which may even take infinite values. We nevertheless show that any minimum u of Φ solves (1.1) in a suitable weak sense (cf.propositions 3.1 and 3.2). This is obtained by some extension of a technique introduced by Hempel [Hem] to control differential quotients in the calculus of variations.

We also consider in some details the case p=N. The space $W_0^{1,p}(\Omega)$ is then imbedded into an Orlicz space defined by an N-function which grows like $\exp|s|^{N/(N-1)}$ at infinity (cf.[Tr],[Mo]). We use this imbedding in orcer to weaken further in that case the one-sided growth condition imposed on $f(x,s)$. When $p > N$, no growth condition on $f(x,s)$ is needed.

Several papers have been concerned with the introduction of strong nonlinearities in the lower order part of an elliptic equation. See e.g. [Br], [Hes], [We], [Br-Br], [Be-Go] when the top order part is nonlinear and some sign condition is imposed on the zero order term, [Ka-Wa], [Br-Ni], [De-Go$_1$] when the top order part is linear and nonresonance or resonance is considered. All these works eventually rely on monotone iteration or degree theory. The variational approach used here leads in a natural way to a clear distinction between nonresonance or resonance conditions on one side and growth conditions on the other side. The relative lack of interaction between these conditions provides some flexibility in the applications, as illustrated in the study of oscillating strong nonlinearities (cf.example 4.6). Moreover the nonresonance or resonance conditions which are expressed in terms of $F(x,s)$ and connected with inequality (1.2) are more general that the analogous ones which are expressed in terms of $f(x,s)$ and connected with the inequality

$$\text{(1.7)} \qquad \limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \le \lambda_1$$

(cf.remark 2.9 and examples 2.10, 2.11). This fact was already noticed in [M-W-W] for the nonresonance condition of proposition 2.2 (with p=2). It is also worth observing in the present context of strong nonlinearities that inequality (1.7) itself already imposes an one-sided growth restriction on $f(x,s)$, which is stronger than (1.5).

Specific references to as well as comparisons with previous works

are given in section 4, after the statement of our existence theorem for (1.1). As indicated there, a large part of this theorem appears to be new even when p=2, i.e. for the semilinear problem

$$\text{(1.8)} \qquad \begin{cases} -\Delta u = f(x,u)+h(x) \text{ in } \Omega \ , \\ u=0 \text{ on } \partial\Omega \ . \end{cases}$$

A particular case of this theorem has recently been used in [De-Go$_3$] as an intermediate step to deal with a situation where no growth restriction at all is imposed on the nonlinearity f (cf.remark 4.4).

Questions similar to those treated in this paper can also be considered for the more general, possibly higher order, quasilinear problem

$$\begin{cases} \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,u,\nabla u,\ldots,\nabla^m u) = f(x,u)+h(x) \text{ in } \Omega \ , \\ D^\alpha u = 0 \text{ on } \partial\Omega \text{ for } |\alpha| \le m-1 \end{cases}$$

(cf.remark 4.2 in the semilinear case). This requires among other things an adequate definition of the first eigenvalue λ_1 and of the associated eigenfunctions. These questions will be studied in a subsequent paper.

2. NONRESONANCE OR RESONANCE CONDITIONS AND COERCIVITY.

In this section we study some conditions which imply that the functional Φ is coercive.

Let Ω be a bounded open subset of $\mathbb{R}^N$, with boundary $\partial\Omega$ of class $C^{2,\beta}$ for some $\beta\in]0,1[$. This regularity is needed only to guarantee the simplicity of λ_1 below; this simplicity itself is used only when $p\neq2$ to give a simple statement of propositions 2.4 and 2.8.

We recall that the first eigenvalue λ_1 of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ is defined by

(2.1) $$\lambda_1=\inf\{\int_\Omega|\nabla v|^p/\int_\Omega|v|^p; v\in W_0^{1,p}(\Omega) \text{ and } v\not\equiv0\}.$$

It is known that λ_1 is >0, that the infinum above is achieved, and that it is achieved at $u\in W_0^{1,p}(\Omega)$ if and only if $u\not\equiv0$ and u satisfies

$$-\Delta_p u=\lambda_1|u|^{p-2}u \text{ in } \Omega.$$

Moreover the eigenvalue λ_1 is simple (i.e. any two corresponding eigenfunctions are multiple one of the other) and the associated eigenspace is generated by an eigenfunction which is >0 in Ω. For these results as well as for other informations about the spectrum of the p-Laplacian, see [An$_1$],[An$_2$] and the references therein. We will denote by φ_1 the normalized positive eigenfunction, where the normalization is taken with respect to the $W_0^{1,p}(\Omega)$ norm

$$\|v\| := (\int_\Omega|\nabla v|^p)^{1/p}.$$

Let $1\leq\alpha\leq p$. It will be convenient below to say that a function $\eta(x)$ belongs to X_α if $\eta\in L^q(\Omega)$ for some $q>(p^*/\alpha)'$ when $p<N$, $\eta\in L^q(\Omega)$ for some $q>1$ when p=N, and $\eta\in L^1(\Omega)$ when $p>N$. Here $(p^*/\alpha)'$ denotes the Hölder conjugate of p^*/α. We will also say that $\eta(x)$ belongs to Y_1 if $\eta\in L^{p^{*\prime}}(\Omega)$ when $p<N$, $\eta\in L^q(\Omega)$ for some $q>1$ when p=N, and $\eta\in L^1(\Omega)$ when $p>N$. The following lemma is an easy consequence of the Sobolev imbedding theorem.

LEMMA 2.1. *Let* $u_n\to u$ *weakly in* $W_0^{1,p}(\Omega)$. *If* $\eta\in X_\alpha$, *then* $\eta|u_n|^\alpha\to\eta|u|^\alpha$ *in* $L^1(\Omega)$. *If* $\eta\in Y_1$, *then* $\eta|u_n|\to\eta|u|$ *in* $L^1(\Omega)$.

Let $F:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function such that

(F_0) $$\sup_{|s|\leq R}|F(x,s)|\in L^1(\Omega)$$

for any $R>0$. We consider the functional

(2.2) $$\Phi(u) := \frac{1}{p}\int_\Omega|\nabla u|^p-\int_\Omega F(x,u)-\langle h,u\rangle,$$

where h is given in $W^{-1,p'}(\Omega)$ and $\langle,\rangle$ denotes the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. The assumptions to be made on F(x,s) will imply that Φ is well-defined on $W_0^{1,p}(\Omega)$.

