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Some Variational Aspects of Hydrodynamics

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In these lectures we touch on some existence and stability properties of equations arising in fluid dynamics in which variational methods play a role. First we look at flows in which vorticity plays a central role. Next we examine the classical problems of solitary waves in fluids with a free boundary. Finally, we show how solitary waves for some model equations can be shown to be stable. The references listed are sparse, but many, in turn, have good bibliographies.

We begin with flows governed by the Euler equations (see [4]):

$$\vec{q}_t + (\vec{q}\cdot\nabla)\vec{q} = -\nabla p \tag{1}$$

or, equivalently,

$$\vec{q}_t = \vec{q}\times \mathrm{curl}\vec{q} - \nabla(p + \frac{1}{2}|\vec{q}|^2) \tag{2}$$

together with

$$\nabla\cdot\vec{q} = 0 \tag{3}$$

where $\vec{q}$ is the fluid velocity, and p is the pressure. The vorticity $\vec{\omega} = \mathrm{curl}\vec{q}$ satisfies

$$\vec{\omega}_t = \mathrm{curl}(\vec{q}\times\vec{\omega}) \equiv (\vec{\omega}\cdot\nabla)\vec{q} - (\vec{q}\cdot\nabla)\vec{\omega}. \tag{4}$$

For the most part we restrict attention to two-dimensional flows. In that case the momentum equations can also be written

$$\begin{aligned} u_t + uu_x + vu_y &= -p_x \\ v_t + uv_x + vv_y &= -p_y \end{aligned} \tag{5}$$

where $\vec{q} = (u, v)$. The fluid under consideration is assumed to be incompressible so that there is a stream function Ψ for which

$$\frac{\partial\Psi}{\partial y} = u, \frac{\partial\Psi}{\partial x} = -v. \tag{6}$$

Suppose there is an ambient flow corresponding to a stream function ψ and that the total flow is governed by a stream function $\Psi = \psi + \tilde{\psi}$, where $\tilde{\psi}$ generates an irrotational flow and ψ, vanishing on the boundary of the flow domain D under consideration, represents the part of the stream function corresponding to the vorticity. If $\vec{k}$ is a vector perpendicular

to the plane of the flow, then $\vec{\omega} = \omega\vec{k}$ where the magnitude of the vorticity is $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ and thus

$$-\Delta\psi = \omega \tag{7}$$

In this case $(\vec{\omega}\cdot\nabla)\mathbf{q} = 0$ and so (4) becomes

$$\omega_t = (\mathbf{q}\cdot\nabla)\omega. \tag{8}$$

which can also be written

$$\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = 0 \tag{9}$$

Using the Jacobian derivative notation the flow equation can be expressed as

$$\frac{\partial\omega}{\partial t} + \frac{\partial(\omega,\Psi)}{\partial(x,y)} = 0 \tag{10}$$

It should be noted that the flow equation (10) can be put in a Hamiltonian form (see [7],[12]). Let G be the Green's function for the Laplacian in the domain D so that (7) has the solution $\psi = G\omega$. Suppose ω is a solution of the evolution equation (10) with $\Psi = G\omega + \tilde{\psi}$ and let $\mathbf{q} = (u(t,x,y), v(t,x,y))$ denote the corresponding flow field. Consider a level set $\Gamma_a(t) = \{(x,y) : \omega(x,y) = a\}$ and suppose it is a closed curve. The curve $\Gamma_a(t)$ evolves with time. According to equation (9), ω does not change in time along an integral curve of the vector field (u, v). Thus points on $\Gamma_a(t)$ merely move on integral curves and since by (3) the field has zero divergence, the area enclosed by $\Gamma_a(t)$ does not change in time. It follows that the associated vorticity functions $\omega(t,x,y)$ at different times are merely rearrangements of each other. Let

$$\begin{aligned} E(\omega) &= \int\int_D \frac{1}{2}|\nabla(\psi+\tilde{\psi})|^2 - |\nabla(\tilde{\psi})|^2 \\ &= \int\int_D [\omega G\omega + \tilde{\psi}\omega] \end{aligned} \tag{11}$$

denote the excess kinetic energy of the flow due to the vorticity ω. Let ϕ be a test function in D (smooth in $\bar{D}$ and vanishing on the boundary ∂D). Consider ϕ as a trial stream function generating a divergence free vector field

$$\tilde{u} = \frac{\partial\phi}{\partial y}, \quad \tilde{v} = -\frac{\partial\phi}{\partial x}.$$

For a given vorticity $\omega(x,y)$ and for τ in a neighborhood of $\tau = 0$ solve the evolution equation

$$\begin{aligned} \frac{\partial\tilde{\omega}}{\partial t} + \frac{\partial(\tilde{\omega},\phi)}{\partial(x,y)} &= 0 \quad \text{in} \quad D\times[-a,a] \\ \tilde{\omega}|_{\tau=0} &= \omega \end{aligned}$$

The distributions $\tilde{\omega}$ corresponding to different trial functions ϕ are rearrangements of ω, that is, 'isovortical' with ω. One finds that

$$\frac{d}{d\tau}E(\tilde{\omega})|_{\tau=0} = \int\int_D \omega\frac{\partial(G\omega+\bar{\psi},\phi)}{\partial(x,y)}dxdy.$$

Thus the requirement that E be stationary with respect to isovortical variations is the weak form

$$0 = \int\int_D \omega\frac{\partial(G\omega+\bar{\psi},\phi)}{\partial(x,y)}dxdy. \tag{12}$$

of the equation (10) for steady solutions. This characterization is due to Arnold (a reference can be found in [13] which we use as a reference for the variational treatment of vortex motion that follows). For smooth ω and Ψ one verifies that

$$\int\int_D \omega\frac{\partial(\Psi,\phi)}{\partial(x,y)}dxdy. = \int\int_D \phi\frac{\partial(\omega,\Psi)}{\partial(x,y)}dxdy. \tag{13}$$

and so the equation (12) expresses the requirement that ω and $\Psi = G\omega+\psi$ be functionally related. We seek a solution for which the relation takes the form

$$\omega = \lambda f_s(G\omega+\bar{\psi}+\text{constant}) \tag{14}$$

where f_s is the derivative of a function $f \in C^2[0,\infty]$ with $f(0) = f_s(0) = 0$, $f_{ss} > 0$ for $s > 0$, and $ms^r \le f(s) \le Ms^r$ for some $1 < r < \infty$, $0 < m \le M < \infty$. Let f^* be the function conjugate to f so that

$$f^*(\sigma) = sup_s[s\sigma - f(s)]$$

For example, if $f(s) = s^r/r$, then $f^*(\sigma) = s^{r'}/r'$ where $1/r + 1/r' = 1$.

