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Shape Optimization for Dirichlet Problems: Relaxed Solutions and Optimality Conditions

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Abstract: We study a problem of shape optimal design for an elliptic equation with Dirichlet boundary condition. We introduce a relaxed formulation of the problem which always admits a solution, and we find necessary conditions for optimality both for the relaxed and the original problem.

Let Ω be a bounded open subset of $\mathbf{R}^n$ $(n \geq 2)$, let $f \in L^2(\Omega)$, and let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function such that

$$|g(x,s)| \leq a_0(x) + b_0|s|^2 \qquad \forall (x,s) \in \Omega \times \mathbf{R}, \tag{1}$$

for suitable $a_0 \in L^1(\Omega)$ and $b_0 \in \mathbf{R}$. We consider the following optimal design problem:

$$\min_{A \in \mathcal{A}(\Omega)} \int_\Omega g(x, u_A(x))\,dx, \tag{2}$$

where $\mathcal{A}(\Omega)$ is the family of all open subsets of Ω, and u_A is the solution of the Dirichlet problem

$$-\Delta u_A = f \ \text{ in } A, \qquad u_A \in H_0^1(A), \tag{3}$$

extended by 0 in $\Omega \setminus A$.

It is well-known that, in general, the minimum problem (2) has no solution (see for instance Example 2). The reason is that, if we try to apply the direct method of the calculus of variations, we find that every minimizing sequence (A_h) has a subsequence such that the corresponding solutions u_{A_h} of (3) converge weakly in $H_0^1(\Omega)$ to a function u. But, in general, we can not find an open subset A of Ω such that u is the solution u_A of problem (3). On the contrary, it can be proved (see [4]) that the limit function u is the solution of a relaxed Dirichlet problem of the form

$$-\Delta u + \mu u = f \ \text{ in } \Omega, \qquad u \in H_0^1(\Omega) \cap L^2(\Omega;\mu), \tag{4}$$

for a suitable nonnegative measure μ which vanishes on all sets of (harmonic) capacity 0, but may take the value $+\infty$ on some non polar subsets of Ω. We may also assume that for every $E \subset \Omega$

$$\mu(E) = \inf\{\mu(A) \,:\, A \text{ finely open}, E \tilde{\subset} A\},$$

where $E \tilde{\subset} A$ means that $E \setminus A$ has capacity 0. For the definition and properties of the fine topology we refer to Doob [5], Part 1, Chapter XI. Following [3], we shall denote by $\mathcal{M}_0^*(\Omega)$ the class of all measures with the properties considered above.

The precise meaning of equation (4) is the following:

$$(5) \qquad \int_\Omega Du D\varphi dx + \int_\Omega u\varphi d\mu = \int_\Omega f\varphi dx \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega;\mu),$$

where the pointwise value of an H^1 function is defined as usual up to sets of capacity 0.

If S is a finely closed subset of Ω, the measure ∞_S defined by

$$(6) \qquad \infty_S(B) = \begin{cases} 0 & \textit{if } B \cap S \textit{ has capacity } 0 \\ +\infty & \textit{otherwise,} \end{cases}$$

belongs to $\mathcal{M}_0^*(\Omega)$. If, in addition, S is closed in Ω, then problem (4) reduces to problem (3) with $A = \Omega \setminus S$ and $\mu = \infty_S$. The relaxed formulation of the optimization problem (2) is then:

$$(7) \qquad \min_{\mu \in \mathcal{M}_0^*(\Omega)} \int_\Omega g(x, u_\mu(x))\,dx,$$

where u_μ is the unique solution of the relaxed Dirichlet problem (4) in the sense given by (5).

The following theorem follows easily from the compactness and density results for relaxed Dirichlet problems proved in [1] (Theorem 2.38) and [4] (Theorem 4.16).

Theorem 1. *Problem (7) admits a solution, and*

$$(8) \qquad \min_{\mu \in \mathcal{M}_0^*(\Omega)} \int_\Omega g(x, u_\mu(x))\,dx = \inf_{A \in \mathcal{A}(\Omega)} \int_\Omega g(x, u_A(x))\,dx.$$

Similar relaxed formulations for different classes of optimal design problems have been considered by Murat and Tartar in [8],[11],[12],[14], and by Kohn, Strang, and Vogelius in [6],[7]. We now give an example where problem (2) has no solution.

Example 2. Assume that $f(x) > 0$ a.e. in Ω, let w be the solution of

$$(9) \qquad -\Delta w = f \ \text{ in } \Omega, \qquad w \in H_0^1(\Omega),$$

and let $g(x,s) = |s - cw(x)|^2$, with $0 < c < 1$. Then the relaxed problem (7) attains its minimum value 0 at the measure μ defined by

$$\mu(B) = \frac{1-c}{c} \int_B \frac{f}{w}\,dx$$

which corresponds to $u_\mu = cw$. On the other hand, it is clear from (3) and (9) that there are no domains A for which $g(x, u_A(x)) = 0$ a.e. in Ω. By (8) this implies that the original problem (2) has no solution.