It will be convenient to write F(x,s) as

$$F(x,s)=\lambda_1|s|^p/p + G(x,s).$$

Thus $G(x,s)$ represents the perturbation of the potential $F(x,s)$ above the level λ_1. Defining

$$G_p^{\pm}(x) := \limsup_{s\to\pm\infty} \frac{G(x,s)}{|s|^p} , \tag{2.3}$$

inequality (1.3) becomes $G_p^{\pm}(x) \le 0$.

PROPOSITION 2.2. Assume (F_0) and

(G'_1) there exists $\eta(x)\in X_p$ such that for any $\varepsilon > 0$, there are $\delta_\varepsilon(x)\in Y_1$ and $\gamma_\varepsilon(x)\in L^1(\Omega)$ with

$$G(x,s) \le \varepsilon\eta(x)|s|^p+\delta_\varepsilon(x)|s|+\gamma_\varepsilon(x)$$

for a.e. $x\in\Omega$ and all $s\in\mathbb{R}$,

(G''_1) there exist $\Omega^+\subset\Omega$ and $\Omega^-\subset\Omega$, of positive measure such that

$$G_p^+ < 0 \text{ on } \Omega^+ \text{ and } G_p^- < 0 \text{ on } \Omega^-.$$

Then Φ is well-defined on $W_o^{1,p}(\Omega)$, takes values in $]-\infty,+\infty]$, is weakly lower semicontinuous and coercive.

Assumptions (G'_1), (G''_1) essentially mean that (1.3) holds a.e., with strict inequality on subsets of positive measure.

PROOF OF PROPOSITION 2.2. Assumption (G'_1), lemma 2.1 and Fatou's lemma imply that Φ is well-defined on $W_o^{1,p}(\Omega)$, takes values in $]-\infty,+\infty]$ and is weakly lower semicontinuous on $W_o^{1,p}(\Omega)$. To prove that Φ is coercive, suppose by contradiction the existence of a sequence $u_n \in W_o^{1,p}(\Omega)$ with $\|u_n\|\to+\infty$ and $\Phi(u_n) \le c$. Write $v_n=u_n/\|u_n\|$. Then, for a subsequence, $v_n \to v$ weakly in $W_o^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω.

We first prove that $v=\pm\varphi_1$. Since $\Phi(u_n) \le c$, we have, by (G'_1),

$$\frac{1}{p}\int_\Omega|\nabla u_n|^p-\frac{\lambda_1}{p}\int_\Omega|v_n|^p-\varepsilon\int_\Omega\eta(x)|v_n|^p-\int_\Omega\delta_\varepsilon(x)\frac{|v_n|}{\|u_n\|^{p-1}}$$
$$-\int_\Omega\frac{\gamma_\varepsilon(x)}{\|u_n\|^p}-\langle h,\frac{v_n}{\|u_n\|^{p-1}}\rangle \le \frac{c}{\|u_n\|^p}.$$

Using $\|v_n\|=1$ and letting first $n\to\infty$ and then $\varepsilon\to 0$, we get

$$1 \le \lambda_1\int_\Omega|v|^p.$$

Consequently, from the definition of λ_1 and the fact that $\|v\|\le 1$, we have

$$\lambda_1\int_\Omega|v|^p \le \int_\Omega|\nabla v|^p \le 1 \le \lambda_1\int_\Omega|v|^p , \tag{2.4}$$

so that equality holds everywhere in (2.4). In particular $\|v\|=1$ and v achieves the infinum in (2.1), which yields the conclusion $v=\pm\varphi$. (Since $\|v_n\|\to\|v\|$, one also derives that $v_n \to v$ in $W_o^{1,p}(\Omega)$). We will assume from now on that $v=\varphi_1$. Similar arguments can be given in the other case.

We now deduce from $\Phi(u_n)\leq c$, by using the definition of λ_1, that

$$(2.5)\qquad \int_\Omega G(x,u_n) \geq -\langle h,u_n\rangle - c.$$

This implies, after division by $\|u_n\|^p$, that

$$(2.6)\qquad \liminf \int_\Omega \frac{G(x,u_n)}{\|u_n\|^p} \geq 0.$$

Denoting by χ_n the characteristic function of the set $\{x\in\Omega;\ |u_n(x)|\geq 1\}$ and using (F_0), we deduce from (2.6) that

$$(2.7)\qquad \liminf \int_\Omega \frac{G(x,u_n)}{|u_n|^p}\, |v_n|^p \chi_n \geq 0.$$

Since $v_n \to \varphi_1$ a.e. in Ω, $u_n \to +\infty$ a.e. in Ω and $\chi_n \to 1$ a.e. in Ω. Moreover, using (G'_1) and lemma 2.1, one sees that Fatou's lemma can be applied to (2.7). This gives

$$\int_\Omega G_p^+(x)\varphi_1(x)^p \geq 0\ ,$$

which contradicts (G''_1). Q.E.D.

We now turn to situations where possibly $F^+(x)\equiv\lambda_1$ and $F^-(x)\equiv\lambda_1$, i.e. $G_p^+(x)\equiv 0$ and $G_p^-(x)\equiv 0$. Density conditions of the type introduced in

[De-Go$_2$] could be considered. However this approach seems to require growth restrictions on $f(x,s)$ which exclude the consideration of strong nonlinearities (cf. [De-Go$_2$] when $p=2$, [An$_2$] when $p\neq 2$). We go here in another direction and impose some control on the speed of the convergence of $pF(x,s)/|s|^p$ towards λ_1. This sort of idea was originally introduced in [De$_1$] and used later in [Go$_1$], [Go$_2$].

Conditions (G'_1), (G''_1) involve the comparison of $G(x,s)$ with the function $|s|^p$ as $s\to\pm\infty$. We will use more general comparison functions.

DEFINITION 2.3. An even continuous function $\psi:\mathbb{R}\to\mathbb{R}^+$ is called a comparison function of order α, $1\leq\alpha\leq p$, if

(C_1) $\quad \psi(s)/|s|^p \to 0$ as $s\to+\infty$,

(C_2) $\quad \psi(s)/|s| \to +\infty$ as $s\to+\infty$,

(C_3) $\quad \psi(s_n)/\psi(t_n) \to r^\alpha$ whenever $s_n/t_n \to r$ with $s_n\to+\infty$ and $t_n\to+\infty$,

(C_4) $\quad$ for any $\beta > \alpha$ there exist t_0, a,b such that $\psi(ts)/\psi(t) \leq as^\beta+b$ for all $t \geq t_0$ and all $s \geq 0$.

Typical examples of comparison functions of order α are $\psi(s)=|s|^\alpha$ for $1 < \alpha < p$, or $\psi(s)=|s|^\alpha/\log|s|$ for $1 < \alpha \leq p$, or $\psi(s)=|s|^\alpha\log|s|$ for $1 \leq \alpha < p$.