Consider the functionals

$$\begin{aligned}\Phi_\lambda(\omega) &\equiv E(\omega) - \int\int_D \lambda f^*(\omega/\lambda)\\ C(\omega) &\equiv \int\int_D \omega.\end{aligned} \tag{15}$$

The functional Φ_λ represents the energy modified by a generalized enstrophy integral and C is the total circulation. These functionals are invariant under the flow. For functionals of the form $\int\int g(\omega)$ this follows from the fact that the evolving vorticity is a rearrangement of the initial one. The invariance is not needed for the existence of stationary solutions, but is relevant to stability considerations. Consider

Problem P: maximize $\Phi_\lambda(\omega)$ subject to $\omega \ge 0$ in D and $C(\omega) = 1$.

The condition that ω be a maximizer gives the variational condition $G\omega + \bar{\psi} - f_\sigma^*(\omega/\lambda) = \mu\cdot 1$ where $\omega > 0$ and $G\omega + \bar{\psi} \le \mu \cdot 1$ where $\omega = 0$. Thus

$$\begin{aligned}G\omega+\bar{\psi}-\mu &= f_\sigma^*(\omega/\lambda), \quad \omega > 0\\ G\omega+\bar{\psi}-\mu &\le 0, \quad \omega = 0\end{aligned} \tag{16}$$

The functions f and f^*, being conjugate, are related through $\sigma = f_s(s)$ and $s = f_\sigma^*(\sigma)$. Thus the last condition can be expressed as

$$\omega/\lambda = f_s((G\omega+\bar{\psi}-\mu)^+)$$

where $s^+ \equiv \max\{s,0\}$. In a typical application the 'vortex core' $\Omega \equiv \{(x,y) : G\omega+\bar{\psi} > \mu$ is a compact subset of D.

We consider an iterative procedure for constructing a maximizer for Problem P. Let

$$K \equiv \{\omega \in L^{r'}(D) : 0 \le \omega, C(\omega) \equiv \int\int_D \omega dxdy = 1\}.$$

Iterative Procedure: Given an arbitrary $\omega^0 \in K$, let

$$\omega^j = \lambda f_s((G\omega^{j-1}+\bar{\psi}-\mu^j)^+) \in K, \quad j = 1,2,..$$

where $\mu^j \in \mathbf{R}$ is chosen so that $C(\omega^j) = 1$. Note that one can view each stage of the iterative procedure as involving two steps: first solve the linear Dirichlet problem $-\Delta\psi^{j-1} = \omega^{j-1}$ in D with $\omega^{j-1} = 0$ on ∂D and then adjust the real parameter μ^j so that $C(\omega^j) = 1$.

The procedure also has a variational formulation. The function ω^j is the maximizer of the problem

$$\max_{\tilde{\omega}\in K}\int\int_D \left[\tilde{\omega}(G\omega^{j-1}+\bar{\psi}) - \lambda f^*\left(\frac{\tilde{\omega}}{\lambda}\right)\right]. \tag{17}$$

with μ^j being the Lagrange multiplier corresponding to $C(\tilde{\omega}) = 1$.

The next lemma exhibits a crucial monotonicity property of the iterations. We use the following 'energy' norm

$$\|\omega\|_G \equiv \{\int\int_D \omega G\omega\}^{1/2} = \{\int\int_D |\nabla G\omega|^2\}^{1/2}$$

Lemma 1. *The sequence ω^j defined by the iterative procedure satisfies*

$$\frac{1}{2}\|\omega^j - \omega^{j-1}\|_G^2 \le \Phi_\lambda(\omega^j) - \Phi_\lambda(\omega^{j-1}), \quad j = 1,2,..$$

Proof. From the definition of Φ_λ

$$\begin{aligned}\Phi_\lambda(\omega^j) - \Phi_\lambda(\omega^{j-1}) &= \int\int_D \left[\frac{1}{2}\omega^j G\omega^j - \bar{\psi}\omega^j - \lambda f^*\left(\frac{\omega^j}{\lambda}\right)\right] \\ &\quad - \int\int_D \left[\frac{1}{2}\omega^{j-1} G\omega^{j-1} - \bar{\psi}\omega^{j-1} - \lambda f^*\left(\frac{\omega^{j-1}}{\lambda}\right)\right] \\ &= \int\int \frac{1}{2}(\omega^j - \omega^{j-1})G(\omega^j - \omega^{j-1}) \\ &\quad + \int\int_D \left[\omega^j(G\omega^{j-1} + \bar{\psi}) - \lambda f^*\left(\frac{\omega^j}{\lambda}\right)\right] \\ &\quad - \int\int_D \left[\omega^{j-1}(G\omega^{j-1} + \bar{\psi}) - \lambda f\left(\frac{\omega^{j-1}}{\lambda}\right)\right] \\ &\geq \frac{1}{2}\|\omega^j - \omega^{j-1}\|_G^2.\end{aligned}$$

The last inequality follows since ω^j is the solution of problem (17) and thus gives the largest value of the integrand in (17).