We now give some optimality conditions for the solutions of problem (7). Let μ be a minimum point of (7) and let $u = u_\mu$. By a general result concerning measures of $\mathcal{M}_0^*(\Omega)$ (see [3]), μ can be decomposed in the form

$$(10) \qquad \mu = \infty_S + \mu_A$$

where A is the set of all $x \in \Omega$ having a fine neighbourhood V such that $\mu(V) < +\infty$, μ_A is the restriction of μ to A, and $S = \Omega \setminus A$. It is clear that A is finely open, hence S is finely closed in Ω. By $\partial^* A$ and $cl^* A$ we denote the fine boundary and the fine closure of A in Ω.

Proposition 3. *There exist a Radon measure $\nu \in \mathcal{M}_0^*(\Omega)$ carried by $\partial^* A$, and a continuous linear map $T : L^2(\Omega) \to L^2(\partial^* A, \nu)$ such that, if $h \in L^2(\Omega)$ and $w \in H_0^1(\Omega) \cap L^2(\Omega; \mu)$ is a solution of*

$$-\Delta w + \mu w = h \quad in\, \Omega$$

in the sense given by (5), then

$$\int_A Dw D\varphi dx + \int_{\partial^* A} T(h) \varphi d\nu + \int_A w\varphi d\mu_A = \int_{cl^* A} h\varphi dx \qquad \forall \varphi \in H_0^1(\Omega).$$

If A is an open set with a smooth boundary and $\mu_A(B) = \int_{B\cap A} \vartheta dx$ with $\vartheta \in L^\infty(\Omega)$, an integration by parts gives that we can take

$$(11) \qquad \nu(B) = -\int_{B\cap\partial A} \frac{\partial W}{\partial n} d\sigma, \qquad T(h) = \frac{\frac{\partial w}{\partial n}}{\frac{\partial W}{\partial n}}$$

where σ denotes the surface measure on the (euclidean) boundary ∂A of A, n is the outer unit normal to A, and W is the solution of the Dirichlet problem

$$(12) \qquad -\Delta W = 1 \quad in\, A, \qquad W \in H_0^1(A).$$

In addition to the previous hypotheses, we assume now that $g(x, s)$ is continuously differentiable with respect to s and that

$$|g_s(x, s)| \le a_1(x) + b_1|s| \qquad \forall (x, s) \in \Omega \times \mathbf{R}$$

for suitable $a_1 \in L^1(\Omega)$ and $b_1 \in \mathbf{R}$.

In order to give our optimality conditions, we introduce the adjoint equation

$$(13) \qquad -\Delta v + \mu v = g_s, \qquad v \in H_0^1(\Omega) \cap L^2(\Omega; \mu)$$

where g_s denotes the function $g_s(x, u(x))$. We denote by v the solution of (13) in the sense given by (5), with f replaced by g_s, and we set

$$(14) \qquad \alpha = T(f), \qquad \beta = T(g_s).$$

Our main result is the following theorem.

Theorem 4. *Let $\mu = \infty_S + \mu_A$ be a solution of problem (7), let $u = u_\mu$ be the corresponding solution of (4), and let v be the solution of the adjoint equation (13). Then $u = v = 0$ on S (up to a set of capacity 0), and*

(a) $\quad uv \le 0$ *a.e. on A,*

(b) $\quad \alpha\beta \ge 0$ *ν-a.e. on $\partial^* A$,*

(c) $\quad f(x) g_s(x, 0) \ge 0$ *a.e. on $\Omega \setminus cl^* A$,*

(d) $\quad uv = 0$ *μ_A-a.e. on A,*

where α and β are given by (14).

Suppose now that there exists an optimal domain A for the original problem (2), and that A has a smooth boundary. By (8) the measure $\mu = \infty_S$ defined by (6) with $S = \Omega \setminus A$ is a minimum point of the relaxed problem (7). Taking (11) into account, the optimality condition of Theorem 4 become:

(a') $uv \leq 0$ *a.e. on* A,

(b') $\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} \geq 0$ σ *-a.e. on* $\Omega \cap \partial A$,

(c') $f(x)g_s(x,0) \geq 0$ *a.e. on* $\Omega \setminus A$,

while condition (d) is trivial because $\mu_A = 0$. From *(a')* and *(b')* we obtain

(b") $\frac{\partial u}{\partial n}\frac{\partial v}{\partial n} = 0$ σ*-a.e. on* $\Omega \cap \partial A$.

The last condition is already known in shape optimization (see for instance [2], [9], [10], [13], [15]), while conditions *(a')* and *(c')* seem to be new.

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