Given such a comparison function ψ, we define

$$G^{\pm}_{\psi}(x) := \limsup_{s\to\pm\infty} \frac{G(x,s)}{\psi(s)} .$$

PROPOSITION 2.4. <u>Assume</u> (F_0) <u>and</u>

(G'_2) <u>there exist a comparison function</u> ψ <u>of order</u> α, $1 \le \alpha \le p$, $\eta(x)\in X_\alpha$, $\delta(x)\in Y_1$ <u>and</u> $\gamma(x)\in L^1(\Omega)$ <u>such that</u>

$$G(x,s) \le \eta(x)\psi(s) + \delta(x)|s| + \gamma(x)$$

<u>for</u> a.e. $x\in\Omega$ <u>and all</u> $s\in\mathbb{R}$,

(G''_2) $$\int_\Omega G^{+}_{\psi}(x)(\varphi_1(x))^\alpha < 0 \text{ and } \int_\Omega G^{-}_{\psi}(x)(\varphi_1(x))^\alpha < 0.$$

<u>Then the conclusion of proposition 2.2 holds</u>.

Assumption (G'_2) essentially means that $G^{\pm}_{\psi}(x)$ is bounded from above by $\eta(x)$ (Recall that $G^{\pm}_{p}(x)$ was ≤ 0 in proposition 2.2).

PROOF OF PROPOSITION 2.4. We first show that (G'_2) implies (G'_1). This is an immediate consequence of (F_0) and (C_1) when $\alpha=p$ or when $p \ge N$. Let us consider the remaining case $1 \le \alpha < p$ and $p < N$. Recall that the function $\eta(x)$ in (G'_2) satisfies $\eta(x)\in L^q(\Omega)$ for some $q > (p^*/\alpha)'$. By (C_4), $\eta(x)\psi(s)$ can be estimated in terms of $\eta(x)|s|^\beta$ for some β with $\alpha < \beta < p$ and $q > (p^*/\beta)'$. Using Young's inequality, we get

$$\eta(x)|s|^\beta \le [\varepsilon^{-1}\eta(x)^{(1-\upsilon)}]^{(p/\beta)'} + [\varepsilon\eta(x)^{\upsilon}|s|^\beta]^{p/\beta}$$

for $0 \le \upsilon \le 1$. Define $\upsilon_0\in]0,1[$ by $(1-\upsilon_0)(p/\beta)'=(p^*/\beta)'$. A simple calculation shows that by taking υ less than but sufficiently near to υ_0, one has $\eta^{(1-\upsilon)(p/\beta)'}\in L^1(\Omega)$ and $\eta^{\upsilon\, p/\beta}\in X_p$. One can thus derive (G'_1) from (G'_2) in this way.

Since (G'_1) holds, the proof of proposition 2.2 can be followed until (2.5) is reached. Dividing by $\psi(\|u_n\|)$, we then obtain by (C_2)

$$\liminf \int_\Omega \frac{G(x,u_n)}{\psi(\|u_n\|)} \ge 0$$

and consequently

$$\liminf \int_\Omega \frac{G(x,u_n)}{\psi(u_n)} \frac{\psi(v_n\|u_n\|)}{\psi(\|u_n\|)} \chi_n \ge 0 , \tag{2.8}$$

where χ_n has the same meaning as before. Using (G'_2), (C_4) with $\beta > \alpha$ and $(p^*/\beta)' < q$, and lemma 2.1, one sees that Fatou's lemma can be applied to (2.8). It then follows from (C_3) that

$$\int_\Omega G^{+}_{\psi}(x)(\varphi_1(x))^\alpha \ge 0 ,$$

which contradicts (G''_2). Q.E.D.

The following example allows some comparison between propositions 2.2 and 2.4 (and proposition 2.8 below). See also remark 2.13 for such a comparison.

EXAMPLE 2.5. Take $1 < \alpha < p$ and

$$F(x,s)=\lambda_1|s|^p/p + \eta(x)|s|^\alpha$$

where $\eta(x)\in X_\alpha$. Then proposition 2.4 applies with $\psi(s)=|s|^\alpha$ provided $\int_\Omega \eta(x)\varphi_1(x)^\alpha < 0$. However proposition 2.2 does not apply. Moreover if η changes sign, no other comparison function with a different growth at infinity can be used and in addition proposition 2.8 below does not apply.

REMARK 2.6. Conditions (G'_2),(G''_2) with $\psi(s)=|s|^p$ (which is not a comparison function in the sense of definition 2.2) do not imply the coercivity of Φ (nor even the solvability of (1.1) when p=2 and F is of the form (1.2) with f(x,s) linear). This can be seen from an example in [Fo-Go]. Condition (1.3) is violated in that example.

REMARK 2.7. It should be interesting to remove the condition that ψ is even. This can be done under some further restriction on the growth of f(x,s), by adapting an argument from [De-Go$_2$] based on the mean value theorem. See also [Go$_2$] for another result involving noneven comparison functions.

The limiting case of proposition 2.4 where $\psi(s)=|s|^p$ is provided by proposition 2.2. The other limiting case, where $\psi(s)=|s|$, is the following proposition, where the functions $G_1^\pm(x)$ are defined by formula (2.3) with p=1.

PROPOSITION 2.8. <u>Assume</u> (F_0) <u>and</u>

(G'_3) <u>there exist</u> $\eta(x)\in Y_1$ <u>and</u> $\delta(x)\in L^1(\Omega)$ <u>such that</u>

$$G(x,s) \le \eta(x)|s|+\delta(x)$$

<u>for</u> a.e. $x\in\Omega$ <u>and all</u> $s\in\mathbb{R}$,

$$(G''_3) \qquad \int_\Omega G_1^-(x)\varphi_1(x) < \langle h,\varphi_1\rangle < -\int_\Omega G_1^+(x)\varphi_1(x).$$

<u>Then the conclusion of proposition</u> 2.2 <u>holds</u>.

In opposition with the previous two propositions, the assumptions of proposition 2.8 involve the forcing term h(x).

PROOF OF PROPOSITION 2.8. Since (G'_3) clearly implies (G'_1), the arguments of the proof of proposition 2.2 can be followed until (2.5) is reached. Dividing by $\|u_n\|$, we then obtain

$$(2.9) \qquad \liminf \int_\Omega \frac{G(x,u_n)}{|u_n|}|v_n|\chi_n \ge -\langle h,\varphi_1\rangle ,$$

where χ_n has the same meaning as before. Using (G'_3) and lemma 2.1, one sees that Fatou's lemma can be applied to (2.9). This gives

$$\int_\Omega G_1^+(x)\varphi_1(x) \ge -\langle h,\varphi_1\rangle ,$$

which contradicts (G''_3). Q.E.D.