It follows that $\Phi_\lambda(\omega^j)$ is increasing as $j \to \infty$. Moreover, it is bounded. To see this first note that estimates for elliptic equations combined with Sobolev inequalities show

$$\|G\omega\|_{L^\infty} \leq C\|\omega\|_{L^{r'}}.$$

Since $\int\int_D \omega = 1$,

$$\begin{aligned}\int\int \omega G\omega &\leq \int\int \omega\|G\omega\|_{L^\infty} \\ &\leq C\|\omega\|_{L^{r'}}.\end{aligned}$$

Using the growth assumption on f and $r' - 1 = r'r^{-1}$ has

$$\begin{aligned}\int\int_D \lambda f^*\left(\frac{\omega}{\lambda}\right) &\geq C'\int\int_D \lambda\left(\frac{\omega}{\lambda}\right)^{r'} \\ &= C'\lambda^{-r'/r}\int\int_D \omega^{r'}.\end{aligned}$$

So

(18) $$\Phi_\lambda(\omega) \leq C\|\omega\|_{L^{r'}} - C'\lambda^{-r'/r}\|\omega\|_{L^{r'}}^{r'},$$

with $1 < r' < \infty$.

Let Ω denote the set of extremals for the maximization problem P and let

$$\text{dist}_G(\omega, \Omega) = \inf_{\tilde{\omega}\in\Omega} \|\omega - \tilde{\omega}\|_G$$

Theorem 1. *$\Omega \neq \phi$ and for any ω^0, $\text{dist}_G(\omega^j, \Omega) \to 0$ as $j \to +\infty$.*

Proof. Since $\Phi_\lambda(\omega^j)$ increases with j, it follows from the bound (18) that $\|\omega^j\|_{L^{r'}} \leq N_0$ for some N_0 depending on the initial ω^0. Hence there is a subsequence j_k so that

$$\omega^{j_k} \rightharpoonup \omega^*, \quad \omega^{j_k - 1} \rightharpoonup \omega^{**}$$

as $k \to \infty$, the half arrow denoting weak convergence in $L^{r'}$. The map G is compact and so we have the strong L^r convergence

$$G\omega^{j_k} \to G\omega^*, \quad G\omega^{j_k-1} \to G\omega^{**}.$$

The weak and strong convergence together give

$$\|\omega^* - \omega^{**}\|_G = \lim((\omega^{j_k} - \omega^{j_k-1}), G(\omega^{j_k} - \omega^{j_k-1})) = 0.$$

So $\omega^* = \omega^{**}$. The function ω^* is the solution of

$$\max_{\tilde{\omega}\in K} \int\int_D \left[\tilde{\omega}(G\omega^* + \bar{\psi}) - \lambda f^*\left(\frac{\tilde{\omega}}{\lambda}\right)\right].$$

To see this note that for any $\tilde{\omega} \in K$

$$\begin{aligned}&\int\int_D \left[\tilde{\omega}(G\omega^* + \bar{\psi}) - \lambda f^*\left(\frac{\tilde{\omega}}{\lambda}\right)\right] \\ &= \lim_{k\to\infty}\int\int_D \left[\tilde{\omega}(G\omega^{j_k-1} + \bar{\psi}) - \lambda f^*\left(\frac{\tilde{\omega}}{\lambda}\right)\right] \\ &\leq \lim_{k\to\infty}\int\int_D \left[\omega^{j_k}(G\omega^{j_k-1} + \bar{\psi}) - \lambda f^*\left(\frac{\omega^{j_k}}{\lambda}\right)\right] \\ &\leq \int\int_D \left[\omega^*(G\omega^* + \bar{\psi}) - \lambda f^*\left(\frac{\omega^*}{\lambda}\right)\right]\end{aligned}$$

using the fact that ω^{j_k} is the maximizer when G is applied to ω^{j_k-1} and the fact that f^* is convex. The limit $\omega^* \in K$ so it is a maximizer.

Finally, it is not possible for any subsequence ω^{j_k} of ω^j to satisfy $\text{dist}_G(\omega^{j_k}, \Omega) \geq \delta > 0$ for we have seen that a further subsequence will converge in the G norm to a maximizer. Hence $\text{dist}_G(\omega^j, \Omega) \to 0$ as $j \to \infty$.

Theorem 2. *If the set of extremals Ω^0 arising from a single initial point ω^0 contains an isolated point ω^*. Then $\|\omega^j - \omega^*\|_G \longrightarrow 0$ as $j \to \infty$.*

Proof. Suppose $\omega^* \in \Omega^0$ is isolated and let N_1, N_2 be disjoint neighborhoods of $\{\omega^*\}$ and $\Omega^0\backslash\{\omega^*\}$. We have seen that for all large j, $\omega^j \in N_1 \cup N_2$. Suppose $\text{dist}_G(N_1, N_2) = \delta > 0$.

For some j_0, $\|\omega^j - \omega^{j-1}\|_G < \delta/2$ for $j \geq j_0$ and so there can be no limit point in N_2.
Variants:

1) To obtain a vorticity function $\omega = \lambda\chi_\Omega$, where χ_Ω is the characteristic of a set, one can pose the problem

$$\text{maximize } E(\omega) \text{ subject to } 0 \leq \omega \leq \lambda,\ C(\omega) = 1.$$

2)Suppose $\eta = \eta(x, y)$ with $\Delta\eta = 0$ is fixed and define a generalized impulse

$$I(\omega) = \int\int_D \eta\omega$$

Then the problem

$$\text{maximize } \Phi_\lambda \text{ subject to } \omega \geq 0 \text{ in } D,\ C(\omega) = 1, I(\omega) = m$$

leads to two Lagrange multipiers μ and c and

$$\omega = \lambda f_s((G\omega - c\eta - \mu)^+)$$

results, so the effective ambient flow is given by $\bar{\psi} = -c\eta$.

3)Analogous formulations are available for axisymmetric flows in three dimensions.

A class of problems which has much in common with steady flows with vorticity are those involving waves which progress without change of shape in fluids with constant density and a free upper surface or in fluids with density stratification. Further details about the physical problem may be found in [2],[3], and [8]. These problems are typically posed in coordinates based in a moving wave so that the flow becomes steady. The result is a nonlinear elliptic equation or an integral equation, with an eigenvalue parameter, the parameter being related to the speed of the wave. Existence theories using methods of global bifurcation have proven very effective in treating these waves and allow one to obtain results on limiting configurations as a bifurcation parameter or a wave amplitude approaches infinity (see [2],[3]). The existence theory using variational methods is not as precise and is less satisfactory than that for vortex flows. In part this is due to the presence of unbounded domains, entailing noncompactness, but is also a reflection of the difficulty of finding function spaces in which the attendant functionals are tractable. There are several different variational schemes (for example [6],[10]) that formally give solutions to the steady equations of motion for the classical surface wave, some of them having the desirable feature of involving invariant functionals, but they do not seem to be easily suited to use in rigorous existence proofs. One can base an existence proof roughly on maximizing potential energy subject to kinetic energy remaining fixed, but as we shall see, this program uses intermediate truncations and results in waves of limited amplitude.

Consider first a fluid of constant density ρ which occupies a flow domain

$$S = \{(x, y) : x \in \mathbf{R}, 0 < y < Y(x)\}$$

where Y, or equivalently the 'fluid-air' interface

$$\Gamma = \{(x, Y(x)) : x \in \mathbf{R}\},$$

is an unknown to be determined as part of an eventual solution. The flow, still being assumed incompressible, is governed by a stream function as in (6). Moreover, in the case of constant density, it is assumed irrotational so that $\Delta\psi = 0$ in the region S. On Γ one assumes ψ = constant, so that Γ is a streamline and further assumes

$$\frac{1}{2}|\nabla\psi|^2 + \rho g y = \text{constant}, \tag{19}$$

where $g > 0$ is the gravitational constant and ρ is the density. The last condition is merely the requirement that the pressure at the upper surface be equal to atmospheric pressure, assumed constant. We assume that the fluid region has a limiting height as $|x| \to \infty$ and take this to be the unit of length so that $Y(x) \to 1$ as $|x| \to \infty$. A possible flow has velocity $\vec{q} = (c, 0)$ where c is an arbitrary real number, the velocity corresponding to the stream function $\Psi(y) \equiv cy$. Such a flow is called *trivial*. A steady vector field $\vec{q} = (u, v)$ which approaches $(c, 0)$ as $|x| \to \infty$ will, in 'laboratory' coordinates yield a traveling wave which progresses to the left with speed c without changing form. Typically problems involving free boundaries are analytically difficult and one may seek an alternate formulation. In this case and for the more general problem of waves in nondiffusive, stratified fluids an alternate formulation obviates the need to confront an unknown boundary.

We shall consider only flows for which no reversal occurs; that is, $u > 0$, or, equivalently, $\psi_y > 0$. For such flows one can solve for y as a function of the spatial variable x and the material coordinate ψ. The advantage of this semi-Lagrangian formulation is that the unknown interface function Y is now $y(x, \psi)$ evaluated at ψ = constant. The disadvantage is that Laplace's equation $\Delta\psi = 0$ is replaced by a singular, quasi-linear equation for $y(x, \psi)$. Let $Y(\psi) = \psi/c$ denote the function inverse to $\Psi = cy$. A final change of variables is performed: let $\eta = Y(\psi)$, and

$$w(x, \eta) = y(x, \Psi(\eta)) - \eta \tag{20}$$

so that w represents the deviation of the streamline height $y(x, \psi)$ from its value $Y(\psi)$ in a trivial flow. For ease of notation, we subsequently write x for x_1 and η for x_2.

To describe the problem in the new coordinates we define

$$f_1(\nabla w) = \frac{w_x}{1 + w_\eta} \quad \text{and} \quad f_2(\nabla w) = \frac{w_\eta}{1 + w_\eta} - \frac{|\nabla w|^2}{2(1 + w_\eta)^2}. \tag{21}$$

The problem becomes the following: find an eigenvalue $\lambda = \frac{g}{c^2}$ and a function $w(x, y)$, continuous in $\Omega = \mathbf{R} \times [-1, 0]$, satisfying

$$\frac{\partial}{\partial x}(f_1(\nabla w)) + \frac{\partial}{\partial \eta}(f_2(\nabla w)) = 0 \quad \text{in} \quad \Omega, \tag{22}$$

$$f_2(\nabla w) - \lambda w = 0 \quad \text{on} \quad \eta = 0, \tag{23}$$

$$\begin{aligned} &w(x,-1) = 0, \quad x \in \mathbf{R}. \\ &w(x,\eta) \to 0, \quad |x| \to \infty \end{aligned} \tag{24}$$

Note that the linearization of (22)-(24) about $w = 0$, omitting the the vanishing of w as $|x| \to \infty$, has a solution $w(x,\eta) = \eta$ with $\lambda = 1$ corresponding to a speed $c = \sqrt{g}$ ($\sqrt{gh}$ if the limiting fluid height at ∞ is h). This is the classical speed of infinitesimal long waves. The following result (from [15]) is stated both for waves of finite period k and for period $k = \infty$, the latter case being understood to be a limiting case providing a solitary wave. Let Ω_k denote the rectangle $[-k,k] \times (-1,0)$.

Theorem 3. *There is a positive constant $\tilde{R}$ and positive $k(R)$ such that for $0 < R \leq \tilde{R}$ and $k(R) < k \leq \infty$ the problem (22)(23) has a solution (λ, w) in $\mathbf{R} \times C^2(\bar{\Omega})$ satisfying*
1) the speed $c = \sqrt{g/\lambda}$ is supercritical: $c > \sqrt{g}(1 - C_1 R^{4/3})^{-1/2}$.
2) w has period $2k$ in x
3) $\int_{-1}^{0} \int_{-k}^{k} \rho(\eta) \frac{|\nabla w|^2}{1+w_\eta} = R^2$
4) $w(x,\eta) > 0$ and $w(-x,\eta) = w(x,\eta)$ in Ω,
$|w(x,\eta)| \leq C exp(-\beta|x|$, and $|\nabla w(x,\eta)| \leq C exp(-\beta|x|$ in Ω_k for constants C, β independent of k.