REMARK 2.9. Let

$$G(x,s)=\int_0^s g(x,t)dt .$$

It is then easily verified that if $\ell:\mathbb{R}\to\mathbb{R}$ is continuous with $\ell(s)s>0$ for $|s|$ large and if the primitive L of ℓ tends to $+\infty$ at $\pm\infty$, then

$$\limsup_{s\to\pm\infty}\frac{G(x,s)}{L(s)}\le\limsup_{s\to\pm\infty}\frac{g(x,s)}{\ell(s)} .$$

In particular, for q=1 or p,

$$qG_c^{\pm}(x)\le g_q^{\pm}(x):=\limsup_{s\to\pm\infty}\frac{g(x,s)}{|s|^{q-2}s}$$

These inequalities show that conditions (G''_1) and (G''_3) follow from analogous conditions involving $g_p^{\pm}(x)$ and $g_1^{\pm}(x)$ respectively. The latter are in general more restrictive. Actually $(G'_1),(G''_1)$ or $(G''_3),(G''_3)$ may hold with $g_p^{\pm}(x)$ or $g_1^{\pm}(x)$ fairly arbitrary, as shown in the following two examples. Similar observations can be made for $(G'_2),(G''_2)$.

EXAMPLE 2.10. Take

$$g(x,s)=a|s|^{p-2}s+b|s|^{p-2}s\,\sin s$$

with $a,b\in\mathbb{R}$. Then $G_p^{\pm}(x)\equiv a/p$ and $g_p^{\pm}(x)\equiv a+|b|$. Proposition 2.2 applies as soon as $a<0$ (with b arbitrary).

EXAMPLE 2.11. Take p=2 and

$$g(x,s)=a\ {}^{2}/_{\pi}\ \operatorname{arctg} s+b\sin s$$

with $a,b\in\mathbb{R}$. Then $G_1^{\pm}(x)\equiv a$ and $g_1^{\pm}(x)\equiv a+|b|$. Proposition 2.9 applies as soon as $a<0$ (with b arbitrary). However the inequalities obtained from (G''_3) by replacing $G_1^{\pm}$ by $g_1^{\pm}$ (which are then the classical Landesman-Lazer conditions) can never be satisfied if $|b|\ge -a$.

REMARK 2.12. By restricting Φ to the line $\mathbb{R}\varphi_1$, one sees that a necessary condition for the coercivity of Φ is that

$$(2.10)\qquad \int_\Omega G(x,r\varphi_1)+\langle h,r\varphi_1\rangle\to-\infty \text{ as } |r|\to+\infty.$$

Condition (2.10) however is not *sufficient in general to guarantee the coercivity of* Φ *(or even the solvability of* (1.1) *when* $p=2$ *and* F is of the form (1.2) with $f(x,s)$ linear). This can be seen from an example in [Fo-Go]. Condition (2.10) corresponds in the present setting to the Ahmad-Lazer-Paul condition [A-L-P]. It is easily seen here, by means of Fatou's lemma, that $(G'_3),(G''_3)$ imply (2.10).

REMARK 2.13. Consider the autonomous case $G(x,s)=G(s)$. If $(G'_1),(G''_1)$ hold, then

$$G(s)\le-\varepsilon|s|^p+c$$

for some $\varepsilon > 0$ and some $c \in \mathbb{R}$, and consequently $(G'_2),(G''_2)$ hold for any ψ (with $G^*_\psi=-\infty$). Similarly, if $(G'_2),(G''_2)$ hold, then $(G'_3),(G''_3)$ hold for any h (with $G^*_1=-\infty$). Proposition 2.8 thus provides the best result in the autonomous case. Example 2.6 shows that this is not so anymore in the nonautonomous case.

REMARK 2.14. The sommability requirement on $\eta(x)$ can be slightly weakened in the three propositions above when $p=N$ by using the Trudinger imbedding theorem [Tr].

3. GROWTH CONDITIONS AND EULER LAGRANGE EQUATION.

In this section we investigate the question whether a minimum u of Φ solves (1.1) in a suitable sense.

Let Ω be any bounded open subset of $\mathbb{R}^N$. We recall the limiting case of the Sobolev imbedding theorem (cf.[Tr],[Mo]): $W_0^{1,N}(\Omega) \subset E_A(\Omega)$ continuously, where $E_A(\Omega)$ is the small Orlicz space associated to the N-function $A(t)=\exp t^{N/(N-1)}-1$. See e.g. [Kr-Ru] for the basic definitions from Orlicz spaces theory: a N-function $B(t)$, the large Orlicz space $L_B(\Omega)$, the small Orlicz space $E_B(\Omega)$, the conjugate N-function $\overline{B}$,....

Let $f:\Omega\times\mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that, for any $R > 0$,

$$(f_0) \qquad \sup_{|s|\leq R} |f(x,s)| \in L^{p^{*\prime}}(\Omega)(\text{resp.} L_{\overline{A}}(\Omega), L^1(\Omega))$$

when $p < N$ (resp. $p=N$, $p > N$). This implies that the primitive $F(x,s)$ is a Caratheodory function which satisfies (F_0). The associated function Φ will be well-defined on $W_0^{1,p}(\Omega)$, with values in $]-\infty,+\infty]$, if there exist $a,b\in\mathbb{R}$, $c(x)\in L^1(\Omega)$ such that

$$(F_*) \qquad F(x,s) \leq \begin{cases} a|s|^{p^*}+c(x) & \text{when } p < N, \\ a\, A(b|s|)+c(x) & \text{when } p=N, \end{cases}$$

for a.e. $x\in\Omega$ and all $s\in\mathbb{R}$; no further condition is needed when $p > N$. Observe that (G'_1) (and consequently (G'_2) or (G'_3)) implies (F_*).

In order to express the growth condition to be imposed on $f(x,s)$ when $p < N$, we need an auxillary function $\theta:\mathbb{R} \to \mathbb{R}$ with the following properties: θ is Lipschitzian, nondecreasing, with $\theta(\pm\infty)=\pm\infty$, $\theta(s)s > 0$

for $s\neq 0$ and $|s|^{p^*}/\theta(s)$ nondecreasing for $|s|$ large. We will refer to these properties as (P). Observe that $\theta(u(x)) \in W_0^{1,p}(\Omega)$ if $u(x) \in W_0^{1,p}(\Omega)$.