The proof follows from several propositions, given below. Formally, a critical point of

$$F(w) \equiv \int_{\Omega_k} f(\nabla w),$$

where

$$f(p_1,p_2) \equiv \frac{1}{2}\frac{p_1^2 + p_2^2}{1 + p_2},$$

satisfies the equation (22) in T. To incorporate the Bernoulli pressure condition at the fluid-air interface we incorporate a term reflecting the drop in density across the interface from a positive value ρ to zero in the air. The general equations for a heterogeneous fluid have a term $\lambda(d\rho/d\eta)w$ on the right side of (22). (see [3]). Let

$$\hat{\rho}(\eta) = \begin{cases} \rho, & 0 < \eta < 1, \\ 0, & \eta = 1 \end{cases}$$

Thus for a constant density of size ρ in the fluid we are led to a functional

$$B(w) \equiv \int\int_{\Omega_k} (-\frac{d}{d\eta}\hat{\rho})\frac{w^2}{2} = \int_{-k}^{k} \rho \frac{w^2}{2} dx$$

corresponding to a Dirac measure on the upper boundary. Now, formally

$$sup_{F(w)=const} B(w)$$

leads to a Lagrange multiplier λ and function w satisfying the equations (22),(23). However, B is unbounded on level sets of F. Note that $f(p_1,p_2)$, apart from being singular at $p_2 = -1$, has only linear growth in p_2 at ∞. To improve the behavior of the 'kinetic energy' we use a truncation. Let $\zeta(t)$ be a smooth, decreasing cutoff function, equal to 1 for $0 \leq t \leq 1$ and equal to zero for $t \geq 2$. We replace f in the functional F by

$$a(\eta,p_1,p_2) \equiv \zeta_r \frac{p_1^2 + p_2^2}{2(1 - sgn(\eta)p_2)} + (1 - \zeta_r)\frac{p_1^2 + p_2^2}{2}$$

where $\zeta_r = \zeta(p_1^2 + p_2^2/r^2)$. For small positive r the function a is globally convex in (p_1,p_2), quadratic near ∞, and coincides with f for $p_1^2 + p_2^2 < r^2$. Let $\tau(\eta)$ be a smooth, non-increasing, function on $(-\infty,0)$ which is zero for $\eta < -1$ and satisfies $\tau(0) = -1$. Let

$$\hat{\rho}_\epsilon(\eta) \equiv \rho(1 + \tau(\eta/\epsilon))$$

so that $\hat{\rho}(\eta) = lim_{\epsilon \to 0}\hat{\rho}_\epsilon(\eta)$ Extend the constant function $\rho(\eta) \equiv \rho$ to $0 < \eta < 1$ as an even function and extend $\hat{\rho}_\epsilon(\eta)$ as an odd function. Now let $\hat{\Omega}_k \equiv (-k,k) \times (-1,1)$ and define the functionals

$$\hat{A}(w) \equiv \int_{\Omega_k} a(\eta, \nabla w)$$

and

$$\hat{B}(w) \equiv \int_{\Omega_k} (-\frac{d}{d\eta}\hat{\rho}_\epsilon)\frac{w^2}{2}.$$

Let $H_k^e(\Omega)$ denote the space of functions in the Sobolev space H^1 which vanish on $\eta = 0$, are even in x, and have period $2k$ in x. Let $H_k^e(\hat{\Omega})$ denote similarly defined functions which are even in η as well.

Proposition 1. *For each $k > 0$ and $R > 0$ the problem*

$$\hat{A}'(w) = \lambda \hat{B}'(w)$$

has a solution (λ, w) satisfying $\lambda > 0$, $w \in H_k^e(\hat{\Omega})$, $A(w) = 2R^2$, and $w > 0$ in $\hat{\Omega}$. the function w is characterized by

$$\hat{B}(w) = sup_{\substack{v \in H_k^e \\ A(v)=2R^2}} \hat{B}(v^+)$$

where $v^+ = \max\{0,v\}$.

Proof. Reverting to $x = x_1, \eta = x_2$ and using the summation convention one sees that a standard variational procedure leads to a w with $A(w) = 2R^2$ satisfying

$$M(\phi) \equiv \int_{\hat{\Omega}_k} \rho a_i(\eta, \nabla w)\frac{\partial \phi}{\partial x_i} + \lambda(\frac{d}{d\eta}\hat{\rho}_\epsilon)w^+\phi = 0$$

for a suitable $\lambda > 0$ and all $\phi \in H_k^e(\hat{\Omega})$ where a_i denotes the partial derivative with resepct to p_i. One finds $a_1(x_2,-p_1,p_2) = -a_1(x_2,p_1,p_2)$ and $a_1(-x_2,-p_1,-p_2) = a_1(x_2,p_1,p_2)$ while a_2 has the opposite parity. Thus M annihilates all test functions which are odd in x_1 or in x_2 or in both and so is zero on all test functions. The weak maximum principle shows $w > 0$.

One can use elliptic estimates to bound the gradient of w in L^∞ in terms of L^2 integrals of the gradient, uniformly in ϵ and so pass to a limit as $\epsilon \to 0$ to obtain

Proposition 2. *For each $k > 0$ and $R > 0$ the problem*

$$A'(w) = \lambda B'(w)$$

has a solution (λ, w) satisfying $\lambda > 0$, $w \in H_k^\epsilon(\Omega) \cap C^2(\Omega) \cap C^{1,\alpha}(\Omega)$, $A(w) = R^2$, $w \geq 0$, and $w(-x,\eta) = w(x,\eta)$ in Ω. The function w is an extremal for by

(25) $$B(w) = \sup_{\substack{v \in H_k^\epsilon \\ A(v) = R^2}} \int_{-k}^{k} \rho \frac{v^2}{2} dx$$

and for any bounded subset $\Omega_1 \equiv [a,b] \times [-1,0]$ of Ω

(26) $$|w|^2_{C^{1,\alpha}} \leq C \int_{\Omega_2} |\nabla w|^2$$

where $\Omega_2 \equiv [a-1, b+1] \times [-1,0]$ of Ω.