Typical examples of functions θ with property (P) are $\theta(s)=s$ for $s\in\mathbb{R}$, or, if $0 < \alpha \leq 1$ and $0 \leq \beta < 1$,

$$\theta(s) = \begin{cases} s^\alpha & \text{for } s \text{ large} \geq 0 , \\ -|s|^\beta \log|s| & \text{for } s \text{ large} \leq 0 , \end{cases}$$

with θ suitably extended on all $\mathbb{R}$ in the second example.

PROPOSITION 3.1. *Assume* (f_0), (F_*) *and*

$$(f_1) \begin{cases} \underline{\text{when}}\ p < N : \theta(s)f(x,s) \leq a|s|^{p^*}+d(x)|\theta(s)| , \\ \\ \underline{\text{when}}\ p=N : \operatorname{sgn}s\, f(x,s) \leq aA(b|s|)+e(x) , \end{cases}$$

for some function θ *with property* (P), *some* $a,b\in\mathbb{R}$, $d(x)\in L^{p^{*\prime}}(\Omega)$, $e(x)\in L_{\overline{A}}(\Omega)$ *and* a.e. $x\in\Omega$ *and all* $s\in\mathbb{R}$. *Then any minimum* u *of* Φ *over* $W_0^{1,p}(\Omega)$ *(if it exists) satisfies*

(3.1) $\qquad F(x,u) \in L^1(\Omega)$,

(3.2) $\qquad f(x,u) \in L^1(\Omega)$,

(3.3) $\qquad \int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi=\int_\Omega f(x,u)\varphi+\langle h,\varphi\rangle$ *for all* $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$,

moreover when $p < N$, $f(x,u)\theta(u) \in L^1(\Omega)$ *and* (3.3) *holds with* $\varphi=\theta(u)$; *when* $p=N$, $f(x,u)u\in L^1(\Omega)$ *and* (3.3) *holds with* $\varphi=u$.

PROPOSITION 3.2. *Assume* (f_0),(F_*) *and*

$$(f_2) \begin{cases} \underline{\text{when}}\ p < N : \theta(s)f(x,s) \geq -a|s|^{p^*}-d(x)|\theta(s)| , \\ \\ \underline{\text{when}}\ p=N : \operatorname{sgn}s f(x,s) \geq -aA(b|s|)-e(x) , \end{cases}$$

for some function θ *with property* (P), *some* $a,b\in\mathbb{R}$, $d(x)\in L^{p^{*\prime}}(\Omega)$, $e(x)\in L_{\overline{A}}(\Omega)$ *and* a.e. $x\in\Omega$ *and all* $s\in\mathbb{R}$. *Then the conclusion of proposition 3.1 holds.*

Conditions (f_1) and (f_2) are the precised forms of the one-sided growth conditions (1.5) and (1.6) from the introduction. Actually it easily follows from a lemma in [Go_2] that in the autonomous case with $p < N$, (f_1) and (f_2) are equivalent to (1.5) and (1.6) respectively. When $p > N$, (f_0) alone implies that Φ is C^1 and consequently the above propositions become trivial.

The proof of proposition 3.1 is based on the following lemma, which provides some extension of a technique introduced in [Hem].

LEMMA 3.3. *Let* $p < N$ *and assume* (f_0),(f_1). *Take* $u\in W_0^{1,p}(\Omega)$, $\varphi\in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and* $0 < \alpha \leq 1/2K$, *where* K *is the Lipschitz constant of* θ, *and denote* $\varphi-\alpha\theta(u)$ *by* v. *Then there exists* $\ell(x)\in L^1(\Omega)$ *such that*

(3.4) $\qquad (F(x,u+tv)-F(x,u))/t \geq \ell(x)$

for a.e. $x \in \Omega$ and all $t \in]0,1[$. Same conclusion when $p=N$ if one takes $v=\varphi-\alpha u$ with $0 < \alpha \le 1/2$.

PROOF. We first deduce some consequence from (f_1). Let us consider the case $p < N$. By (f_0), (f_1) and (P), one has, for some $L^{p^{*'}}$ function $\tilde{d}$,

$$f(x,s) \le a\frac{|s|^{p^*}}{\theta(s)} + d(x) \le a\frac{|r|^{p^*}}{\theta(r)} + \tilde{d}(x)$$

for $0 < s \le r$, and consequently

(3.5) $$\theta(r)f(x,s) \le a|r|^{p^*} + \tilde{d}(x)|\theta(r)|$$

for $s \in [0,r]$. The same argument yields (3.5) for $s \in [r,0]$ when $r < 0$. A similar relation can also be deduced from (f_1) when $p=N$. Indeed one then has

$$sf(x,s) \le aA(\tilde{b}|s|) + \tilde{e}(x)|s|$$

for some $\tilde{b} \in \mathbb{R}$ and $\tilde{e}(x) \in L_{\bar{A}}(\Omega)$; consequently, by the argument above, one obtains, for some $\hat{e}(x) \in L_{\bar{A}}(\Omega)$,

(3.6) $$rf(x,s) \le aA(\hat{b}|r|) + \hat{e}(x)|r|$$

for $s \in [0,r]$ or $s \in [r,0]$ depending on the sign of r.

We now consider the differential quotient (3.4). Suppose $p < N$. By the mean value theorem, one has

(3.7) $$(F(x,u+tv)-F(x,u))/t = f(x,u+\tau v)v$$

for some $\tau = \tau(x,u,t,\varphi,\alpha,\theta) \in]0,t[$. Let

$$\Omega_1 = \{x \in \Omega;\ |\alpha\theta(u(x))| \le |\varphi(x)|\}.$$

Since $|\theta(s)| \to +\infty$ as $|s| \to +\infty$, $u \in L^\infty(\Omega_1)$. It then follows from (3.7) and (f_0) that there exists $\ell(x) \in L^1(\Omega_1)$ such that

$$|(F(x,u+tv)-F(x,u))/t| \le \ell(x)$$

on Ω_1. Let us now consider $x \in \Omega \setminus \Omega_1$. Thus

(3.8) $$|\varphi(x)| < |\alpha\theta(u(x))|.$$

We claim that

(3.9) $$0 \le u(x)+\tau v(x) \le u(x) \text{ if } u(x) > 0,$$

(3.10) $$u(x) \le u(x)+\tau v(x) \le 0 \text{ if } u(x) < 0.$$

Indeed, omitting to write x for simplicity, we have, by (3.8) and the relation $|\theta(s)| \le K|s|$,