The estimate (26) enables one to restrict the 'energy' R so that $A'(w) = F'(w)$. Moreover, one can, as with a Dirichlet integral, replace w by its symmetric, decreasing rearrangement $\hat{w}$ and still have an extremal.

Proposition 3. *There is an $R_0 > 0$ such that for each $k > 0$ and $0 < R < R_0$ the problem*

$$F'(w) = \lambda B'(w)$$

has a solution (λ, w) satisfying $\lambda > 0$, $w \in H_k^\epsilon(\Omega) \cap C^2(\Omega) \cap C^{1,\alpha}(\hat{\Omega})$, $F(w) = R^2$, $w > 0$, and $w(-x,\eta) = w(x,\eta)$ in Ω. Moreover, $w = \hat{w}$ (it is nonincreasing in x on $0 \leq x \leq k$ and satisfies (26).

Proof. For a suitable $R_0 > 0$ the inequality (26) guarantees that for $R \leq R_0$, $w_x^2 + w_\eta^2 < r^2$ so that $F(w) = A(w)$ and $F'(w) = A'(w)$ so that w solves $F'(w) = \lambda B'(w)$. If w_n, $n = 1, 2, 3, ..$ is a maximizing sequence for (25) satisfying $|w - w_n|_{W^{1,\infty}} < 1/n$, then $F(w_n) = A(w_n)$ for large n and one can, after a slight renormalisation, use a result on symmetrization (see [14]) to conclude that, without loss of generality, w_n can be replaced by its symmetrisation $\hat{w}_n$ and still yield a maximising sequence. Thus the solution of $A'(w) = \lambda B'(w)$ can be assumed symmetrized and as (26) holds, the solution of $F'(w) = \lambda B'(w)$, as well. The strict positivity of w on Ω_k follows from the strong maximum proneiple. On the boundary where $\eta = 0$ we know w is nonincreasing in x so it suffices to establish that $w(0,k) > 0$. On the top boundary

$$f_2(\nabla w) \equiv \frac{w_\eta}{1 + w_\eta} - \frac{1}{2} \frac{\nabla w}{(1 + w_\eta)^2} = \lambda w.$$

By evenness and periodicity $w_x(0,k) = 0$. If $w(0,k) = 0$, then since f is strictly convex in its variables (for $p_2 > 0$), $w_\eta = 0$ would follow and this would again violate the strong maximum principle.

To establish the remaining assertions of the main theorem one first uses a trial function in the variational principle to show that the Lagrange multiplier λ satisfies $\lambda \leq \sqrt{g}(1 - CR^{4/3})$. With this one can show that $|w|_{L^\infty} \geq C' R^{4/3}$, both of these inequalities being independent of k. The fact that the parameter λ is below the lowest point of the spectrum

of the equation linearized about $w = 0$ allows one to use the exponential decay inherent in the corresponding Green's function to obtain exponential decay for the nonlinear problem. The decay estimates, as well, are independent of k so that by taking limits as $k \to \infty$ one obtains the results for solitary waves. The regularity up to the boundary where $\eta = 0$ follows from results of Lewy [11] (see also [2] where a bootstrap technique is used in a similar situation).

While there are many existence results for traveling wave solutions of the equations for incompressible, inviscid flows, a complementary stability theory is still in its infancy. Here we will show how stability properties of traveling waves relate to the variational structure of equations from hydrodynamics. We shall examine equations generalizing the Korteveg-deVries type. For variational treatment of stability of Euler and related equations we refer the reader to [1].

Consider an evolution equation of the form

(27) $$u_t + u_x - Mu_x + f(u)_x$$

where subscripts denote partial differentiation, $u(x,t)$ is a real function of the real variables x and t, M is a constant coefficient pseudodifferential operator of order $\mu \geq 1$, and f is a smooth function. More precisely, letting $\hat{}$ denote the Fourier transofrm, $\widehat{Mu}(\xi) = |\xi|^\mu \hat{u}(\xi)$. Further $f(0) = f'(0) = 0$, and, in case $\mu = 1$, $|f(s)| \leq O(|s|^p)$ as $|s| \to \infty$ for some $p < \infty$. Such models arise in various areas of mathematical physics. A classical example is that derived by Korteveg and deVries:

$$u_t + u_x + u_{xxx} + (u^2)_x = 0.$$

For the discussion of stability we require an existence result for the equation (27) with initial data $u(x,0) = u_0(x)$. (cf. [9])

Theorem. Let $s > \frac{3}{2}$ and $f \in C^{s+1}$. For each $u_0 \in H^s(\mathbf{R})$, there is a unique solution $u \in C([0,t_*); H^s)$ of (1) with $u(\cdot, 0) = u_0$. Either $t_* = \infty$ or $\|u\|_{H^{\mu/2}} \to \infty$ as $t \to t_*$.

Here solution is understood in the weak sense though for $s > \mu + \frac{3}{2}$ it is classical. In the existence theorem one also establishes the time invariance of certain functionals along solution orbits. The functionals are:

(28) $$\begin{aligned} E(g) &= \int_{-\infty}^{\infty} \left\{ \frac{1}{2} g M g - \frac{1}{2} g^2 - F(g) \right\} dx \\ V(g) &= \frac{1}{2} \int_{-\infty}^{\infty} g^2 dx \\ I(g) &= \int_{-\infty}^{\infty} g dx \end{aligned}$$

where $F' = f$ and $F(0) = 0$. Formally the invariance of E follows from

$$\begin{aligned} \frac{\partial}{\partial t} E(g) &= \int_{-\infty}^{\infty} [Mg - g - f(g)] g_t \\ &= \int_{-\infty}^{\infty} [Mg - g - f(g)] \frac{\partial}{\partial x} (-g + Mg + f(g)) \\ &= \int_{-\infty}^{\infty} \frac{1}{2} \frac{\partial}{\partial x} [Mg - g - f(g)]^2 \\ &= 0. \end{aligned}$$

The functionals V and I are treated similarly.

The search for a solution of the evolution equation (1) in the form $U(x,t) = \phi(x-ct)$ leads to the equation

$$-c\frac{d}{dx}\phi - \frac{d}{dx}M\phi + \frac{d}{dx}\phi + \frac{d}{dx}f(\phi) = 0 \tag{29}$$

in one variable. In the case of KdV it is an ordinary differential equation and one finds a solution $\phi(x) = 3/2(c-1)\text{sech}^2\{1/2(c-1)^{1/2}[x+a]\}$ for $c>1$ any real a. Assuming ϕ and $M\phi$ decay to zero as $|x| \to \infty$ in the general case the last equation implies

$$M\phi + c\phi - \phi - f(\phi) = 0. \tag{30}$$

Let $\langle\cdot,\cdot\rangle$ denote the L^2 inner product as well as its extension to a pairing between $X = H^{\frac{1}{2}\mu}(\mathbf{R})$ and $X^* = H^{-\frac{1}{2}\mu}(\mathbf{R})$. As a mapping from X to $\mathbf{R}$, V is smooth, $V'(u) = u$ and $V''(v) =$ identity, primes denoting Frechet derivatives. With the growth conditions on f one shows $E'(u) \in X^*$, $E''(u)$ exists, and

$$E'(u) = Mu - u - f(u), \quad E''(u) = M - 1 - f'(u).$$

Hence the equation (30) satisfied by the solution ϕ may be expressed as

$$E'(\phi) + cV'(\phi) = 0 \tag{31}$$

which characterizes ϕ as a stationary point of E subject to V being constant. We shall often write ϕ_c to denote a solution of (30) corresponding to a particular value of c. Among our hypotheses is:

H_1 : There is an inteval (c_1, c_2) with $1 \le c_1 < c_2 \le \infty$ such that for every $c \in (c_1, c_2)$ there exists a solution ϕ_c of (30). The curve $c \to \phi_c$ is C^1 into $H^{1+\frac{1}{2}\mu}(\mathbf{R})$, $\phi_c(x) > 0$, $\phi_c \in H^{3+\frac{1}{2}\mu}(\mathbf{R})$, and $(1+|x|)^{1/2}\frac{d\phi_c}{dx} \in L^1(\mathbf{R})$.

An important object is the linearization $\mathcal{L}_c = \mathcal{L}$ of $E' + cV'$ around ϕ_c:

$$\mathcal{L}_c = \mathcal{L} = M + c - 1 - f'(\phi) = E''(\phi) + cV''(\phi)$$

Regarding $\mathcal{L}_c$ we assume

H_2 : The operator $\mathcal{L}_c$ has a unique negative, simple eigenvalue with eigenfunction χ_c. The zero eigenvalue (with eigenfunction $\frac{d}{dx}\phi$) is simple and all the rest of the spectrum of $\mathcal{L}$ is positive and bounded away from zero. Further $c \to \chi_c$ is continuous into $H^{1+\frac{\mu}{2}}(\mathbf{R})$,

It is shown in [4] that H_2 holds for KdV; see the references in [9] for the general equation (27). The main result, under the aforementioned hypotheses, is that ϕ_c is stable if and only if

$$d(c) \equiv E(\phi_c) + cV(\phi_c)$$

is convex in c. We concentrate on the stability and begin with a discussion of the notion of stability. One can see that a definition of stability must be tailored to the particular situation at hand for, in general if $\tilde{c}$ is close to c, and thus $\phi_c(x)$ close to $\phi_{\tilde{c}}(x)$ at $t=0$, the two orbits will not remain close, for the waves travel at different speeds.

Let $(\tau_s f)(x) = f(x+s)$ denote the translation by s of a function on the real line. Define the pseudometric

$$\text{dist}(u,w) \equiv \inf_s \|u - \tau_s w\|_X$$

and, for $\epsilon > 0$ the 'tube'

$$U_\epsilon = \{u \in X : \text{dist}(u, \phi_c) < \epsilon\}$$

consisting of functions 'near' some translate of ϕ_c.

Definition: ϕ_c is stable $\Leftrightarrow \forall\epsilon > 0, \exists\delta > 0$ such that if $u_0 \in U_\delta$, then $u(\cdot,t) \in U_\epsilon$ for all $t \ge 0$.

We are thus using a notion of 'orbital stability'.

Lemma 2. *Suppose $d''(c) > 0$. If $y \in X$ is orthogonal to both ϕ and $\frac{d\phi}{dx}$. Then $\langle\mathcal{L}y, y\rangle > 0$.*

Proof. Since $E'(\phi_c) + cV'(\phi_c) = 0$

$$\begin{aligned} d'(c) &= \langle E'(\phi_c) + cV'(\phi_c), \frac{d\phi_c}{dc}\rangle + V(\phi_c) \\ &= V(\phi_c) = \frac{1}{2}\int \phi_c^2 \end{aligned}$$

and so

$$\begin{aligned} 0 < d''(c) &= \langle \phi_c, \frac{d}{dc}\phi_c\rangle \\ &= -\langle \mathcal{L}\frac{d}{dc}\phi_c, \frac{d}{dc}\phi_c\rangle, \end{aligned}$$

the equation $\mathcal{L}\frac{d\phi_c}{dc} = -\phi_c$ being obtained by differerentiating (30). Write

$$\frac{d\phi}{dc} = a_0\chi + b_0\frac{d\phi}{dx} + p_0$$

where p_0 is in the positive subspace of $\mathcal{L}$. Then

$$\begin{aligned} 0 &< -\langle\mathcal{L}(a_0\chi + b_0\frac{d\phi}{dx} + p_0), a_0\chi + b_0\frac{d\phi}{dx} + p_0\rangle \\ &= -a_0^2(-\lambda^2) - \langle\mathcal{L}p_0, p_0\rangle \end{aligned}$$