(3.11) $$|\varphi-\alpha\theta(u)| \le 2|\alpha\theta(u)| \le |u|,$$

which yields the first inequality in (3.9) and the second in (3.10). By considering each possibility $\varphi \ge 0$ or $\varphi < 0$, $\alpha\theta(u) > 0$ or $\alpha\theta(u) < 0$, one easily deduces from (3.8) that

(3.12) $$\operatorname{sgn}(\varphi-\alpha\theta(u)) = -\operatorname{sgn} u,$$

which yields the remaining inequalities of (3.9),(3.10). Now, using (3.12) and (3.11), one gets

$$f(x,u+\tau v)v \geq -[f(x,u+\tau v)v]^- \geq -[f(x,u+\tau v)(-2\alpha\theta(u))]^-.$$

Moreover it follows from (3.9),(3.10) that (3.5) can be applied to the right-hand side of the above inequality (with $r=u(x)$ and $s=u(x)+\tau v(x)$). One obtains in this way

$$f(x,u(x)+\tau v(x))v(x) \geq -2\alpha a|u(x)|^{p^*}-2\alpha\tilde{d}(x)|\theta(u(x))|.$$

This shows that (3.7) is on $\Omega\backslash\Omega_1$ greater than some L^1 function independant of t. Similar argument when p=N: one replaces $\theta(u)$ by u and uses (3.6). Q.E.D.

PROOF OF PROPOSITION 3.1. Let us consider the case p<N. Let u be a minimum of Φ over $W_0^{1,p}(\Omega)$. This implies that $\Phi(u)$ is finite and consequently that $F(x,u)\in L^1(\Omega)$. Take $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $0<\alpha\leq 1/2K$ and write $v=\varphi-\alpha\theta(u)$. One has, for $t\in]0,1[$,

$$0 \leq \frac{\Phi(u+tv)-\Phi(u)}{t}$$
$$= \frac{1}{p}\int_\Omega \frac{|\nabla(u+tv)|^p-|\nabla u|^p}{t} - \int_\Omega \frac{F(x,u+tv)-F(x,u)}{t} - \langle h,v\rangle .$$

By lemma 3.3, Fatou's lemma can be applied to the second term of the right-hand side when $t\to 0$. One deduces in this way $f(x,u)v\in L^1(\Omega)$ and

(3.13) $$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v \geq \int_\Omega f(x,u)v+\langle h,v\rangle .$$

Taking $\varphi=0$, one gets $f(x,u)\theta(u) \in L^1(\Omega)$. This implies, using (f_0), that $f(x,u) \in L^1(\Omega)$. Letting now $\alpha\to 0$ in (3.13) and replacing φ by $-\varphi$, one gets (3.3). Since $f(x,u) \in L^1(\Omega)$ and $f(x,u)\theta(u) \in L^1(\Omega)$, it follows from a lemma by Brézis-Browder [Br-Br] that (3.3) also holds for $\varphi=\theta(u)$, which concludes the proof when p<N. The argument is similar when p=N. One simply replaces $\theta(u)$ by u. Q.E.D.

The proof of proposition 3.2 is based on the following variant of lemma 3.3.

LEMMA 3.4. <u>Same statement as lemma</u> 3.3 <u>except that</u> (f_1) <u>is replaced by</u> (f_2) <u>and that</u> α <u>is taken with</u> $-1/2K \leq \alpha < 0$.

PROOF. It is similar to that of *lemma* 3.3 and we simply indicate here the successive steps in the case p<N. One first deduces from (f_2) that

$$\theta(r)f(x,s) \geq -2^{p^*}a|r|^{p^*}-\tilde{d}(x)|\theta(r)|$$

for $s \in [r,2r]$ if $r > 0$ and for $s \in [2r,r]$ if $r < 0$. The study of (3.7) on Ω_1 is identical to that in the proof of lemma 3.3. Now, on $\Omega\backslash\Omega_1$, (3.9) and (3.10) become

$$u(x) \leq u(x)+\tau v(x) \leq 2u(x) \text{ if } u(x) > 0 ,$$
$$2u(x) \leq u(x)+\tau v(x) \leq u(x) \text{ if } u(x) < 0 .$$

The inequalities (3.11) are maintanied but (3.12) becomes

$$\mathrm{sgn}(\Upsilon-\alpha\theta(u))=\mathrm{sgn}\,u\ .$$

The rest of the proof is easily adapted. Q.E.D.

PROOF OF PROPOSITION 3.2. Identical to that of proposition 3.1 except that one takes $-1/2K \leq \alpha < 0$ and uses lemma 3.4. Q.E.D.

REMARK 3.5. It clearly suffices in the two propositions above that u be a local minimum of Φ, or more generally that $\Phi(u)$ be finite and that for each $w \in W_0^{1,p}(\Omega)$, $\Phi(u+tw) \geq \Phi(u)+o(|t|)$ when $|t|\to 0$. Condition (f_1) can also be replaced in proposition 3.1 by the requirement that, when $p<N$,

$$(3.14)\qquad \theta(r)f(x,s) \leq a|r|^{p^*}+d(x)|\theta(r)|+|F(x,r)|$$

for $s \in [0,r]$ if $r > 0$ and for $s \in [r,0]$ if $r<0$ (compare with (3.5)). Similar observation when $p=N$ and in proposition 3.2. A condition like (3.14) with $\theta(s)=s$, an exponent p in the right-hand side and $f(x,s)$ of the form $q(x)f(s)$ was considered in [Hem].

REMARK 3.6. Suppose $p<N$. Condition (1.4) with $b(x) \in L^{p^{*\prime}}(\Omega)$ implies that Φ is C^1 (cf.e.g.[De_3]). Any minimum u of Φ thus solves (1.1) in the usual variational sense. More generally if (F_*) holds and if

$$(3.15)\qquad |f(x,s)| \leq a|s|^{p^*}+b(x)$$

for some $a\in\mathbb{R}$, $b(x)\in L^1(\Omega)$, then Φ can be differentiated in the direction

of any testing function at any u where $\Phi(u)$ is finite. Consequently any minimum u of Φ solves (1.1) in the distribution sense. (By using Gronwall's inequality, one can show that the same is true if (3.15) is replaced by

$$(3.16)\qquad |f(x,s)| \leq a|s|^{p^*}+b(x)+c|F(x,b)|\ ;$$

this was pointed out to us by M.Willem; cf.[Wi] for an argument of this type). It is not clear whether the one-sided growth conditions (f_1) and (f_2) can be weakened further so as to reach, as in the two-sided growth condition (3.15), the limiting exponent p^*. Observe that when $p=N$, the limiting growth is reached in (f_1) and (f_2).

REMARK 3.7. Here is another situation where any minimum of Φ solves (1.1). Suppose f autonomous and $F(s)/|s|\to-\infty$ as $|s|\to+\infty$, in a monotonic way for $|s|$ large. Let $h(x)\in L^\infty(\Omega)$. Then any minimum u of Φ (which exists, e.g. by proposition 2.2) is essentially bounded and consequently solves (1.1) in the usual variational sense. Indeed assume by contradiction u unbounded. Let s_0 be such that $F(s)+h(x)s <$ $F(s_0)+h(x)s_0$ for a.e. x and all $|s|>s_0$ and denote by u_{s_0} the truncated function defined by $u_{s_0}(x)=u(x)$ if $-s_0 \leq u(x) \leq s_0$, $u_{s_0}(x)=s_0$ if $u(x) > s_0$, $u_{s_0}(x)=-s_0$ if $u(x) < -s_0$. It easily follows from the unboundedness of u that $\Phi(u_{s_0}) < \Phi(u)$, a contradiction.

4. EXISTENCE THEOREM FOR (1.1).

The following theorem is a direct consequence of the results of the previous two sections.

THEOREM 4.1. *Let Ω be as in section 2. Suppose (f_0). Suppose also either $(G'_1),(G''_1)$ or $(G'_2),(G''_2)$ or $(G'_3),(G''_3)$. Finally suppose (f_1) or (f_2). Then there exists $u \in W_0^{1,p}(\Omega)$ which solves (1.1) (in the sense that the properties stated in the conclusion of proposition 3.1 hold) and which minimizes Φ over $W_0^{1,p}(\Omega)$.*

REMARK 4.2. When $p=2$, theorem 4.1 remains true in an arbitrary bounded open set Ω and for an uniformly elliptic linear operator of the form

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+a_0(x)u$$

with $a_{ij}(x)=a_{ji}(x) \in L^{\infty}(\Omega)$ and $a_0(x) \in L^{N/2}(\Omega)$. See e.g.[De$_2$] for the corresponding properties of λ_1,φ_1. Extension to a higher order symmetric uniformly elliptic linear operator in divergence form can also easily be given. The nonresonance or resonance conditions however must be slightly modified in order to take into account among other things the fact that the eigenfunctions corresponding to λ_1 may change sign in Ω.

Theorem 4.1 provides improvements of several known results. Let us first make some comparisons in the semilinear case $p=2$ i.e. for

problem (1.8). In the case $(G'_1),(G''_1)$, with $\Omega_+=\Omega_-$, and when $(f_1),(f_2)$ are replaced by a strenghthened form of the two-sided growth condition (1.4), theorem 4.1 reduces to a result of [M-W-W]. In the case of nonresonance or resonance conditions analogous to $(G'_1),(G''_1)$ or $(G'_3),(G''_3)$ but bearing on f and when (f_1) holds with $\theta(s)=s$ and with $|s|^2$ with right-hand side, theorem 4.1 reduces to results of [De-Go$_1$] (see also [Br-Ni]). Under the same strenghthened form of (f_1) and a nonresonance condition analogous to $(G'_2),(G''_2)$ but bearing on f, with a comparison function ψ of the power type, theorem 4.1 reduces to a result in [Go$_1$].

There have been rather few works dealing with problem (1.1) near resonance in the quasilinear case $p\neq2$. Jumping nonlinearities are considered in [Dr] in the O.D.E. case N=1. For arbitrary N's, the nonlinear Fredholm alternative for homogenous operators (cf.[F-N-S-S]) implies that if $f(x,s)=\lambda|s|^{p-2}s$ with $\lambda < \lambda_1$ (or more generally with λ different of an eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$), then (1.1) is solvable for any $h \in W_0^{-1,p'}(\Omega)$. The only result which goes beyond this general one seems to be that of [B-D-G-K]. It is shown there that if (1.4) holds with $|s|^{p-1}$ in the right-hand side and if, for some $\beta < \lambda_1$,

$$\limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \beta ,$$

then (1.1) is solvable for any $h \in L^{p'}(\Omega)$. Theorem 4.1 clearly improves this result from several respects. The case where f(x,s) lies asymtotically between two consecutive eigenvalues of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ is also considered in [B-D-G-K]. We observe in this respect that for $p \neq 2$ and $N \geq 2$, the existence of two such consecutive eigenvalues has been established only for λ_1 and λ_2 (cf. [An_1],[An_2]).

REMARK 4.3. In a preliminary version of theorem 4.1 we used a different method of proof which was based on a special truncation of the nonlinearity $f(x,s)$ (as in [An_2]) and a theorem of De La Vallee Poussin (as in [$De\text{-}Go_1$]). One interest of this truncation (which bears on f) is that it preserves the nonresonance conditions (which bear on F).

REMARK 4.4. A particular case of theorem 4.1 has recently been used in [$De\text{-}Go_3$] as an intermediate step to show that under the sole assumption

$$\limsup_{s \to \pm\infty} 2F(s)/s^2 < \lambda_1 \quad ,$$

the autonomous problem

$$\begin{cases} -\Delta u = f(u) + h(x) \text{ in } \Omega , \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

where $h(x) \in L^\infty(\Omega)$ always has a solution. The solution constructed in [$De\text{-}Go_3$] however may not correspond to a minimum of the functional.

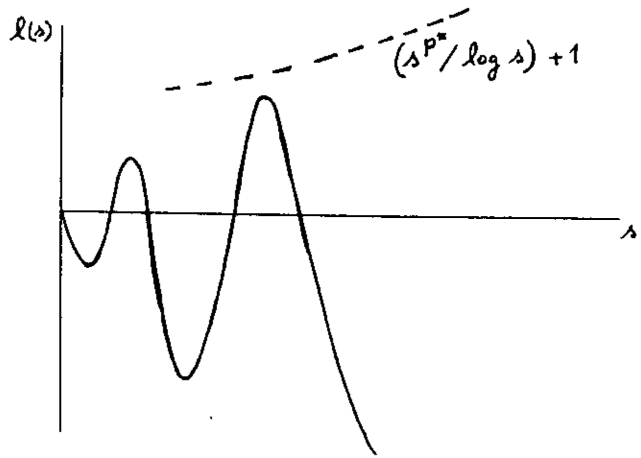
We conclude this section with some examples.

EXAMPLE 4.5. Theorem 4.1 applies in all the situations considered in examples 2.5, 2.10 and 2.11. No strong nonlinearity is involved there.