where $-\lambda^2$ is the negative eigenvalue of $\mathcal{L}$. Thus

$$\langle\mathcal{L}p_0, p_0\rangle < a_0^2\lambda^2.$$

Since y is orthogonal to $\frac{d\phi}{dx}$, $y = a\chi + p$ with p in the positive subspace of $\mathcal{L}$. Further

$$0 = -\langle\phi, y\rangle = \langle\mathcal{L}\frac{d\phi}{dc}, y\rangle = -a_0 a\lambda^2 + \langle\mathcal{L}p_0, p\rangle$$

implies that

$$\begin{aligned}\langle \mathcal{L}y, y\rangle &= -a^2\lambda^2 + \langle \mathcal{L}p, p\rangle \\ &\geq -a^2\lambda^2 + \langle \mathcal{L}p, p_0\rangle^2/\langle \mathcal{L}p_0, p_0\rangle \\ &> -a^2\lambda^2 + \frac{(a_0 a\lambda^2)^2}{a_0^2\lambda^2} \\ &= 0.\end{aligned}$$

Using the implicit function theorem one shows

Lemma 3. *There exists $\epsilon > 0$ and a unique C^1 map $\alpha : U_\epsilon \to \mathbf{R}$ such that for every $u \in U_\epsilon$ and $r \in \mathbf{R}$*

(i) $\langle u(\cdot + \alpha(u)), \frac{d\phi}{dx}\rangle = 0.$

(ii) $\alpha(u(\cdot + r)) = \alpha(u) - r.$

Lemma 4. *Suppose $d''(c) > 0$. Then there are constants $C > 0$ and $\epsilon > 0$ such that*

$$E(u) - E(\phi) \geq C\|u(\cdot + \alpha(u)) - \phi\|^2_{H^{\mu/2}}$$

for all $u \in U_\epsilon$ which satisfy $V(u) = V(\phi)$.

Proof. Write $u(\cdot + \alpha(u)) = (1 + a)\phi + y$ when $\langle y, \phi\rangle = 0$ and a is a scalar. Since V is translation invariant

$$\begin{aligned}V(\phi) = V(u) &= V(u(\cdot + \alpha(u))) \\ &= V(\phi) + \langle \phi, v\rangle + \frac{1}{2}\|v\|^2_{L^2}\end{aligned}$$

where $v = u(\cdot + \alpha(u)) - \phi = a\phi + y$. Since $\langle \phi, v\rangle = a\|\phi\|^2_{L^2}$, $a = 0(\|v\|^2_{L^2})$. Hereafter let $\|.\|$ denote the norm in X. Using $L \equiv E + cV$ as a functional on $X = H^{\mu/2}(\mathbf{R})$ another expansion yields

$$L(u) = L(u(\cdot + \alpha(u))) = L\phi + \frac{1}{2}\langle \mathcal{L}v, v\rangle + o(\|v\|^2)$$

since the linear term in v is $\langle E' + cV', v\rangle = 0$. Since $V(v) = V(\phi)$

$$E(u) - E(\phi) = \frac{1}{2}\langle \mathcal{L}v, v\rangle + o(\|v\|^2).$$

But

$$\begin{aligned}\langle \mathcal{L}v, v\rangle &= \langle \mathcal{L}(y + a\phi), y + a\phi\rangle \\ &= \langle \mathcal{L}y, y\rangle + 2\langle \mathcal{L}(y + a\phi), a\phi\rangle - \langle \mathcal{L}(a\phi), a\phi\rangle \\ &= \langle \mathcal{L}y, y\rangle + O(a\|v\|_X) + O(a^2) \\ &= \langle \mathcal{L}y, y\rangle + o(\|v\|^2_X).\end{aligned}$$

Since y is orthogonal to ϕ and $\frac{d\phi}{dx}$, by Lemma 2

$$E(u) - E(\phi) \geq 2C\|y\|^2 + o(\|v\|^2)$$

and since $\|y\| = \|v - a\phi\| \geq \|v\| - O(\|v\|^2)$ one obtains

$$E(u) - E(\phi) \geq C\|v\|^2.$$

Theorem 5. *Suppose $d''(c) > 0$. Then ϕ_c is stable.*

Proof. Given $\epsilon > 0$, let $u_n^0 \in X$ be any sequence such that

$$\text{dist}(u_n^0, \phi) \to 0 \quad \text{as} \quad n \to \infty$$

and let u_n be the solution of the evolution equation with initial data u_n^0. If ϕ is not atable then for each n there is a time t_n such that

$$\text{dist}(u_n(\cdot, t_n), \phi) = \frac{\epsilon}{2}. \tag{32}$$

Since E and V are continuous on X and translation invariant

$$E(u_n(\cdot, t_n)) = E(u_n^0) \to E(\phi), V(u_n(\cdot, t_n)) = V(u_n^0) \to V(\phi)$$

as $n \to \infty$. Next choose $w_n \in U_\epsilon$ so that $V(w_n) = V(\phi)$ and $\|w_n - u_n(\cdot, t_n)\| \to 0$ as $n \to \infty$. By the previous lemma

$$\begin{aligned}O \leftarrow E(w_n) - E(\phi) &\geq C\|w_n(\cdot + \alpha(w_n)) - \phi\|^2 \\ &= C\|w_n - \phi(\cdot - \alpha(w_n))\|^2\end{aligned}$$

and so $\|u_n(\cdot, t_n) - \phi(\cdot - \alpha(w_n))\| \to 0$. This contradicts (32), showing that ϕ is stable.

The solution of (30) with $f(u) = u^p$ can be written

$$\phi_c(x) = (c-1)^{2/(p-1)}v((c-1)1/\mu x)$$

where v solves

$$Mv + (c-1)v - v^p = 0.$$

In [9] it is shown that

$$d(c) = \frac{1}{2}\mu(c-1)^{2/(p-1)+1-1/\mu}\langle Mv, v\rangle$$

from which one obtains

Theorem 6. *For $f(u) = u^p$ the solitary wave solution of (27) is stable if $1 < p < 2\mu + 1$.*

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