EXAMPLE 4.6. Let $1 < p < N$ and consider a continuous function $\ell : \mathbb{R}^+ \to \mathbb{R}$ with $\ell(0)=0$, $\ell(s) \leq (s^{p^*}/\log s)+1$ for s large and

$$\limsup_{s \to +\infty} \frac{p}{s^p} \int_0^s \ell(t)dt < \lambda_1 .$$

The latter condition is verified for instance if ℓ is oscillating as in the following picture



and if the area of each negative bump is greater in absolute value than the area of the following positive bump. Note that the growth of ℓ is unrestricted from below. Let k be another function on $\mathbb{R}^+$ satisfying the same conditions. Define

$$f(x,s) = \begin{cases} \ell(s) \text{ is } x \in \Omega^+ \text{ and } s \geq 0 , \\ -k(-s) \text{ if } x \in \Omega^- \text{ and } s \leq 0 , \\ \lambda_1 |s|^{p-2}s \text{ otherwise} , \end{cases}$$

where Ω^+ and Ω^- are subsets of Ω of positive masure. Then theorem 4.1 applies. This is still the case if the growth restriction on ℓ and k is from below instead of from above : $\ell(s) \geq -(s^{p^*}/\log s)-1$ and similarly for k. When p=N, the function $(s^{p^*}/\log s)+1$ which controls the one-sided growth of the nonlinearity is replaced by $(\exp s^{N/(N-1)})+1$.

REFERENCES

[A-L-P] S.AHMAD, A.C.LAZER and J.L.PAUL, Elementary critical point theory and perturbations of elliptic boundary value problems at resonance, Ind.Univ.Math.J., 25 (1976), 933-944.

[An_1] A.ANANE, Simplicité et isolation de la première valeur propre du p-laplacien avec poids, C.R.Acad.Sc.Paris, 305 (1987), 725-728.

[An_2] A.ANANE, Etude des valeurs propres et de la résonance pour l'opérateur p-laplacien, Thèse doct., Université Libre de Bruxelles, 1987.

[Be-Go] A.BENKIRANE and J.-P.GOSSEZ, An approximation theorem in higher-order Orlicz-Sobolev spaces and applications, Studia Mat., to appear.

[B-D-G-K] L.BOCCARDO, P.DRABEK, D.GIACHETTI and M.KUCERA, Generalization of Fredholm alternative for nonlinear differential operators, Nonli.An.Th., Meth.,Appl., 10 (1986), 1083-1103.

[Br-Br] H.BREZIS and F.BROWDER, Some properties of higher-order Sobolev spaces, J.Math.Pures Appl., 61 (1982), 245-259.

[Br-Ni] H.BREZIS and L.NIRENBERG, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, Ann.Sc.Norm.Sup.Pisa, 5 (1978), 225-326.

[Br] F.BROWDER, Existence theory for boundary value problems for quasilinear elliptic systems with strongly nonlinear lower order terms, Proc.Symp.Pure Math., Amer.Math.Soc., 23 (1971), 269-286.

[De_1] D.DE FIGUEIREDO, The Dirichlet problem for nonlinear elliptic equations: a Hilbert space approach, Springer Lect. Notes Math., 445 (1974), 144-165.

[De_2] D.DE FIGUEIREDO, Positive solutions of semilinear elliptic problems, Springer Lect.Notes Math., 957 (1981), 34-87.

[De_3] D.DE FIGUEIREDO, Lectures on the Ekeland variational principle with applications and détours, I.C.T.P., Trieste, 1988.

[$De\text{-}Go_1$] D.DE FIGUEIREDO and J.-P.GOSSEZ, Nonlinear perturbations of a linear elliptic problem near its first eigenvalue, J.Diff.Equat., 30 (1978), 1-19.

[$De\text{-}Go_2$] D.DE FIGUEIREDO and J.-P.GOSSEZ, Nonresonance below the first eigenvalue for a semilinear elliptic problem, Math.Ann., 281 (1988), 589-610.

[De-Go$_3$] D.DE FIGUEIREDO and J.-P.GOSSEZ, Un problème elliptique semilinéaire sans condition de croissance, C.R.Ac.Sc.Paris, à paraître.

[Dr] P.DRABEK, Remarks on nonlinear noncoercive problems with jumping nonlinearities, Com.Math.Univ.Carol., 25 (1984), 373-399.

[Fo-Go] A.FONDA and J.-P.GOSSEZ, Semicoercive variational problems at resonance: an abstract approach, to appear.

[F-N-S-S] S.FUCIK, J.NECAS, J.SOUCEK and S.SOUCEK, Spectral analysis of nonlinear operators, Springer Lect.Notes Math., 346 (1973).

[Go$_1$] J.-P.GOSSEZ, Some nonlinear differential equations with resonance at the first eigenvalue, Conf.Sem.Mat.Univ.Bari, 167 (1979), 355-389.

[Go$_2$] J.-P.GOSSEZ, Nonresonance near the first eigenvalue of a second order elliptic problem, Springer Lect.Notes Math., 1324 (1988), 97-104.

[Ha] A.HAMMERSTEIN, Nichtlineare Integralgleichungen nebst Anwendungen, Acta Math., 54 (1930), 117-176.

[Hem] R.HEMPEL, Eine Variationsmethode für elliptische Differentialoperatoren mit strengen Nicht linearitäten, J.Math., 333 (1981), 179-190.

[Hes] P.HESS, A strongly nonlinear elliptic boundary value problem, J.Math.Anal.Appl., 43 (1973), 241-249.

[Ka-Wa] J.KAZDAN and F.WARNER, Remarks on some quasilinear elliptic equations, Com.Pure Appl.Math. 28 (1975), 567-597.

[Kr-Ru] M.KRASNOSELSKII and Y.RUTICKII, Convex functions and Orlicz spaces, Noordhoff, 1961.

[Li] L.LICHTENSTEIN, Uber einige Existenzprobleme der Variationsrechnung, Methode der unendlichvielen Variablen, J.Reine Angew.Math., 145 (1915), 24-85.

[Ma] J.MAWHIN, Nonlinear variational two-point boundary value problems, Proc.Conf.Variat.Methods, Paris, 1988, à paraître.

[M-W-W] J.MAWHIN, J.WARD and M.WILLEM, Variational methods and semilinear equaitons, Arch.Rat.Mech.Anal., 95 (1986), 269-277.

[Mo] J.MOSER, A sharp form of an inequality by N.Trudinger, Ind.Univ.Math.J., 20 (1971), 1077-1092.

[Tr] N.TRUDINGER, On imbedding into Orlicz spaces and some applications, J.Math.Mech., 17 (1967), 473-484.

[We] J.WEBB, Boundary value problems for strongly nonlinear elliptic equations, J.London Math.Soc., 21 (1980), 123-132.

[Wi] M.WILLEM, Analyse convexe et optimisation, Ciaco, 1987.